Fall 2007 Micro
Prelim Solutions

(Use at own risk. The graduate students who derived these had no answer keys.)
Does not include II.2

Fall 2007, Question I.1

Let $\succeq$ be a reflexive, transitive, complete and strictly increasing (i.e., strongly monotone) preference relation on the consumption set $X = \mathbb{R}^L_+$. Preference relation $\succeq$ is said to be *homoethetic* if the following holds for every $x, x' \in \mathbb{R}^L_+$ and every $\lambda > 0$:

$$\text{if } x \sim x' \text{ then } \lambda x \sim \lambda x'$$

where $\sim$ is the indifference relation of $\succeq$.

Prove that $\succeq$ is homothetic if and only if there exists a **utility representation** u of $\succeq$ such that u is homogeneous of degree 1.

Solution

Before answering all parts of the question I need the following definitions:

Definition 1. $u(\cdot)$ is a utility representation of $\succeq$ if for all $x, x' \in X$, $x \succeq x' \Leftrightarrow u(x) \geq u(x')$.

Theorem 1. *If the utility function is homogenous of degree 1 (HD1), then $\succeq$ is homothetic.*

Proof. By the utility function representing preferences, we know that $u(x) = u(x') \Rightarrow x \sim x'$. By the utility function being HD1 we know that $\lambda u(x) = u(\lambda x)$, which gives:

$$\begin{aligned} u(x) &= u(x') \\ \lambda u(x) &= \lambda u(x') \\ u(\lambda x) &= u(\lambda x') \\ &\Rightarrow \lambda x \sim \lambda x' \end{aligned}$$

Putting all these together gives

$$\begin{array}{rcl} u(x) = u(x') & \text{then} & u(\lambda x) = u(\lambda x') \\ \text{therefore} \quad x \sim x' & \text{then} & \lambda x \sim \lambda x', \end{array}$$

which is the desired result. □

Theorem 2. *If $\succeq$ is homothetic, there exists a utility function $u(\cdot)$ such that $u(\cdot)$ is HD1.*

NOTE: There is an important subtlety in this question. We DON'T need to show that a $u(\cdot)$ which is HD1 is the ONLY utility function that can represent $\succeq$. We only need to show that a HD1 utility function CAN represent it.

Proposition 1. *There exists a utility function that represents $\succeq$.*

NOTE: Here we need to assume continuity!

Proof. Let $e \equiv \{1, 1, \ldots, 1\} \in \mathbb{R}^L_+$. Define:

$$A^+(x) = \{y : e \cdot y \succeq x\} \quad \text{and} \quad A^-(x) = \{y : x \succeq e \cdot y\}.$$

By continuity of preferences, we know that the upper and lower contour set of $\succeq$ is closed $\Rightarrow A^+(x)$ and $A^-(x)$ are closed. By completeness of $\succeq$, $A^+(x) \bigcup A^-(x) = X = \mathbb{R}^L_+$. By connectedness of $\mathbb{R}^L_+$, $A^+(x) \bigcap A^-(x) \neq \emptyset$. Let $\alpha(x) \in A^+(x) \bigcap A^-(x)$. Then, consider the following

Claim 1. *$\alpha(x)$ is unique.*

Proof. Suppose not. Then $\exists$ $\alpha'(x)$ such that $\alpha'(x) \neq \alpha$, but $\alpha'(x) \cdot e \sim x$. WLOG, let $\alpha'(x) > \alpha(x)$. Then $\alpha'(x) \cdot e > \alpha(x) \cdot e \Rightarrow \alpha'(x) \cdot e \succeq \alpha(x) \cdot e$ by monotonicity of preferences. But then $x \sim \alpha'(x) \cdot e \succeq \alpha(x) \cdot e \sim x$, a contradiction. □

Claim 2. *$u(x) = \alpha(x)$ represents $\succeq$.*

Proof. We do it in steps:

(i) $x \succeq x' \Rightarrow u(x) \geq u(x')$

(ii) $x \succeq x' \Rightarrow e \cdot \alpha(x) \succeq e \cdot \alpha(x')$

(iii) $\Rightarrow e \cdot \alpha(x) \succeq e \cdot \alpha(x')$ by monotonicty of preferences

(iv) $\Rightarrow \alpha(x) \succeq \alpha(x')$

(v) $\Rightarrow u(x) \geq u(x')$,

so $u(x) = \alpha(x)$ represents $\succeq$. □

Then, we have that

$$\begin{aligned} u(x) \geq u(x') &\Rightarrow x \succeq x' \\ u(x) \geq u(x') &\Rightarrow \alpha(x) \geq \alpha(x') \\ &\Rightarrow e \cdot \alpha(x) \geq e \cdot \alpha(x') \\ &\Rightarrow e \cdot \alpha(x) \succeq e \cdot \alpha(x') \text{ (by monotonicity of preferences)} \end{aligned}$$

so $e \cdot \alpha(x) \succeq e \cdot \alpha(x') \Rightarrow x \succeq x'$. □

Proposition 2. *Of all the $u(\cdot)$ that represent $\succeq$, at least one is HD1.*

Proof. ($\Rightarrow$) Let $\succeq$ be homothetic. WTS: $u(\cdot)$ can be HD1. Let $x \sim x'$ and $\lambda x \sim \lambda x'$. Then $u(x) = u(x')$ and $u(\lambda x) = u(\lambda x')$. Clearly, by the work shown above, $u(\cdot)$ being HD1 satisfies this. ($\Leftarrow$) Let $u(\cdot)$ be HD1. WTS: $\succeq$ is homothetic. Let $u(x) = u(x')$, and $u(\lambda x) = u(\lambda x')$. Then $x \sim x' \Rightarrow \lambda x \sim \lambda x'$. □

Fall 2007, Question I.2

Consider two real-valued random variables Y and Z on a probability space. You may think about Y and Z as two contingent claims on a state space. You may assume that the state space is finite. Suppose that Y can take only one of two possible values y_1, y_2 with respective probabilities $\pi_1 > 0$ and $\pi_2 > 0$ such that $\pi_1 + \pi_2 = 1$. Suppose further that the expectations of Z conditional on $\{Y = y_1\}$ and $\{Y = y_2\}$ are zero, that is, $E[Z|Y = y_1] = 0$ and $E[Z|Y = y_2] = 0$.

(a) Prove that $Y + Z$ is <u>**more risky**</u> (in the sense of second-order stochastic dominance) than Y.

(b) Prove that $Y + 2Z$ is more risky than $Y + Z$.

Solution

First I'll give the definition:

Definition 2. Let A and B be two random variables on a state space. Random variable A is more risky than random variable B if there exists a random variable ξ such that

$$A - E(A) =^d B - E(B) + \xi \quad \text{and} \quad E(\xi|B) = E(\xi) = 0,$$

where "$=^d$" means equality in distribution and $E(\xi|B) = E(\xi) = 0$ states that ξ is mean independent from B (this is, the expectation of ξ conditional on (any realization of) B does not depend on B). If the equation above holds and ξ is not the zero random variable, then A is strictly riskier than B.

Given the assumptions of the question, Z is mean-independent from Y. To solve this problem, I'll use the following Lemma, which I state without proof:

Lemma 1. *Let A and B be two random variables with the same expectation. Random variable A is strictly riskier than random variable B iff every strictly risk-averse agent strictly prefers B to A.*

To prove the statement, I'll assume $E(Z) = 0$ and I'll also need the following

Proposition 3. *For any random variable Q, if $\varepsilon \neq 0$ is mean independent of Q and $E(\varepsilon) = 0$, then $Q + \lambda\varepsilon$ is strictly riskier than $Q + \gamma\varepsilon$ for every $\lambda > \gamma \geq 0$.*

Proof. Let $a = \gamma/\lambda$. Then $Q + \gamma\varepsilon = a(Q + \lambda\varepsilon) + (1 - a)Q$. Since $0 \leq a < 1$, for every strictly concave utility function v we have

$$v(Q + \gamma\varepsilon) \geq av(Q + \lambda\varepsilon) + (1 - a)v(Q) \tag{1}$$

(note this is a vector inequality). Taking expectations on both sides of (1), we get

$$E[v(Q + \gamma\varepsilon)] \geq aE[v(Q + \lambda\varepsilon)] + (1 - a)E[v(Q)]. \tag{2}$$

Since $Q + \lambda\varepsilon$ is strictly riskier than Q, from Lemma 1 we have that $E[v(Q)] > E[v(Q + \lambda\varepsilon)]$. Using this in (2),

$$E[v(Q + \gamma\varepsilon)] > E[v(Q + \lambda\varepsilon)].$$

Then, using Lemma 1 again, $Q + \lambda\varepsilon$ is strictly riskier than $Q + \gamma\varepsilon$. □

Remark 1. Because expectations do not matter in orderings by riskiness, Proposition 3 remains true for any $\varepsilon \neq 0$ that is mean independent of Q even if $E(\varepsilon) \neq 0$.

The solution now follows directly:

(a) Follows from Proposition 3 with $Q = Y$, $\varepsilon = Z$, $\lambda = 1$ and $\gamma = 0$.

(b) Follows from Proposition 3 with $Q = Y$, $\varepsilon = Z$, $\lambda = 2$ and $\gamma = 1$.

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I.2 (a) $E[v(Y+Z)] = \pi_1 E[v(Y+Z)|Y=y_1] + \pi_2 E[v(Y+Z)|Y=y_2]$

$\forall$ concave v by Jensen's Inequality

$$v[E(Y+Z|Y=y_1)] \geq E[v(Y+Z)|Y=y_1]$$

$$v(y_1+E(Z|Y=y_1)) \geq E[v(Y+Z)|Y=y_1]$$

$$v(y_1) \geq E[v(Y+Z)|Y=y_1]$$

similarly $v(y_2) \geq E[v(Y+Z)|Y=y_2]$

$$\pi_1 v(y_1) + \pi_2 v(y_2) \geq \pi_1 E[v(Y+Z)|Y=y_1] + \pi_2 E[v(Y+Z)|Y=y_2]$$

$$E[v(Y)] \geq E[v(Y+Z)] \quad \forall \text{ concave } v$$

Y SSD $Y+Z$ so $Y+Z$ is more risky than Y

(b) $Y+Z = \frac{1}{2}(Y+2Z) + \frac{1}{2}Y$

$v(Y+Z) \geq \frac{1}{2}v(Y+2Z) + \frac{1}{2}v(Y)$ $\forall$ concave v

$E[v(Y+Z)] \geq \frac{1}{2}E[v(Y+2Z)] + \frac{1}{2}E[v(Y)]$

$$\frac{1}{2}E[v(Y+Z)] + \underbrace{\frac{1}{2}E[v(Y+Z)] - \frac{1}{2}E[Y]}_{>0 \text{ by part a}} \geq \frac{1}{2}E[Y+2Z]$$

$$\frac{1}{2}E[v(Y+Z)] \geq \frac{1}{2}E[Y+2Z]$$

$$E[v(Y+Z)] \geq E[Y+2Z] \quad \forall \text{ concave } v$$

$Y+Z$ SSD $Y+2Z$ so $Y+2Z$ is more risky than $Y+Z$

Microeconomics Prelim

Fall 2007

February 8, 2008

1 Question I.1

The statement asked to prove is false.

Consider lexicographic preferences:

$$x \succeq y \text{ if either } x_1 > y_1 \text{ or } x_1 = y_1 \text{ and } x_2 \geq y_2$$

Clearly, these preferences are reflexive, transitive, complete, and strictly increasing. They are also homothetic. Consider x, x'such that$x \sim x'$. It must be that $x_1 = x_1'$ and $x_2 = x_2'$. So we have $\lambda x_1 = \lambda x_1'$ and $\lambda x_2 = \lambda x_2'$ so $\lambda x \sim \lambda x'$.

We have shown that lexicographic preferences hold all of the assumptions. Yet lexicographic preferences are not continuous and have no utility representation.

Proof Suppose a utility representation does exist.

$\forall x_1$, we can choose $r(x_1) \in \mathbb{Q}$ such that $u(x_1, 2) > r(x_1) > u(x_1, 1)$.

If $x_1 > x_1', r(x_1) > r(x_1')$ since $r(x_1') > u(x_1, 1) > u(x_1', 2) > r(x_1')$.

Therefore, r(.) is a one-to-one function from $\mathbb{Q} \rightarrow \Re \quad \rightarrow\leftarrow$

So we have found an example of a preference relation for which all of the assumptions hold, but does not have utility representation of any type.

2 Question I.2

Definition: $y + z$ is strictly riskier than y iff every strictly risk averse agent strictly prefers y to $y + z$

Lemma For any plan y, if $\epsilon \neq 0$ is mean independent of y and $E(\epsilon) = 0$, then $y + \lambda\epsilon$ is strictly riskier than $y + \gamma\epsilon \quad \forall \lambda > \gamma \geq 0$.

Proof Let $a = \gamma/\lambda$.

Then

$$y + \gamma\epsilon = a(y + \lambda\epsilon) + (1 - a)y \tag{2.1}$$

Because $0 \leq a \leq 1$, for strict concave utility (strictly risk averse agents):

$$u(y + \gamma\epsilon) > aE(u(y + \lambda\epsilon)) + (1 - a)E(u(y)) \tag{2.2}$$

Taking expectations

$$E(u + \gamma\epsilon) > aE(u(y + \lambda\epsilon)) + (1 - a)E(u(y)) \tag{2.3}$$

We know $y + \lambda\epsilon$ is riskier than y, so $E[u(y)] > E[u(y + \lambda\epsilon)]$. So, certainly,

$$E[u(y + \gamma\epsilon)] > E[u(y + \lambda\epsilon)] \tag{2.4}$$

(a) Using the lemma, let $z = \epsilon$ $(E(z|y) = 0, E(z) = 0)$. Let $\lambda = 1, \gamma = 0$

(b) Using lemma, let $z = \epsilon, \lambda = 2, \gamma = 1$.

3 Question III.1

(a)

Just see Ariel's Page (c)

Consider the following two-player infinite horizon game:

Let $A^i = [0, \frac{1}{2}]$ and $i = 2$ and u^i be defined as follows, where $a^i = x$ and $a^{-i} = y$:

$u^i(a) = x$ if $x < \frac{y}{2}$

$= \frac{y(1-x)}{2-y}$ if $x \geq \frac{y}{2}$

Then $(0, 0)$ is the only Nash Equilibrium and it is weakly dominated. I'm feeling lazy now and don't want to TeX the proof for this.

Fall 2007, Question II.1

This question applies to pure exchange economies with ℓ commodities and n traders, $i = 1, \ldots, n$, each having initial endowment vector $e_i \in \mathbb{R}^\ell$ and preferences $\succeq_i$ which are assumed throughout to be continuous complete preorders on the consumption set $X_i \subseteq \mathbb{R}^\ell$.

(a) State the first welfare theorem.

(b) Prove the first welfare theorem.

(c) Does the conclusion of the first welfare theorem hold when the following complications separately (one at a time) are present?

Justify each one by one of the following methods: pointing out that it doesn't affect your proof, explaining how your proof can be modified to encompass the complication, providing a counterexample to the conclusion of the first welfare theorem when this one complication is present, or explaining precisely how the complication prevents your proof from being modified to demonstrate that the first welfare theorem holds despite the complication. The complications are as follows:

(i) preferences that are convex but not strictly/strongly convex

(ii) preferences that are weakly convex but not convex

(iii) preferences that are nonsatiated but not locally nonsatiated

(iv) preferences that are strictly monotone but consumption sets X_i are not necessarily convex

(d) Briefly discuss (i.e., in an essay of 50-300 words) the economic significance of the first welfare theorem.

Solution

Before answering all parts of the question I need the following definitions:

Definition 3. Define an allocation as a vector $x \in \mathbb{R}^{\ell n}$. An allocation x is Pareto efficient if it is feasible (i.e., if $\sum_i x_i \leq \sum_i e_i$) and if there is no other feasible allocation x' such that $x'_i \succeq_i x_i$ for all i and $x'_i \succ_i x_i$ for some i.

Definition 4. A competitive equilibrium is a price vector $p^* \in \mathbb{R}^\ell \backslash \{0\}$ and an allocation $x^* \in \mathbb{R}^{\ell n}$ such that

(i) $x_i^* \in \mathbb{R}^\ell$ for all i,

(ii) $x_i^* \in x_i(p^*, e_i, \succeq_i)$ for all i, where $x_i(p^*, e_i, \succeq_i)$ represents the trader's demand, and

(iii) $\sum_i x_i^* \leq \sum_i e_i$.

Definition 5. Let $\succeq_i$ be a continuous complete preorder on the consumption set $X_i \in \mathbb{R}^\ell$. Preferences $\succeq_i$ are locally nonsatiated (lns) if for all $x_i \in X_i$ and all $\varepsilon > 0$ there exists some $x'_i \in X_i$ such that $\|x_i - x'_i\| \leq \varepsilon$ and $x'_i \succ_i x_i$.

(a) Here I state the first welfare theorem.

Theorem 3. *Consider a pure exchange economy with ℓ commodities and n traders, $i = 1, \ldots, n$, each having initial endowment vector $e_i \in \mathbb{R}^\ell$ and preferences $\succeq_i$ which are assumed throughout to be continuous complete locally nonsatiated preorders on the consumption set $X_i = \mathbb{R}^\ell$. Let $x \in \mathbb{R}^{\ell n}$ be a competitive equilibrium allocation. Then x is Pareto efficient.*

(b) *Proof.* Suppose x^* is not Pareto efficient. Then there exists another feasible allocation $x' \in \mathbb{R}^{\ell n}$ such that $x'_i \succeq_i x^*_i$ for all i and $x'_i \succ_i x^*_i$ for some i. Using this information, we get

$$p^* \cdot x'_i \geq p^* \cdot x^*_i = p^* \cdot e_i \qquad \text{for all } i, \tag{3}$$

$$p^* \cdot x'_i > p^* \cdot x^*_i = p^* \cdot e_i \qquad \text{for some } i, \tag{4}$$

where (3) follows from continuity and lns of $\succeq_i$, and (4) follows from the utility maximization properties of the competitive equilibrium allocation. Adding up (3) and (4) gives

$$\sum_i p^* \cdot x'_i > \sum_i p^* \cdot e_i$$

which, given $p^* \neq 0$, is a contradiction to the feasibility of x'. □

(c) Here I analyze each of the four complications:

(i) The conclusion still holds. Strict convexity of $\succeq_i$ is not an assumption that plays any role on the proof of the theorem.

(ii) The conclusion need not hold. Consider the following definition:

Definition 6. Preferences $\succeq$ are weakly convex if for all $x, y \in \mathbb{R}^\ell$ with $x \neq y$ and all $\lambda \in (0, 1)$, if $x \succeq y$ then $\lambda x + (1 - \lambda)y \succeq y$.

While it is true that convexity of $\succeq_i$ does not play a role in the first welfare theorem, preferences that are weakly convex cannot rule out "thick" indifference curves, in which case lns does not hold (see more details below).

(iii) The conclusion need not hold. Under nonsatiated but not locally nonsatiated preferences, $x'_i \succeq x^*_i$ for all i does not necessarily imply $p^* \cdot x'_i \geq p^* \cdot x^*_i$ for all i. (Think again of "thick" indifference curves: Preferences are nonsatiated but depending on the position of the budget line and the indifference curve, allocation x'_i may even be cheaper than x^*_i.)

(iv) The conclusion need not hold.

First, note that nonconvex consumption sets cause difficulties for existence of a competitive equilibrium. If the equilibrium exists, then the conclusions of the theorem pass through. However, if the nonconvexity does not allow for an equilibrium to exist, then, not too much can be made out of the theorem.

Now, if a CE does exist, a simple Edgeworth box argument can provide an example of a CE that is not PO. Thus, consider an Edgeworth box economy with $e \gg 0$ and a price vector $p \gg 0$, as the one shown in Figure 1 below.

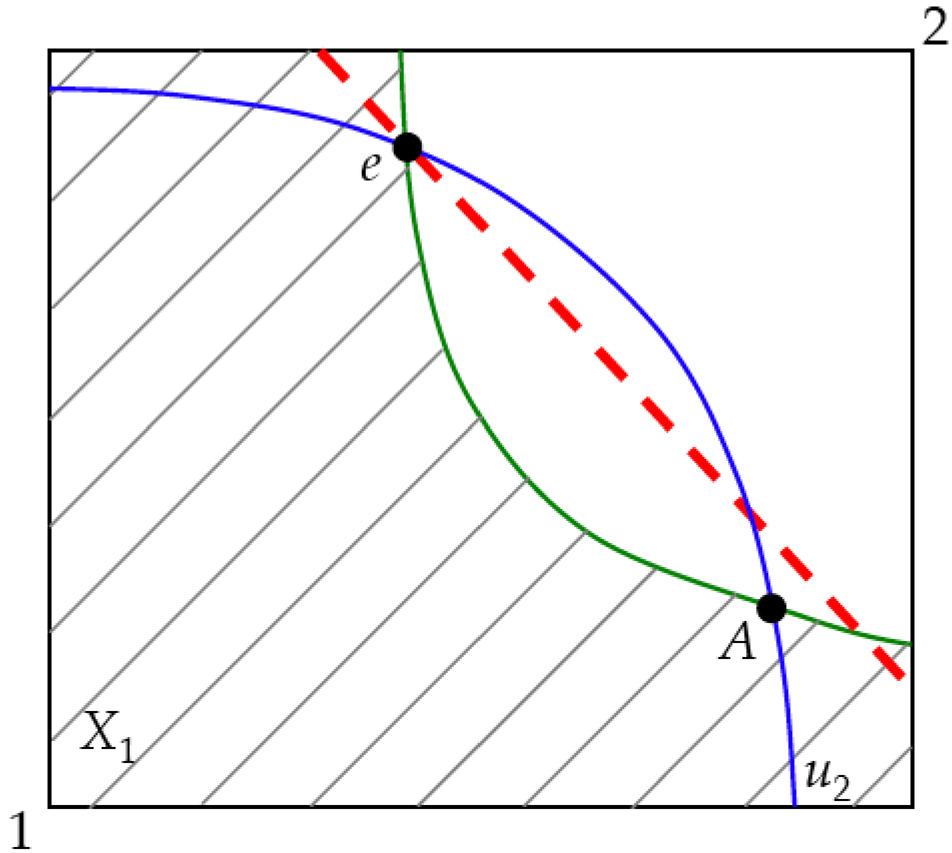

Figure 1: An Edgeworth box economy where the CE is not PO.

In this case, the point e represents the endownment, and the red (dashed) line is the price line. Consumer 2 has the entire Edgeworth box as his consumption set, and his indifference curves are labeled in blue. For consumer 1, his consumption set X_1 is nonconvex, and is represented by the dashed area. Consumer 1's indifference curves are labeled in green.

Clearly, the endowment point is a CE; however, any allocation in the segment $\overline{eA}$ over the green line is a Pareto improvement for consumer 2. Thus, the CE is not PO.

(d) I guess it's just about writing up some nice poetry!

Fall 2007 Prelim - Micro (Majors)

Q2.1 (a) Consider a pure economy with l commodities and n traders, each having an initial endowment $e_i \in \mathbb{R}^l$ and continuous, complete preferences $\succsim_i$ on the consumption set $X_i \subseteq \mathbb{R}^l$. If $\succsim_i$ are locally nonsatiated, then any competitive equilibrium is Paerto optimal.

(b) Let x^* be a competitive equilibrium allocation. For contradiction suppose x^* is not Paerto efficient. Then $\exists \hat{x} \in \{x \mid \sum_i x_i \leq \sum_i e_i\}$ such that $\hat{x}_i \succsim x_i^* \; \forall i$ and $\hat{x}_j \succ x_j^*$ for some j

Since $x_i^* \succsim_i x_i$ for any $x_i \in \{x_i \in X_i \mid p^* \cdot x_i \leq p^* \cdot e_i\}$, $\hat{x}_j \notin \{x_i \in X_i \mid p^* \cdot x_i \leq p^* e_i\}$. Thus $p^* \cdot \hat{x}_j > p^* \cdot x_j^*$.

Subclaim $p^* \cdot \hat{x}_i \geq p^* \cdot x_i^* \; \forall i$

Suppose for contradiction $p^* \cdot x_i^* > p^* \cdot \hat{x}_i$. Since x^* is a competitive equilibrium allocation and $\hat{x}_i$ is in the budget set of trader i $x_i^* \succsim_i \hat{x}_i$. By supposition $\hat{x}_i \succsim x_i^* \; \forall i$. Thus, $x_i^* \sim \hat{x}_i \; \forall i$. By local nonsatiation $\forall \varepsilon \; \exists x_i^{(\varepsilon)}$ such that $\|\hat{x}_i - x_i^{(\varepsilon)}\| < \varepsilon$ and $x_i^{(\varepsilon)} \succ_i \hat{x}_i$. For small enough ε, $p^* \cdot x_i^{(\varepsilon)} < p^* \cdot x_i^*$ and $x_i^{(\varepsilon)} \succ_i \hat{x}_i \sim_i x_i^*$, so x^* is not a competitive equilibrium allocation.

CONTRADICTION, so $p^* \cdot \hat{x}_i \geq p^* \cdot x_i^*$.

Since $p^* \cdot \hat{x}_i \geq p^* \cdot x_i^* \; \forall i$, $\sum_{i \neq j} p^* \cdot \hat{x}_i \geq \sum_{i \neq j} p^* \cdot x_i^*$. Since $p^* \cdot \hat{x}_j > p^* x_j^*$, $\sum_i p^* \hat{x}_i > \sum_i p^* x_i^*$. CONTRADICTION, so x^* is Paerto efficient.

(c) i. } Yes, preferences are assumed to be continuous, complete, and locally nonsatiated,
ii. } but not necessarily convex.

iii. No, local nonsatiation is critical for $p^* \cdot \hat{x}_i \geq p^* x_i^*$. Nonsatiation alone would not imply $x_i^{(\varepsilon)} \succ_i x_i^*$.

Fall 2007, Question III.1

(a) Show that every finite game possesses a Nash equilibrium in which no player places a strictly positive probability on a weakly dominated strategy.

(b) Improve this result from (a) by showing that every finite game possesses a Nash equilibrium σ in which for every player i, σ_i is not dominated.

(c) Show by an example that the result in the previous point (b) requires finiteness.

Solution

(a) Only need to show that there is a NE in which no player places a strictly positive probability on a weakly dominated strategy. Now I claim that any PE has such a property, then since any PE is NE, so we done.

Suppose $\hat{s}$ is a PE, then there exists a sequence $s_{\epsilon_n} \to \hat{s}$, s.t. $\forall \epsilon_n > 0, s_{\epsilon_n} \in NE(G_{\epsilon_n})$. If a_k is weakly dominated and $\hat{s}^i(a_k) > 0$, we have $s^i_{\epsilon_n}(a_k) \to \hat{s}^i(a_k) > 0$ and $\epsilon^i_n(a_k) \to 0$, so for n large enough, $s^i_{\epsilon_n}(a_k) > \epsilon^i_n(a_k)$, i.e. $s^i_{\epsilon_n}(a_k) - \epsilon^i_n(a_k) > 0$.

Now suppose s^i weakly dominates a_k, which means

$$\begin{array}{lccc} \forall\, a^{-i}, & u^i(s^i, a^{-i}) & \geqslant & u^i(a_k, a^{-i}) \\ \exists\, b^{-i}, & u^i(s^i, b^{-i}) & > & u^i(a_k, b^{-i}). \end{array}$$

Then I will construct a new mixed strategy t^i as follows: Other things equal to the strategy $s^i_{\epsilon_n}$ but give the strategy a_k probability of $\epsilon^i_n(a_k)$, and give the left probability $s^i_{\epsilon_n}(a_k) - \epsilon^i_n(a_k) > 0$ to the strategy s^i. Because s^i weakly dominates a_k, we have:

$$\begin{array}{lccc} \forall\, a^{-i}, & u^i(t^i, a^{-i}) & \geqslant & u^i(s^i_{\epsilon_n}, a^{-i}) \\ \exists\, b^{-i}, & u^i(t^i, b^{-i}) & > & u^i(s^i_{\epsilon_n}, b^{-i}), \end{array}$$

so

$$\sum_{a^{-i} \in A^{-i}} s^{-i}_{\epsilon_n}(a^{-i}) u^i(t^i, a^{-i}) > \sum_{a^{-i} \in A^{-i}} s^{-i}_{\epsilon_n}(a^{-i}) u^i(s^i_{\epsilon_n}, a^{-i})$$

i.e., $u^i(t^i, s^{-i}_{\epsilon_n}) > u^i(s^i_{\epsilon_n}, s^{-i}_{\epsilon_n})$. So $s^i_{\epsilon_n}$ is not a best response (BR) and $s^i_{\epsilon_n} \notin NE(G_{\epsilon_n})$. Contradiction.

(b) Suppose σ is a Nash equilibrium(NE) and for some i, σ^i is dominated, then

$$\exists\, s^i \in S^i \text{ s.t. } \forall a^{-i} \in A^{-i}, \quad u^i(s^i, a^{-i}) > u^i(\sigma^i, a^{-i}),$$

so

$$\sum_{a^{-i} \in A^{-i}} s^{-i}(a^{-i}) u^i(s^i, a^{-i}) > \sum_{a^{-i} \in A^{-i}} s^{-i}(a^{-i}) u^i(\sigma^i, a^{-i}).$$

This means that $u^i(s^i) > u^i(\sigma^i)$. So σ^i is not a best response (BR) for i and it is not a NE. Contradiction.

(c) Here we need to show a counterexample that for all NE, for some player i, σ^i is dominated. In fact, I find that if there is NE in a infinite game, then I can only find some of NE is dominated. So the only way to do this, in my opinion, is to find a infinite game with no NE. The following is an example:

	1
1	$(1, 0)$
2	$(1\frac{1}{2}, 0)$
3	$(1\frac{3}{4}, 0)$
4	$(1\frac{7}{8}, 0)$
$\vdots$	$\vdots$

In this example, player 2 has only one action and always gets 0 payoff. Player 1 will get a payoff of $(1 + 2^{1-i})$ for action i. So every strategy is dominated by the one below and there is no NE in this game.

Fall 2007, Question III.2

(a) Define the set of correlated strategies and correlated equilibria for a finite game.

(b) Define an augmented game, and show how the BNE of the augmented game and the correlated equilibrium are related.

(c) Prove that the set of correlated equilibrium payoffs is a closed, convex, non-empty set.

Solution

(a) First, consider

Definition 7 (Correlated strategy)**.** A correlated strategy is a probability distribution $\mu \in \Delta(A)$, where $A = \times_{i \in I} A^i$.

Example:

	L	R
T	$\mu(TL)$	$\mu(TR)$
B	$\mu(BL)$	$\mu(BR)$

where $\mu(ij) \geq 0$, $\forall\, i, j$, $\sum_{ij} \mu(ij) = 1$.

Definition 8 (Correlated equilibrium)**.** A $\hat{\mu} \in \Delta(A)$ is a correlated equilibrium iff:

$$\forall\, i, a^i \in A^i, b^i \in A^i \sum_{a^{-i} \in A^{-i}} \hat{\mu}(a^{-i}|a^i) u^i(a^{-i}, a^i) \geq \sum_{a^{-i} \in A^{-i}} \hat{\mu}(a^{-i}|a^i) u^i(a^{-i}, b^i) \qquad (*)$$

Notes:

- $a^i \in A^i$ is a recommendation,
- $b^i \in A^i$ is any action of player i,
- Also,

$$\mu(a^{-i}|a^i) = \frac{\mu(a^{-i}, a^i)}{\sum_{b^{-i} \in A^{-i}} \mu(b^{-i}, a^i)} = \frac{\mu(a^{-i}, a^i)}{\mu(a^i)}.$$

We define the set of correlated strategies as:

$$\mathbb{CS} = \{\mu : \mu \in \Delta(A)\},$$

and the set of correlated equilibria as:

$$\mathbb{CE} = \{\hat{\mu} : \hat{\mu} \in \Delta(A) \quad \text{and} \quad (*) \quad \text{is satisfied}\}.$$

(b) Since:

$$\mu(a^{-i}|a^i) = \frac{\mu(a^{-i}, a^i)}{\mu(a^i)},$$

we can rewrite condition (*) as:

$$\forall\, i, a^i \in A^i, b^i \in A^i \sum_{a^{-i}\in A^{-i}} \mu(a^{-i}, a^i)u^i(a^{-i}, a^i) \geq \sum_{a^{-i}\in A^{-i}} \mu(a^{-i}, a^i)u^i(a^{-i}, b^i). \quad (**)$$

First step towards GC

Lemma 2. *A correlated strategy is a correlated equilibrium iff:*

$$\sum_{a\in A} \mu(a)u^i(a) \geq \sum_{a\in A} \mu(a)u^i(\delta^i(a^i), a^{-i}),$$

for all $i \in I$ and all $\delta^i : A^i \to A^i$.

Note: We can interpret δ^i as a mechanism giving us any action in player's i action set A^i, so we can treat it as a way to generate b^i.

Proof

$\Longrightarrow$ Suppose μ is a CE. Fix i and $\delta^i : A^i \to A^i$. Then by definition (just rewrite (**) in terms of δ^i):

$$\sum_{a^{-i}\in A^{-i}} \mu(a)u^i(a) \geq \sum_{a^{-i}\in A^{-i}} \mu(a)u^i(\delta^i(a^i), a^{-i})$$

for all $a^i \in A^i$. Summing the above inequality over $a^i \in A^i$ gives the condition in the statement of the lemma.

$\Longleftarrow$ Suppose μ is not a CE, so that there exists $i \in I$, $\hat{a}^i \in A^i$, $\hat{b}^i \in A^i$ such that (**) is violated:

$$\sum_{a^{-i}\in A^{-i}} \mu(a^{-i}, \hat{a}^i)u^i(a^{-i}, \hat{a}^i) < \sum_{a^{-i}\in A^{-i}} \mu(a^{-i}, \hat{a}^i)u^i(a^{-i}, \hat{b}^i).$$

Define $\delta^i : A^i \to A^i$ by:

$$\delta^i(a^i) = \begin{cases} \hat{b}^i & \text{if } a^i = \hat{a}^i \\ a^i & \text{if } a^i \neq \hat{a}^i \end{cases}$$

Then:

$$\sum_{a\in A} \mu(a)u^i(\delta^i(a^i), a^{-i}) = \sum_{a^{-i}\in A^{-i}} \mu(a^{-i}, \hat{a}^i)u^i(a^{-i}, \hat{b}^i) + \sum_{a^i\neq\hat{a}^i} \sum_{a^{-i}\in A^{-i}} \mu(a)u^i(a)$$

$$> \sum_{a\in A} \mu(a)u^i(a),$$

which violates the statement of the lemma.

Definition and description of augmented game (GC)

Definition 9. A game with communication, GC, is a Bayesian game:

$$\mathbb{GC} = (I, (A^i)_{i \in I}, (\Theta^i)_{i \in I}, (u^i)_{i \in I}),$$

in which $u^i : A \to \mathbb{R}$ for all $i \in I$ (i.e. payoffs do not depend on types).

Description

- $(\Theta, \mathcal{F}, P)$ is a probability space,
- $\Theta = \times_{i \in I} \Theta^i$, where Θ^i denotes signal space of player i,
- P is the probability distribution (assumed to be common knowledge among players).

(a) a vector of signals $\theta \in \Theta$ is drawn according to distribution $P \in \Delta(\Theta)$,

(b) $\theta^i \in \Theta^i$ is told to i,

(c) each i selects $a^i \in A^i$,

(d) game is over, payoff is paid.

Second step towards GC

We define a behavioral strategy in GC as:

$$\sigma^i : \Theta^i \to \Delta(A^i),$$

so we have $\sigma^i(\theta^i) \in \Delta(A^i)$ for all i. Then:

$$\sigma^i(\theta^i)[a^i]$$

gives us probability that player i will choose action a^i given that his signal was θ^i.

Next we want to find the probability of action $a = (a^1, a^2, \dots, a^I)$, where each player chooses exactly given component a^i. By the law of total probability this is given by:

$$\sum_{\theta \in \Theta} P(\theta) \prod_{i \in I} \sigma^i(\theta^i)[a^i].$$

Our conclusion here is that **GC induces correlated strategy μ via σ**, since $\mu(a)$ gives us also probability that action a will occur. So we have:

$$\mu(a) = \sum_{\theta \in \Theta} P(\theta) \prod_{i \in I} \sigma^i(\theta^i)[a^i].$$

This motivates the following definition:

Definition 10. A game with communication induces a correlated strategy μ if there exists a strategy profile, σ, such that:

$$\mu(a) = \sum_{\theta \in \Theta} P(\theta) \prod_{i \in I} \sigma^i(\theta^i)[a^i]$$

for all $a \in A$.

First approach to relation GC–CE

First we define a special case of GC, in which the signals are actions, in which case they may be viewed as *recommendations* as to what they should do.

Definition 11. A canonical game with communication, CCG, is a game with communication in which $\Theta^i = A^i$ for all $i \in I$.

Interpreting signals as recommendations in such games motivates the following name for a certain type of strategy:

Definition 12. A randomized strategy for player i of a canonical game with communication $\sigma^i : A^i \to \Delta(A^i)$ is obedient if:

$$\sigma^i(a^i)[a^i] = 1, \qquad \forall a^i \in A^i.$$

A strategy profile $\sigma = (\sigma^i)_{i \in I}$ is obedient if every σ^i is obedient.

Similarily as we proved Lemma 2, we can show:

Claim 3. *A correlated strategy μ is a correlated equilibrium iff it is induced by the canonical game with communication $(I, (A^i)_{i \in I}, (A^i)_{i \in I}, \mu, (u^i)_{i \in I})$ via the obedient strategy profile.*

Proof

$\Longrightarrow$ Suppose μ is a CE. This means that all players always follow recommendations. So in a canonical game with communication all players have obedient strategies, σ^i. We want to find probability of action a in this game to show that it is equal $\mu(a)$ for all $a \in A$.

Then CE will be induced by CCG via strategy profile σ which is obedient. By definition we have:

$$\prod_{i\in I}\sigma^i(a^i)[a^i] = \begin{cases} 1 & \text{if } a^i = a^i, \quad \forall i \in I \\ 0 & \text{if } \exists i \text{ s.t. } a^i \neq a^i \end{cases}$$

Then probability of action a is:

$$\sum_{a\in A}\mu(a)\prod_{i\in I}\sigma^i(a^i)[a^i] = \sum_{a\in A}\mu(a)\mathbf{1}_a = \mu(a), \qquad \forall a \in A.$$

$\Longleftarrow$ Suppose μ is not a CE, so that there exists $i \in I$, $\hat{a}^i \in A^i$, $\hat{b}^i \in A^i$ such that:

$$\sum_{a^{-i}\in A^{-i}}\mu(a^{-i},\hat{a}^i)u^i(a^{-i},\hat{a}^i) < \sum_{a^{-i}\in A^{-i}}\mu(a^{-i},\hat{a}^i)u^i(a^{-i},\hat{b}^i).$$

Then in CCG for this player we have:

$$\sigma^i(\hat{a}^i)[\hat{a}^i] < \sigma^i(\hat{a}^i)[\hat{b}^i] \quad \Rightarrow \quad \sigma^i(\hat{a}^i)[\hat{a}^i] < 1,$$

which violates condition for obedient strategy σ^i, so strategy profile σ cannot be obedient.

Relation between BNE and CE

Theorem 4. *A correlated strategy μ is a correlated equilibrium iff it can be induced by a game with communication.*

Proof

First suppose that μ is a CE. Then by Claim 3 it is induced by the canonical game with communication $(I,(A^i)_{i\in I},(A^i)_{i\in I},\mu,(u^i)_{i\in I})$ via the obedient strategy profile.

Next suppose that μ is induced by a game with communication $(I,(A^i)_{i\in I},(\Theta^i)_{i\in I},P,(u^i)_{i\in I})$. Let σ be the corresponding BNE. Fix $i \in I$ and $\delta^i : A^i \to A^i$. By the best response condition of σ we have:

$$\sum_{\theta\in\Theta}\sum_{a\in A}P(\theta)u^i(a)\prod_{j\in I}\sigma^j(\theta^j)[a^j] \geq \sum_{\theta\in\Theta}\sum_{a\in A}P(\theta)u^i(a)\tilde{\sigma}^i(\theta^i)[a^i]\prod_{j\neq i}\sigma^j(\theta^j)[a^j]$$

for all strategies $\tilde{\sigma}^i : \Theta^i \in \Delta(A^i)$. So in particular we can take $\tilde{\sigma}^i$ so that for any θ^i it plays $\delta^i(a^i)$ with probability $\sigma^i(\theta^i)[a^i]$. This gives:

$$\sum_{\theta\in\Theta}\sum_{a\in A}P(\theta)u^i(a)\prod_{j\in I}\sigma^j(\theta^j)[a^j] \geq \sum_{\theta\in\Theta}\sum_{a\in A}P(\theta)u^i(\delta^i(a^i),a^{-i})\prod_{j\in I}\sigma^j(\theta^j)[a^j].$$

But by the formula in Definition 10:

$$\text{LHS} = \sum_{\theta\in\Theta}\sum_{a\in A} P(\theta)u^i(a)\prod_{j\in I}\sigma^j(\theta^j)[a^j] = \sum_{a\in A}\mu(a)u^i(a),$$

$$\text{RHS} = \sum_{\theta\in\Theta}\sum_{a\in A} P(\theta)u^i(\delta^i(a^i), a^{-i})\prod_{j\in I}\sigma^j(\theta^j)[a^j] = \sum_{a\in A}\mu(a)u^i(\delta^i(a^i), a^{-i}),$$

so by Lemma 2, μ is a CE.

All the steps above give us the following alternative definition of correlated equilibrium revealing relation between BNE and CE:

Definition 13. A correlated equilibrium is a correlated strategy that can be induced by a game with communication.

(c) First we show that the set of correlated equilibrium is non-empty, closed and convex:

Set of CE is non-empty

Here we show that NE is a CE. Let $\hat{s} \in S$ be a NE. Define correlated strategy as:

$$\mu_{\hat{s}}(a) = \hat{s}(a) = \prod_{i\in I}\hat{s}^i(a^i).$$

To check the condition for correlated equilibrium, we need to find $\mu_{\hat{s}}(a^{-i}|a^i)$. But:

$$\mu_{\hat{s}}(a^{-i}|a^i) = \frac{\mu_{\hat{s}}(a^{-i}, a^i)}{\mu_{\hat{s}}(a^i)}.$$

By above:

$$\mu_{\hat{s}}(a^{-i}, a^i) = \prod_{j\neq i}\hat{s}^j(a^j)\hat{s}^i(a^i).$$

Using note from part (a) we have:

$$\mu_{\hat{s}}(a^i) = \sum_{a^{-i}\in A^{-i}}\mu_{\hat{s}}(a^{-i}, a^i) = \sum_{a^{-i}\in A^{-i}}\prod_{j\neq i}\hat{s}^j(a^j)\hat{s}^i(a^i) = \hat{s}^i(a^i)\underbrace{\sum_{a^{-i}\in A^{-i}}\prod_{j\neq i}\hat{s}^j(a^j)}_{=1} = \hat{s}^i(a^i).$$

So we have:

$$\mu_{\hat{s}}(a^{-i}|a^i) = \frac{\mu_{\hat{s}}(a^{-i}, a^i)}{\mu_{\hat{s}}(a^i)} = \frac{\prod_{j\neq i}\hat{s}^j(a^j)\hat{s}^i(a^i)}{\hat{s}^i(a^i)} = \prod_{j\neq i}\hat{s}^j(a^j).$$

Last thing we need to check is whether CE condition (*) holds. If we plug in the obtained above values we have:

$$\forall i, a^i \in A^i, b^i \in A^i \sum_{a^{-i} \in A^{-i}} \prod_{j \neq i} \hat{s}^j(a^j) u(a^{-i}, a^i) \geq \sum_{a^{-i} \in A^{-i}} \prod_{j \neq i} \hat{s}^j(a^j) u(a^{-i}, b^i),$$

but since we define utility of i as:

$$u^i(s) = \sum_{a^i \in A^i} \prod_{i \in I} s^i(a^i) u^i(a),$$

our CE condition is equivalent to:

$$\forall i, a^i \in A^i, b^i \in A^i \qquad u^i(\hat{s}^{-i}, a^i) \geq u^i(\hat{s}^{-i}, b^i),$$

but we know this is satisfied, since this is the best response condition for NE. Defining correlated strategy as $\hat{s}$, where $\hat{s}$ is a NE, guarantees that condition for CE is satisfied. So set of CE is non-empty: NE belongs to this set.

Set of CE is closed and convex

We rewrite condition for CE (*) as:

$$\sum_{a^{-i} \in A^{-i}} \mu(a^{-i}|a^i) \left[u(a^{-i}, a^i) - u(a^{-i}, b^i) \right] \geq 0,$$

so set of CE is the set of solutions to a vector of linear inequalities. So it is closed and convex.

Once we have that the set of CE is non-empty, closed and convex note that we define the set of CE payoffs as:

$$\mathbb{CEP} = \{x \in \mathbb{R}^n : x = \sum_{a \in A} \hat{\mu}(a) u(a), \quad \hat{\mu} \in \mathbb{CE}\},$$

so we obtain the set of CE payoffs from the set of CE by linear operation, which preserves non-emptiness, convexity and closedness of initial set.

Fall 2007, Question IV.1

(a) Define the Nash equilibrium operator NE^δ and the Sub-game perfect Equilibrium operator SP^δ from subsets of the set of feasible and incentive compatible payoffs to subsets of the same set of payoffs.

(b) Define a self-generated set for each of the two operators in part (a).

(c) Show an example where the two operators are different.

Solution

(a) First, I need the following definitions:

Definition 14. Set of feasible payoffs:

$$F = \{x \in \mathbb{R}^I : x = \{u^i(a)_{i\in I}\}_{a\in A}\}.$$

Definition 15. Minimax payoffs:

$$v^i = \min_{s^{-i}} \max_{s^i} u^i(s^i, s^{-i}).$$

Definition 16. Set of feasible and incentive-compatible payoffs:

$$F^* = F^{co} \bigcap \{x \in \mathbb{R}^I : x^i \geq v^i\}.$$

Then,

Definition 17. Nash Equilibrium operator for $A \subseteq F^*$:

$$\begin{aligned} NE^\delta(A) \quad = \quad & \{x \in \mathbb{R}^I : \exists(\hat{s}, y) \in (S \times A) \text{ such that} \\ & (1)\ x = (1-\delta)u(\hat{s}) + \delta y \\ & (2)\ \forall\ i, x^i \geq (1-\delta)\max_{s^i \in S^i} u^i(s^i, \hat{s}^{-i}) + \delta v^i\} \end{aligned}$$

Definition 18. Subgame Perfect equilibrium operator for $A \subseteq F^*$:

$$\begin{aligned} SP^\delta(A) \quad = \quad & \{x \in \mathbb{R}^I : \exists(\hat{s}, w) \in (S \times \{W : S \to A\}) \text{ such that} \\ & (1)\ x = (1-\delta)u(\hat{s}) + \delta y \\ & (2)\ \forall\ i, x^i \geq (1-\delta)\max_{s^i \in S^i}(u^i(s^i, \hat{s}^{-i}) + \delta w(s^i, \hat{s}^{-i}))\} \end{aligned}$$

(b) The definitions are:

Definition 19. Set A is a self-generated set for $NE^\delta(A)$ if $A \subseteq NE^\delta(A)$.

Definition 20. Set A is a self-generated set for $SP^\delta(A)$ if $A \subseteq SP^\delta(A)$.

(c) To show an example where the two operators are different we first need a game. Thus, consider the following:

	L	R
T	$10, 1$	$10, 0$
B	$15, 15$	$0, 30$

Now we need to prove

Theorem 5. *$\exists A \subseteq F^*$ such that $SP^\delta(A) \neq NE^\delta(A)$.*

(*Note:* I had to change the strategy from the one I presented. I originally assigned the punishment to be a payoff outside of F^*. The new payoff is easier and it leads to a slightly different contradiction. Let me know if there are problems. - Justin)

Define a strategy (μ) as:

$$\begin{aligned}
\mu_1^1(T) &= 0 \\
\mu_2^1(T) &= \begin{cases} 0 & \text{if } \hat{s}(L)_1^2 = 1 \\ 1 & \text{if } \hat{s}(L)_1^2 \neq 1 \end{cases} \\
&\vdots \\
\mu_t^1(T) &= \begin{cases} 0 & \text{if } \{\hat{s}_\tau^2(L)\}_{\tau=1}^{t-1} = \mathbf{1} \\ 1 & \text{if } \{\hat{s}_\tau^2(L)\}_{\tau=1}^{t-1} \neq \mathbf{1} \end{cases} \\
\mu_1^2(L) &= 1 \\
\mu_2^2(L) &= 1 \\
&\vdots \\
\mu_t^1(L) &= 1
\end{aligned}$$

Payoffs are

$$u^i(\mu) = (1-\delta)\sum_{t=0}^{\infty} \delta^t u^i(\mu(\hat{s})) = 15 \quad \text{for } i = 1, 2.$$

Let's check that μ is Nash.

Claim 4. *$\mu \to x \in NE^\delta(A)$ for $\delta = 0.75$.*

Proof. To show: $\nexists$ profitable deviation for each payer for δ big enough. We know player 1 will not deviate. His payoff cannot be higher than 15. Suppose player 2 deviates optimally:

(He takes his 30 for 1 period, then plays left forever knowing that player 1 will be playing top.)

$$\phi_1^2(L) = 0, \quad \phi_t^2(L) = 1 \quad \text{for } t = 2, 3, \dots$$

Payoff from ϕ^2:

$$\begin{aligned} u^2(\phi) &= (1-\delta)u^2(\mu_1^1, \phi_1^2) + (1-\delta)\delta \sum_{t=2}^{\infty} \delta^{t-2} u_t^2(\mu_t^1(\mu_1^1, \phi_1^2), \phi_t^2) \\ &= (1-\delta)(30) + (\delta)(1) \\ &= 30 - 29\delta \end{aligned}$$

Player does not deviate if:

$$\begin{aligned} 30 - 29\delta &\leq 15 \\ -29\delta &\leq -15 \\ \delta &\geq 15/29 \end{aligned}$$

Since we consider $\delta = 0.75$, we are fine. □

Claim 5. *$\mu \to x \notin SP^{\delta}(A)$ for $\delta = 0.75$.*

Proof. Consider the subgame that begins after a player deviates at t-1. Payoff from μ_t for player 1 $= 10 =: x_t^1$; payoff from μ_t for player 2 $= 01 =: x_t^2$. Define $x_t = (x_t^1, x_t^2)$. We need to show that $\mu_t \to x_t \notin NE_t^{\delta}(A)$ for $\delta = 0.75$. Thus,

$$\begin{aligned} x_t \in NE^{\delta}(A) \quad &\Leftrightarrow \quad x_t^1 \geq (1-\delta) \max_{s^1 \in S^1} u^1(s^1, \mu_t^2) + \delta v^1 \\ &\Leftrightarrow \quad x_t^1 \geq (1-\delta) \max_{s^1 \in S^1} u^1(s^1, L) + \delta v^1 \\ &\Leftrightarrow \quad 10 \geq (1-\delta)(15) + \delta(10) \\ &\Leftrightarrow \quad -5 \geq -5\delta \\ &\Leftrightarrow \quad 1 \leq \delta \\ &\Leftrightarrow \quad 1 \leq 0.75 \end{aligned}$$

a contradiction. □

Fall 2007, Question IV.2

(a) Define a simple strategy profile.

Let $G = \left\{N, \left(S^i\right)_{i\in N}, \left(u^i\right)_{i\in N}\right\}$ be the normal form game that is played in each period where S^i is the set of mixed strategies and u^i is the per-period utility function. $S = \times_{i\in N} S^i$ is a strategy profile for period game.

Let $\Gamma(\delta)$ be the repeated game with discount factor δ.

For any t, $h_t = (s_1, ..., s_{t-1})$ be the history up to time t. Let $H_t = S^{t-1}$ be the set of histories and let $H_1 = \varnothing$.

A behavioral strategy for player i in the repeated game is a function $\sigma_t^i : H_t \to S^i$ for all t and a behavioral strategy profile (BSP) is $\sigma = \left(\sigma^1, ..., \sigma^n\right)$. Let Σ^i be the set of all strategies for player i and let $\Sigma = \times_{i\in N}\Sigma^i$ be the set of all strategy profiles.

$w = (s_1, s_2, ...) \in \Omega$ is called a path. Every BSP σ induces a path $(s_1, s_2, ...)$ in the following natural way:

$\sigma_1 = s_1$, $\sigma_1(s_1) = s_2$, $\sigma_1(s_1, s_2) = s_3$, ...

With the help of induced paths, the payoff for each $i \in N$ from playing BSP σ is defined as $U^i(\sigma) = (1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}u^i(s_t)$ where $(s_1, s_2, ...)$ is the path induced by σ and δ is the discount factor.

Given two histories h_t and $\widehat{h}_k$, we wrote $\left(h_t, \widehat{h}_k\right)$ for $\left(h_t, \widehat{h}_k\right) = (s_1, ..., s_{t-1}, \widehat{s}_1, ..., \widehat{s}_{k-1})$. Given t and h_t, the conditional (continuation) stategy $\sigma|_{h_t} \in \Sigma$ is defined as follows $(\sigma|_{h_t})(h_k) = \sigma(h_t, h_k)$ for any $k \in \mathbb{N}$ and $h_k \in H_k$.

Definition 1. A BSP $\sigma \in \Sigma$ is SPE iff $\forall i$, $\forall t$ and $\forall h_t$ we have $U^i(\sigma|_{h_t}) \geq U^i\left(\widetilde{\sigma}^i|_{h_t}, \sigma^{-i}|_{h_t}\right)$ for all $\widetilde{\sigma}^i \in \Sigma^i$.

Note that if σ is a SPE, then for any t and h_t, $\sigma|_{h_t}$ is also a SPE.

A path $w \in \Omega$ is called SPE path if it is induced by a SPE $\sigma \in \Sigma$. Let Ω^{SP} represent all subgame equilibrium paths.

Let $(w(i))_{i=0}^n$ be a collection of paths where $w(i) \in \Omega$ is $w(i) = (s(i)_1, s(i)_2, ...)$.

Definition 2. A simple stategy profile associated with $(w(i))_{i=0}^n$ is a $\sigma \in \Sigma$ which is contructed as follows[1]

1. Fix $(s_1, s_2, ...) \in S^\infty$

2. Define the sequence of states $(i_t, r_t)_{t\geq 1} \in \{0, 1, ..., n\} \times \mathbb{N}$ as follows[2]

$$(i_1, r_1) = (0, 1)$$

[1] $w(0)$ is the path that players are agreed and $w(i)$ is the punishment path of player i.

[2] i_t is the name of player that is going to be punished after time t, where 0 stands for no punishment. r_t represents the number of periods of punishment of some player.

Given (i_t, r_t), define (i_{t+1}, r_{t+1}) as follows

$(i_{t+1}, r_{t+1}) = (j, 1)$ if $\exists! j \in N$ s.t. $s_t^j \neq s(i_t)_{r_t}^j$ [3]

$(i_{t+1}, r_{t+1}) = (i_t, r_{t+1})$ if $s_t = s(i_t)_{r_t}$ (no one deviated) or $s_t^j \neq s(i_t)_{r_t}^j$ for more that one j

3. Set $\sigma = (\sigma_1, \sigma_2(s_1), \sigma_3(s_1, s_2), ...) = \left(s(i_t)_{r_t}\right)_{t\geq 1}$
4. Repeat this process for all $(s_1, s_2, ...) \in S^\infty$ to make σ well-defined, that is, it will be defined for all histories.

(b) State the necessary and sufficient condition insuring that a simple strategy profile is a perfect equilibrium

Theorem 1. *A simple strategy profile $\sigma \in \Sigma$ associated with $(w(i))_{i=0}^n$ is SPE iff $\forall i$, $\forall t$ and $\forall j \in \{0, 1, ..., n\}$ and $\forall \widehat{s}^i \in S^i$, we have*

$$u^i(s(j)_t) + \delta \sum_{T=t+1}^{\infty} \delta^{T-(t+1)} u^i(s(j)_T) \geq u^i\left(\widehat{s}^i, s(j)_t^{-i}\right) + \delta \sum_{T=t+1}^{\infty} \delta^{T-(t+1)} u^i\left(s(i)_{T-t}\right) \quad (*)$$

The left-hand side of * is the utility of playing the path $s(j)_t$ and right-hand side of * is the player i's payoff if he deviates to $\widehat{s}^i$ from $s(j)_t^i$. If player i deviated, then all palyers swith to the punishment stategy $s(i)_t$ for player i.

Theorem states that whether we are on the aggred-upon path or we are on some punsihment path, no player has any incentive to deviate at any time. Hence $\sigma \in \Sigma$ induces agreed-upon path $w(0)$ as SPE path.

(c) Prove your statement from (b)

Sketch of the proof: On top of every thing, we assume σ is a simple strategy. For $\Longrightarrow$ part, we are going to use that σ satisfies the definition of SPE that is given in part (a). So we are going to construct some history h_t and deviation strategy $\widetilde{\sigma}^i|_{h_t}$ that is inline with punishment paths of simple strategies and show that condition (*) has to hold. For $\Longleftarrow$ part, we assume the paths associated with the simple strategy σ satisfies condition (*) and show that there is no profitable deviation strategy $\widetilde{\sigma}^i$ (not necessarily one-shot deviation) for any player i. To show this, we define set of strategies $\left\{\widetilde{\sigma}^{i,k}\right\}_{k\in\mathbb{N}}$ that approximate deviation strategy $\widetilde{\sigma}^i$ and used the fact that that U is bounded and continuous.

($\Longrightarrow$) Suppose $\sigma \in \Sigma$ is a simple strategy profile associated with $(w(i))_{i=0}^n$ and suppose σ is SPE.[4] Let $i \in N$ and $t \in \mathbb{N}$ and $j \in \{0, 1, ..., n\}$ and $\widehat{s}^i \in S^i$.

Construct $h_t = \left(s(j)_1, ..., s(j)_{t-2}, s(j)_{t-1}\right)$.[5] Hence $\sigma|_{h_t}$ induces the path $w(j) = \left(s(j)_t, s(j)_{t+1}, ...\right)$. So we can write

$$U^i(\sigma|_{h_t}) = u^i(s(j)_t) + \delta \sum_{T=t+1}^{\infty} \delta^{T-(t+1)} u^i(s(j)_T)$$

[3] There is a unique player j that is supposed to play $s(i_t)_{r_t}^j$ but instead played s_t^j.

[4] that is, $\forall i$, $\forall t$ and $\forall h_t$ we have $U^i(\sigma|_{h_t}) \geq U^i\left(\widetilde{\sigma}^i|_{h_t}, \sigma^{-i}|_{h_t}\right)$ for all $\widetilde{\sigma}^i \in \Sigma^i$ from definition of SPE.

[5] If $j = 0$,then we are on the agreed path. If $j \neq 0$, for simplicity, we are assuming player j deviated from agreed path in the first step and after that there was no deviation from j's punishment path.

For i, define conditional strategy $\widetilde{\sigma}^i|_{h_t}$ as follows: $\left(\widetilde{\sigma}^i|_{h_t}\right)_1 = \widehat{s}^i$.[6] $\left(\widetilde{\sigma}^i|_{h_t}\right)\left(\widehat{s}^i\right) = s(i)_1^i$, $\left(\widetilde{\sigma}^i|_{h_t}\right)\left(\widehat{s}^i, s(i)_1\right) = s(i)_2^i$,...[7] that is, in the conditional strategy, i plays $\widehat{s}^i$ and then plays $s(i)_t^i$. Hence $\left(\widetilde{\sigma}^i|_{h_t}, \sigma^{-i}|_{h_t}\right)$ induce the path $\left(\left(\widehat{s}^i, s(j)_t^{-i}\right), s(i)_1, s(i)_2, ...\right)$. So we can write

$$U^i\left(\widetilde{\sigma}^i|_{h_t}, \sigma^{-i}|_{h_t}\right) = u^i\left(\widehat{s}^i, s(j)_t^{-i}\right) + \delta \sum_{T=t+1}^{\infty} \delta^{T-(t+1)} u^i\left(s(i)_{T-t}\right)$$

Since σ is SPE, we have $U^i(\sigma|_{h_t}) \geq U^i\left(\widetilde{\sigma}^i|_{h_t}, \sigma^{-i}|_{h_t}\right)$ which in turn establishes the condition (*) in the theorem

$$u^i\left(s(j)_t\right) + \delta \sum_{T=t+1}^{\infty} \delta^{T-(t+1)} u^i\left(s(j)_T\right) \geq u^i\left(\widehat{s}^i, s(j)_t^{-i}\right) + \delta \sum_{T=t+1}^{\infty} \delta^{T-(t+1)} u^i\left(s(i)_{T-t}\right)$$

for any arbitrary choice of $i \in N$ and $t \in \mathbb{N}$ and $j \in \{0, 1, ..., n\}$ and $\widehat{s}^i \in S^i$.

($\Longleftarrow$) Suppose $\sigma \in \Sigma$ is a simple strategy profile associated with $(w(i))_{i=0}^n$ that satisfies the condition (*) in the theorem. Suppose for a contradiction, σ is not SPE, so $\exists i \in N$, $\exists t$, $\exists h_t$ and $\exists \widetilde{\sigma}^i \in \Sigma^i$ s.t. $U^i\left(\widetilde{\sigma}^i|_{h_t}, \sigma^{-i}|_{h_t}\right) \geq U^i(\sigma|_{h_t})$.[8]

Note that since σ is a simple strategy profile, we must have $(\sigma|_{h_t})_1 = s(j)_k$ for some $j \in \{0, 1, ..., n\}$ and $k \in N$.[9] Hence $\sigma|_{h_t}$ induces the path $\left(s(j)_k, s(j)_{k+1}, ...\right)$. Then, we can write $U^i(\sigma|_{h_t}) = u^i\left(s(j)_k\right) + \delta \sum_{T=k+1}^{\infty} \delta^{T-(k+1)} u^i\left(s(j)_T\right)$. Define the path for individual i that is induced by $\left(\widetilde{\sigma}^i|_{h_t}, \sigma^{-i}|_{h_t}\right)$ as $\left(\widehat{s}_1^i, \widehat{s}_2^i, ...\right)$.

For $k \in N$, define the strategy profile $\widetilde{\sigma}^{i,k} \in \Sigma^i$ that follows $\widetilde{\sigma}^i|_{h_t}$ for first k periods and then turn back to $\sigma|_{h_t}$. Note that as $k \to \infty$, $\widetilde{\sigma}^{i,k}$ coincide with $\widetilde{\sigma}^i|_{h_t}$. That is as $k \to \infty$, $\left|U^i\left(\widetilde{\sigma}^i|_{h_t}, \sigma^{-i}|_{h_t}\right) - U^i\left(\widetilde{\sigma}^{i,k}, \sigma^{-i}|_{h_t}\right)\right| \to 0$.

For $k = 1$, $\widetilde{\sigma}^{i,1}$ is one-shot deviation at time t and after t, $\widetilde{\sigma}^{i,1}$ follows $\sigma|_{h_t}$, hence follows the punishment path for player i. We can write $U^i\left(\widetilde{\sigma}^{i,1}, \sigma^{-i}|_{h_t}\right) = u^i\left(\widehat{s}_1^i, s(j)_k^{-i}\right) + \delta \sum_{T=t+1}^{\infty} \delta^{T-(t+1)} u^i\left(s(i)_{T-t}\right)$. By condition (*), we have $U^i(\sigma|_{h_t}) \geq U^i\left(\widetilde{\sigma}^{i,1}, \sigma^{-i}|_{h_t}\right)$.

For $k = 2$, we have
$U^i\left(\widetilde{\sigma}^{i,2}, \sigma^{-i}|_{h_t}\right) = u^i\left(\widehat{s}_1^i, s(j)_k^{-i}\right) + \delta\left[u^i\left(\widehat{s}_2^i, s(i)_1^{-i}\right) + \delta \sum_{T=t+2}^{\infty} \delta^{T-(t+2)} u^i\left(s(i)_{T-(t+1)}\right)\right]$.[10]
We are going to compare player i's payoff with $\widetilde{\sigma}^{i,1}$ and $\widetilde{\sigma}^{i,2}$.

$$\begin{aligned}
& U^i\left(\widetilde{\sigma}^{i,1}, \sigma^{-i}|_{h_t}\right) - U^i\left(\widetilde{\sigma}^{i,2}, \sigma^{-i}|_{h_t}\right) \\
&= \delta \sum_{T=t+1}^{\infty} \delta^{T-(t+1)} u^i\left(s(i)_{T-t}\right) - \delta u^i\left(\widehat{s}_2^i, s(i)_1^{-i}\right) - \delta^2 \sum_{T=t+1}^{\infty} \delta^{T-(t+1)} u^i\left(s(i)_{T-t}\right) \\
&= \delta u^i\left(s(i)_1\right) + \delta^2 \sum_{T=t+2}^{\infty} \delta^{T-(t+2)} u^i\left(s(i)_{T-(t+1)}\right) - \delta u^i\left(\widehat{s}_2^i, s(i)_1^{-i}\right) - \delta^2 \sum_{T=t+2}^{\infty} \delta^{T-(t+2)} u^i\left(s(i)_{T-(t+1)}\right) \\
&= u^i\left(s(i)_1\right) + \delta \sum_{T=2}^{\infty} \delta^{T-2} u^i\left(s(i)_T\right) - \delta u^i\left(\widehat{s}_2^i, s(i)_1^{-i}\right) - \delta \sum_{T=2}^{\infty} \delta^{T-2} u^i\left(s(i)_{T-1}\right)
\end{aligned}$$

[6] Player i deviated at time t.

[7] After deviation, he plays the path $w(1) = \left(s(1)_1, s(1)_2, ...\right)$.

[8] Note $\widetilde{\sigma}^i$ is not necessarily one-shot deviation.

[9] That is, we have to be on the kth period of some path $s(j)$.

[10] The first term in right hand side is time t deviation. The next term is time $t+1$ deviation. Since other players plays $\sigma|_{h_t}$, they play the punishment $s(i)_1$ at time $t+1$. For the rest of the periods, player i turn back to $\sigma|_{h_t}$.

Since the condition (*) holds for any $i \in N$, t, $j \in \{0, 1, ..., n\}$ and any $\widehat{s}^i \in S^i$, we have $U^i\left(\widetilde{\sigma}^{i,1}, \sigma^{-i}|_{h_t}\right) \geq U^i\left(\widetilde{\sigma}^{i,2}, \sigma^{-i}|_{h_t}\right)$.[11]

Applying same argument gives $U^i\left(\sigma|_{h_t}\right) \geq U^i\left(\widetilde{\sigma}^{i,k}, \sigma^{-i}|_{h_t}\right)$ for all $k \in \mathbb{N}$.

$$\left|U^i\left(\sigma|_{h_t}\right) - U^i\left(\widetilde{\sigma}^i|_{h_t}, \sigma^{-i}|_{h_t}\right)\right|$$

$$= \left|\underbrace{U^i\left(\sigma|_{h_t}\right) - U^i\left(\widetilde{\sigma}^{i,k}, \sigma^{-i}|_{h_t}\right)}_{\geq 0 \text{ for all } k \in \mathbb{N}} + \underbrace{U^i\left(\widetilde{\sigma}^{i,k}, \sigma^{-i}|_{h_t}\right) - U^i\left(\widetilde{\sigma}^i|_{h_t}, \sigma^{-i}|_{h_t}\right)}_{\to 0 \text{ as } k \to \infty}\right|$$

Since U is continuous and bounded, we have $U^i\left(\sigma|_{h_t}\right) \geq U^i\left(\widetilde{\sigma}^i|_{h_t}, \sigma^{-i}|_{h_t}\right)$ contradicting σ is not SPE.

[11] In words, it is true because every punishment path has to be SPE itself.