Microeconomics I

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1 Summary

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\begin{array}{cl}
\text{Commodity Space}\left(R^L\right) & \\
\downarrow & \\
\text{Consumption Space}(X_i) & \leftarrow\text{Physical constraints} \\
X_i = R_+^L \subset R^L & \\
\downarrow & \\
\text{Budget Set}(B_i\left(p, e_i\right)) = \text{Set of feasible consumption bundle} & \leftarrow\text{Economic Constraints} \\
B_i\left(p, e_i\right) = \{x_i \in X_i \mid p \cdot x_i \leq p \cdot e_i\} & \\
\downarrow & \\
\text{Consumer's maximization programming problem:} & \\
\text{choosing } \succsim -\text{maximal consumption bundles from } B_i\left(p, e_i\right) & \leftarrow\text{we can summarize information} \\
\text{given } p \text{ and } e_i \in R_+^L \text{ for } \forall i & \text{of preference using utility function} \\
\downarrow & \\
\text{Demand Correspondence}(x_i \in D_i\left(p, e_i\right) \text{ for } \forall i) & \leftarrow\text{Finite intersection property} \\
\text{Consumer's maximal utility value} & \leftarrow\text{Maximum theorem} \\
\text{Value function(indirect utiliity function)} &
\end{array}
$$

2 Preference

2.1 Preference-based approach(Rationality)

1. Preorder $\boxed{\text{Sp 1999 III.2.(a)}}$

   On a set $X = R_{+}^{L}$, $\succsim$ is a preorder on X, that is, a binary relationship satisfying

   (a) reflexivity : for $\forall\ x \in X$, $x \succsim x$

   (b) transitivity : for $\forall\ x, y, z \in X$, $[x \succsim y$ and $y \succsim z] \rightarrow x \succsim z$

2. Complete Preorder $\boxed{\text{Sp 1999 III.2.(b)}}$

   On a set $X = R_{+}^{L}$, a preorder $\succsim$ is complete on X,

   if for $\forall\ x, y \in X$, $x \succsim y$ or $x \precsim y$ or $x \sim y$

3. Continuous Complete Preorder $\boxed{\text{Sp 1999 III.2.(c)}}$

   $\Longrightarrow$ (with monotonicity) $\underline{\text{Continuous Utility Function}}$

   On a set $X = R_{+}^{L}$, a complete preorder $\succsim$ is continuous on X,

   - if for $\forall\ y \in X$, $\{x \in X \mid x \succsim y\}$ and $\{x \in X \mid x \precsim y\}$ are closed subsets of X
   - if [for any sequence of pairs $\{x^n, y^n\}_{n=1}^{\infty}$ with $x^n \succsim y^n$ for $\forall n$ and $x = \lim_{n\to\infty} x^n$, $y = \lim_{n\to\infty} y^n$], $x \succsim y$

4. Monotonicity $\Longrightarrow$ $\underline{\text{Increasing Utility Function}}$ $(u(x) > u(y)$ if $x >> y)$

   (a) local nonsatiation
   On a set $X = R_{+}^{L}$, a preorder $\succsim$ is locally nonsatiated on X,
   if for $\forall\ x \in X, \forall \varepsilon > 0$, $\exists y$ s.t. $||y - x|| < \varepsilon$ and $y \succ x$

   (b) monotone
   On a set $X = R_{+}^{L}$, a preorder $\succsim$ is monotone on X,

   - if for $\forall\ x \in X, \forall y \in R_{++}^{L}$, $x + y \succ x$
   - if for $\forall\ x, y \in X$, $[y >> x] \rightarrow [y \succ x]$

   (c) strictly monotone $\Longrightarrow$ $\underline{\text{Strictly Increasing Utility Function}}$ $(u(x) > u(y)$ if $x \geq y$ and $x \neq y)$
   On a set $X = R_{+}^{L}$, a preorder $\succsim$ is strictly monotone on X,

   - if for $\forall\ x \in X, \forall y \in R_{+}^{L} \backslash \{0\}$, $x + y \succ x$
   - if for $\forall\ x, y \in X$, $[y \geq x$ and $y \neq x] \rightarrow [y \succ x]$

(d) weakly monotone

On a set $X = R_+^L$, a preorder $\succsim$ is weakly monotone on X,
if for $\forall\ x, y \in X$, $[y \geq x] \rightarrow [y \succsim x]$

5. Convexity $\Longrightarrow$ <u>Quasiconcave Utility Function</u>
{(The set $\{x \in X \mid u(x) \geq u(y)\}$ is convex for $\forall\ y \in X$)
or $(u(\lambda x + (1-\lambda) y) \geq \min\{u(x), u(y)\}$ for any $x, y \in X$, $\forall \lambda \in [0,1]$

   (a) convex set
   A set $S \subseteq R^n$ is convex if for $\forall a, b \in S$, $\lambda a + (1-\lambda) b \in S$ for $\forall \lambda \in [0,1]$;
   that is, if $a_1, a_2, ..., a_k \in S$ and $\lambda_1 \geq 0, \lambda_2 \geq 0, ..., \lambda_k \geq 0$ s.t. $\sum_{k=1}^{K} \lambda_k = 1$, then $\sum_{k=1}^{K} \lambda_k a_k \in S$.

   (b) weakly convex
   On a set $X = R_+^L$, a preorder $\succsim$ is weakly convex on X,
   - if for $\forall\ x, y \in X$ with $x \neq y$, $x \succsim y$, $\lambda \in [0,1]$, $\lambda x + (1-\lambda) y \succsim y$
   - if for $\forall\ y \in X$, $\{x \in X \mid x \succsim y\}$ is a convex set on X
   - if for $\forall\ y \in X$, $\{x \in X \mid x \succ y\}$ is a convex set on X

   (c) convex $\boxed{\text{Sp 1999 III.2.(f)}}$
   On a set $X = R_+^L$, a preorder $\succsim$ is convex on X,
   - if for $\forall\ x, y \in X$ with $x \neq y$, $x \succ y$, $\lambda \in (0,1)$, $\lambda x + (1-\lambda) y \succ y$

   (d) strictly convex $\boxed{\text{Sp 1999 III.2.(g)}}$ $\Longrightarrow$ <u>Strictly Quasiconcave Utility Function</u>
   {(The set $\{x \in X \mid u(x) > u(y)\}$ is convex for $\forall\ y\,(\neq x) \in X$) or
   $(u(\lambda x + (1-\lambda) y) > \min\{u(x), u(y)\}$ for any $x, y\,(x \neq y) \in X$, $\forall \lambda \in (0,1)$
   On a set $X = R_+^L$, a preorder $\succsim$ is strictly convex on X,
   - if for $\forall\ x, y \in X$ with $x \neq y$, $x \succsim y$, $\lambda \in (0,1)$, $\lambda x + (1-\lambda) y \succ y$

6. Homotheticity{(with continuity) $\Longleftrightarrow$ $u(x)$ is <u>homogenous of degree 1</u>, that is, $u(\alpha x) = \alpha u(x)$ for $\alpha > 0$}
On a set $X = R_+^L$, a monotone preorder $\succsim$ is homothetic on X,
if, for $\forall\ x, y \in X$, $[x \sim y] \rightarrow [\alpha x \sim \alpha y]$ for $\forall \alpha > 0$

7. Quasilinearity{(with continuity) on $R \times R_+^{L-1}$ with respect to the first commodity $\Longleftrightarrow$ <u>$u(x) = x_1 + \Phi(x_2, ..., x_L)$</u>}
On a set $X = R_+^L$, a preorder $\succsim$ is quasilinear on X,
if i) for $\forall\ x, y \in X$, $[x \sim y] \rightarrow [x + \alpha e_1 \sim y + \alpha e_1]$ for $\forall \alpha \in R$ and $e_1 = \{1, 0, 0, ..., 0\}$ and
ii) Good 1 is desirable; that is, $x + \alpha e_1 \succ x$ for $\forall x, \alpha > 0$ and $e_1 = \{1, 0, 0, ..., 0\}$

8. Lexicographic Preorder

   On a set $X = R_+^2$, a preorder $\succsim$ is lexicographic on X,

   $x \succ y$ if $[x_1 > y_1]$ or $[x_1 = y_1$ and $x_2 > y_2]$

2.2 Choice-based approach(The weak axiom of revealed preference)

1. Choice structure($\mathfrak{B}$,$C(\cdot)$)

   $\mathfrak{B}$: a set of nonempty subsets of X s.t. B(an element of $\mathfrak{B}$) $\subset X$ is a budget set
   $C(\cdot)$: a choice rule(correspondence) $C : B \to C(B) \subset B$

2. Weak axiom of revealed preference

   For some $B \in \mathfrak{B}$ with (x_1, x_2), $(x_1', x_2') \in B$ s.t. $(x_1, x_2) \neq (x_1', x_2')$,

   $[[(x_1, x_2) \in C(B)]$ and $[$for any B' $(x_1', x_2') \in C(B')]] \to (x_1, x_2) \in C(B')$

3. Revealed preference relationship

   $(x_1, x_2) \succsim (x_1', x_2') \to \exists$ some $B \in \mathfrak{B}$ s.t. (x_1, x_2), $(x_1', x_2') \in B$ and $(x_1, x_2) \in C(B)$

   $\succsim$ does not have to either complete or transitive.

4. Relation between preference-based and choice-based

   - $[\succsim$ is complete and transitive$] \to$ ($\mathfrak{B}$,$C(\cdot)$) satisfies the weak axiom
   - $[$($\mathfrak{B}$,$C(\cdot)$) satisfies the weak axiom$]$ and $[\mathfrak{B}$ includes all subsets of X up to 3 elements$]$
     $\to [\succsim$ is complete and transitive$]$

5. Demand of Law $\boxed{\text{ProS IV-1}}$

   Suppose an agent with ($\mathfrak{B}$,$C(\cdot)$) which satisfies the weak axiom faces a decision between (x_1, x_2) and(x_1', x_2').

   Those satisfy budget constraints $p_1 \cdot x_1 + p_2 \cdot x_2 \leq I$ and $p_1' \cdot x_1' + p_2' \cdot x_2' \leq I$

   By the weak axiom,

   $p_1 \cdot x_1 + p_2 \cdot x_2 < p_1 \cdot x_1' + p_2 \cdot x_2' \to p_1(x_1 - x_1') + p_2(x_2 - x_2') < 0$

   $p_1' \cdot x_1' + p_2' \cdot x_2' < p_1' \cdot x_1 + p_2' \cdot x_2 \to -p_1'(x_1 - x_1') - p_2'(x_2 - x_2') < 0$

   $(p_1 - p_1')(x_1 - x_1') + (p_2 - p_2')(x_2 - x_2') < 0$

   Suppose $p_2 = p_2'$.

   Then $(p_1 - p_1')(x_1 - x_1') < 0$

   It means that along the indifference curve, changes in the price of the first good and changes in the quantity of the goods move in opposite directions. The weak axiom means the negative substitution effects.

3 Relationship between Preorders

1. If $\succsim$ is strictly monotone, then it is monotone $\boxed{\text{ProS I.5}}$

   Assume that $\succsim$ is strictly monotone.

   $[x >> y] \rightarrow [x \geq y \text{ and } x \neq y]$

   By the definition of strict monotonicity, $[x \geq y \text{ and } x \neq y] \rightarrow [x \succ y]$.

   Therefore, $[x >> y] \rightarrow [x \succ y]$; that is, it is monotone.

2. If $\succsim$ is monotone, then it is locally nonsatiated.

   Assume that $\succsim$ is monotone.

   Let $\left[x \in X, \varepsilon > 0, e = (1,1,1,...1) \in R^L, \text{ and } \exists y = x + \frac{\varepsilon}{\sqrt{L}} e\right] \rightarrow [y >> x]$

   Then it implies $y >> x$ and by the definition of monotonicity, $y \succ x$.

   From the assumption, we also have $||y - x|| = \frac{\varepsilon}{\sqrt{L}} < \varepsilon$.

3. If $\succsim$ is weakly monotone, l.n.s., and transitive, then it is monotone.

   Suppose that $x >> y$.

   Let $\varepsilon = \min\{x_1 - y_1, ..., x_L - y_L\} > 0$, then for all $z \in X$, if $||y - z|| < \varepsilon$, then $x >> z$.

   By l.n.s., $\exists\ z^* \in X$ s.t.$||y - z^*|| < \varepsilon$ and $z^* \succ y$.

   By $x >> z^*$ and the weak monotonicity,$x \succsim z^*$.

   $x \succsim z^*$and $z^* \succ y$ implies $x \succ y$. Thus it is monotone.

4. If $\succsim$ is monotone and continuous, then it is weakly monotone.

5. $\succsim$ is weakly convex,

   if for $\forall\ x, y \in X$ with $x \neq y, x \succsim y, \lambda \in [0,1]$, $\lambda x + (1-\lambda) y \succsim y$

   (a) $\leftrightarrow$if for $\forall\ y \in X$, $S_y = \{x \in X \mid x \succsim y\}$ is a convex set on X

   - Take any $y \in X$, $a, b \in S_y$(w.l.o.g. $a \succsim b$), $\lambda \in (0,1)$.
     $\lambda a + (1-\lambda) b \succsim b$ by weak convexity and
     $b \succsim y$ by the definition of S_y.
     Then $[\lambda a + (1-\lambda) b \succsim b \succsim y] \rightarrow [\lambda a + (1-\lambda) b \succsim y] \rightarrow \lambda a + (1-\lambda) b \in S_y$
   - Take any $a, b \in X$(w.l.o.g. $a \succsim b$), $\lambda \in (0,1)$.
     $[a \succsim b, b \succsim b] \rightarrow [a \in S_b, b \in S_b]$ by the definition of S_b.
     $[\lambda a + (1-\lambda) b \in S_b] \rightarrow [\lambda a + (1-\lambda) b \succsim b]$
     Therefore, $\succsim$ is weakly convex.

   (b) $\leftrightarrow$if for $\forall\ y \in X$, $S'_y = \{x \in X \mid x \succ y\}$ is a convex set on X

- Take any $y \in X$, $a, b \in S'_y$(w.l.o.g. $a \succsim b$), $\lambda \in (0,1)$.
  $\lambda a + (1-\lambda)\, b \succsim b$ by weak convexity and
  $b \succ y$ by the definition of S'_y.
  Then $[\lambda a + (1-\lambda)\, b \succsim b \succ y] \to [\lambda a + (1-\lambda)\, b \succ y] \to \lambda a + (1-\lambda)\, b \in S'_y$
- Use contradiction.
  Suppose that for $\forall\, y \in X$, $S'_y = \{x \in X \mid x \succ y\}$ is a convex set on X but $\succsim$ is <u>not</u> weakly convex.
  Then $\exists x, y \in X$ with $x \neq y$, $x \succsim y$, and $\exists \lambda \in [0,1]$, $\lambda x + (1-\lambda)\, y \prec y$
  On the other hand, we have $\lambda x + (1-\lambda)\, y \succ y$ for $x, y \in X$ by the assumption.
  $\lambda x + (1-\lambda)\, y \succ \lambda x + (1-\lambda)\, y$ which is a contradiction.

6. If $\succsim$ is strictly convex, (and continuous?), then it is convex $\boxed{\text{ProS I.6}}$.

   Assume $\succsim$ is strictly convex. We have to show that for $\forall\, x, y \in X$ with $x \neq y$, $x \succ y, \lambda \in (0,1)$, $\lambda x + (1-\lambda)\, y \succ y$.

   $x \succ y \to x \succsim y$

   From the fact $\succsim$ is strictly convex, we have $\lambda x + (1-\lambda)\, y \succ y$ for all $\lambda \in (0,1)$.

   Thus, from the definition, it is convex.

7. If $\succsim$ is convex and continuous, then it is weakly convex.

   (a) Case $x \succ y$ which implies $x \succsim y$.
   Then by the definition of convexity of $\succsim$, we have $\lambda x + (1-\lambda)\, y \succ y$ for all $\lambda \in (0,1)$.
   And $[\lambda x + (1-\lambda)\, y \succ y] \to [\lambda x + (1-\lambda)\, y \succsim y]$
   Therefore, it is weakly convex.

   (b) Case $x, y \in X$ s.t. $x \sim y$
   By contradiction, $\exists \lambda \in (0,1)$ such that $y \succ \lambda x + (1-\lambda)\, y$
   By the definition of $S_{\lambda x + (1-\lambda) y}$, $\lambda x + (1-\lambda)\, y \notin S_{\lambda x + (1-\lambda) y}$
   Suppose $\exists z_1, z_2 \in X$ s.t. $z_1 = \lambda_1 x + (1-\lambda_1)\, y$ and $z_2 = \lambda_2 x + (1-\lambda_2)\, y$ s.t. $z_1 \neq z_2$.
   If $x \sim y \succ z_1$, then $\alpha_1 x + (1-\alpha_1)\, z_1 = z_2 \succ z_1$ by the definition of convexity.
   If $x \sim y \succ z_2$, then $\alpha_2 y + (1-\alpha_2)\, z_2 = z_1 \succ z_2$ by the definition of convexity.
   This is a contradiction.
   Therefore, there is no $\lambda \in (0,1)$ such that $y \succ \lambda x + (1-\lambda)\, y$

8. If $\succsim$ is weakly convex, continuous, <u>and l.n.s</u>., then it is convex.

9. Note that $\succsim$ weakly convex, continuous, <u>but not l.n.s</u>., it is not convex

   Counter example) Cobb-Douglas with a thick preference curve.

10. If $\succsim$ is strictly convex and <u>weakly monotone</u>, then it is strictly monotone.

    By contradiction, suppose $\exists \succsim$ strict convex, weakly monotone, but not strictly monotone; that is, $\exists x, y \in X$, $[x \geq y, x \neq y]$ but $[x \precsim y]$.

    Let $z = \frac{x}{2} + \frac{y}{2}$. By strict convexity, $[z \succ x]$.

    $[x \geq y] \rightarrow [x \geq z]$. By weak monotonicity, $[x \geq z] \rightarrow [x \succsim z]$ which is a contradiction.

11. Even if $\succsim$ is strictly convex and <u>monotone</u>, it is <u>not</u> strictly monotone.

    Counter Example)

    $$[x \succ y] \leftrightarrow \begin{cases} x_1 > y_1 \\ or \\ x_1 = y_1, x_2 < y_2 \end{cases}$$

12. Lexicographic preorder is complete, transitive, reflexive, strictly monotone, and strictly convex, but not continuous.

    - Lexicographic preference is not continuous.
      Consider a sequence of pairs $\{x^n, y^n\}_{n=1}^{\infty}$ s.t. $x^n = \left(\frac{1}{n}, 0\right), y^n = (0, 1)$ for all n.
      Then we know $[x^n \succ y^n]$ for all n. But at the limit, $x = \lim_{n \to \infty} x^n = (0, 0) \prec (0, 1) = \lim_{n \to \infty} y^n = y$.
      Therefore, it is not continuous.
    - Lexicographic preference can not be represented by a continuous utility function.
      By contradiction, suppose not. Then $\exists$a utility function $u(x)$ which represents lexicographic preference.
      For all x, we can pick a rational number $r(x)$ s.t. $u(x, 2) > r(x) > u(x, 1)$.; $\exists$a rational number between two real numbers.
      Let $x' > x$. Then by lexicographic preference,
      $[x' > x] \rightarrow [u(x', 2) > r(x') > u(x', 1) > u(x, 2) > r(x) > u(x, 1)] \rightarrow [r(x') > r(x)]$.
      Thus, $r(\cdot)$ provides one-to-one function from the set of real number which is uncountable to the set of rational numbers which is countable. This is a mathematical impossibility.

4 Preference Examples

1. Cobb-Douglas is a complete, continuous, monotone(not strict monotone), convex(not strictly convex), preorder.
2. Leontief is a complete, continuous, monotone(not strict monotone), convex(not strictly convex), preorder.
3. Linear indifference curve is convex, strictly monotone, but not strictly convex.
4. Reversed lexicographic is strictly convex and monotone, but not strictly monotone.
5. Exercise 3.B.2 is convex, l.n.s., but not monotone.
6. Reverse lexicographic is convex(not weak convex), monotone(not strict monotone)
7. Cobb-Douglas with a thick preference curve(weakly convex, continuous, but not l.n.s) is not convex(not strictly convex)
8. Weird lexicographic is weakly convex, but not convex.

   Counter) $[x \succ y] \leftrightarrow \begin{cases} x_1 > y_1 \\ \textit{or} \\ x_1 = y_1, x_2 \geq 1 \text{ and } y_2 < 1 \end{cases}$
9. monotone, but not convex

   $(x_1, y_1) \succsim (x_2, y_2)$ if $\sqrt{x_1^2 + y_1^2} \geq \sqrt{x_2^2 + y_2^2}$
10. not l.n.s

    $(x_1, y_1) \succ (x_2, y_2)$ if $y_1 = -x_1 + a$ and $y_2 = -x_2 + b$ s.t. $a > 1 \geq b > 0$

    $(x_1, y_1) \sim (x_2, y_2)$ if $y_1 = -x_1 + a$ and $y_2 = -x_2 + b$ if $a, b > 1$ and $a, b \leq 1$
11. $x \succ y$ if $|x_2 - y_2| > 2$

    $x \sim y$ if $|x_2 - y_2| \leq 2$

    shows a preference which is not transitive.

5 Preference and Utility function

1. Utility function

   A function $u : R_{+}^{L} \to R$ is a utility function

   if it represents individual's preference on consumption allocation;

   for $\forall x_1, x_2 \in X = R_{+}^{L}$, $x_1 \succ x_2 \Leftrightarrow u\left(x_1\right) > u\left(x_2\right)$ and $x_1 \sim x_2 \Leftrightarrow u\left(x_1\right) = u\left(x_2\right)$

2. Summary of Relation

   Restrictions on preferences translate into restrictions on the form of utility functions.

   (a) Continuous Complete Preorder $\Longrightarrow$ (with monotonicity) ✠Continuous Utility Function

   (b) Monotonicity $\Longrightarrow$ Increasing Utility Function($u\left(x\right) > u\left(y\right)$ if $x >> y$)
   strictly monotone $\Longrightarrow$ Strictly Increasing Utility Function($u\left(x\right) > u\left(y\right)$ if $x \geq y$ and $x \neq y$)

   (c) Convexity $\Longrightarrow$ Quasiconcave Utility Function{(The set $\{x \in X \mid u(x) \geq u\left(y\right)\}$ is convex for $\forall\ y\left(\neq x\right) \in X$)}
   strictly convex $\Longrightarrow$ Strictly Quasiconcave Utility Function{(The set $\{x \in X \mid u(x) > u\left(y\right)\}$ is convex for $\forall\, y\left(\neq x\right) \in X$) or ($u\left(\lambda x + \left(1-\lambda\right) y\right) > \min\left\{u\left(x\right), u\left(y\right)\right\}$ for any $x, y\left(x \neq y\right) \in X$, $\forall \lambda \in (0,1)$)}

   - Concavity: for$\forall \lambda \in (0,1)$ and $\forall x, y\left(x \neq y\right) \in X$, $\lambda u\left(x\right) + \left(1-\lambda\right) u\left(y\right) \leq u\left(\lambda x + \left(1-\lambda\right) y\right)$
     Convexity of $\succsim$ does not imply the stronger property that $u\left(\cdot\right)$ is concave.
     When is the strictly increasing transformation f of a quasiconcave function u s.t. $f \circ u$ concave?
     It depends on the curvature of the indifference surfaces.

   (d) Homotheticity{(with continuity) $\Longleftrightarrow$ $u\left(x\right)$ is homogenous of degree 1, that is, $u\left(\alpha x\right) = \alpha u\left(x\right)$ for $\alpha > 0$}

   (e) Quasilinearity{(with continuity) on $R \times R_{+}^{L-1}$ with respect to the first commodity $\Longleftrightarrow$ $\underline{u\left(x\right) = x_1 + \Phi(x_2, ..., x_L)}$}

3. Continuity of Function

   (a) u.s.c.

   - A complete preorder $\succsim$ on R_{+}^{L} is uppersemicontinuous if for any $x' \in X = R_{+}^{L}$, $\{x \in X \mid x \succsim x'\}$ is closed($\Leftrightarrow$ $\{x \in X \mid x \succ x'\}$ is open)
   - A function $u : R_{+}^{L} \to R$ is u.s.c. if for all $\alpha \in R$, $\{x \in X \mid u\left(x\right) < \alpha\}$ is open in R_{+}^{L}.
     $\leftrightarrow$if for all $\alpha \in R$, $\{x \in X \mid u\left(x\right) \geq \alpha\}$ is closed in R_{+}^{L}

   (b) l.s.c.
   A function $u : R_{+}^{L} \to R$ is l.s.c. if for all $\alpha \in R$, $\{x \in X \mid u\left(x\right) > \alpha\}$ is open in R_{+}^{L}.
   $\leftrightarrow$if for all $\alpha \in R$, $\{x \in X \mid u(x) \leq \alpha\}$ is closed in R_{+}^{L}.

(c) If it is u.s.c. and l.s.c., a function $u : R^L_+ \to R$ is continuous $\boxed{\text{ProS III-5(a)}}$

Proof

NTS: for any open(closed) subset of a, $u^{-1}(a)$ is an open(closed) subset of R^L_+

Divide R by three parts;$(-\infty, \alpha], (\alpha, \beta), [\beta, \infty)$

For an open set $(\alpha, \beta) \in R$, $u^{-1}((\alpha, \beta)) = \{x \in X \mid \alpha < u(x)\} \cap \{x \in X \mid u(x) < \beta\}$ which is an intersection of two open sets known from l.s.c. and u.s.c. of u so that $u^{-1}((\alpha, \beta))$ is open.

For a relatively closed set $(-\infty, \alpha] \in R$, $u^{-1}((-\infty, \alpha]) = \{x \in X \mid u(x) \leq \alpha\}$ is closed by l.s.c. of u

For a relatively closed set $[\beta, \infty) \in R$, $u^{-1}([\beta, \infty)) = \{x \in X \mid \beta \leq u(x)\}$ is closed by u.s.c. of u.

Therefore, $u : R^L_+ \to R$ is continuous

(d) Relationship between u.s.c $\succsim$ and u $\boxed{\text{Sp 1997 II.1}}$

i. If $\succsim$ is u.s.c., then a function $u : R_+ \to R$ exists.

Proof

For $\forall x \in X = R_+$, $\exists\, \alpha(x) \in R_+$ s.t. $\alpha(x) e \sim x$

Because $\succsim$ is upper semi continuous, for $\forall\, x' \in X$, $\{x \in X \mid x \succsim x'\}$ is closed.

Hence, for $\forall\, x' \in X$, $A_+ = \{\alpha \in R_+ \mid \alpha(x) e \succsim x' \text{ and } \alpha(x) e \in Z\}$ is nonempty and closed.

Therefore, for $\forall\, x' \in X$, $\exists\, \alpha(x') \in R_+$ s.t. $\alpha(x') e \sim x'$

Furthermore, because $\succsim$ is monotone, $(\alpha_1 > \alpha_2) \to (\alpha_1 e \succ a_2 e)$.

Take $\alpha(x)$ as the utility function; that is, for $\forall x \in X = R^L_+$, let a utility value be $u(x) = \alpha(x)$

This utility function does not have to be continuous.

ii. If $\succsim$ is u.s.c., does a function $u : R_+ \to R$ exist which takes values only between -3 and -2?

Yes,

$[x \succ x'] \leftrightarrow x \geq 0$ and $y < 0$

$[x \sim y] \leftrightarrow x, y \geq 0$ or $x, y < 0$

$u(x) = \begin{cases} -2 \text{ if } x \geq 0 \\ -3 \text{ if } x < 0 \end{cases}$ can represent the above u.s.c. preference relation.

iii. If $\succsim$ is u.s.c., does a continuous utility function $u : R_+ \to R$ exist?

No, the same as 7.

iv. If $\succsim$ is u.s.c., $p \in R_{++}$, $e \in R_{++}$,, does a competitive demand function $D : R_+ \to R$ exist?

No, if the utility function is not continuous, demand correspondence does not have to exist. Even $\succsim$ is not strictly convex so that we do not know whether demand correspondence is single-valued or not(the same as Deman Correspondence chapter 4).

4. Existence of Continuous Utility Function $\boxed{\text{Sp 1998 II.3.(c)}}$

   Theorem

   Suppose that $\succsim$ is a (monotone), continuous, complete preorder(reflexive and transitive) on $X = R^L_+$.

   Then there exists a continuous utility function $u(x)$ representing $\succsim$.

   Proof

   Denote the diagonal ray in R^L_+ by Z and $e = (1, 1, 1, ..., 1)^T \in R^L_{++}$. Then $\alpha e \in Z$ for all $\alpha \geq 0$.

   - Claim 1: For $\forall x \in X = R^L_+$, $\exists$ a unique $\alpha(x) \in R_+$ s.t. $\alpha(x) e \sim x$

     Because $\succsim$ is continuous, for $\forall\ x' \in X$, $\{x \in X \mid x \succsim x'\}$ and $\{x \in X \mid x \precsim x'\}$ are closed subsets of X.

     Hence, for $\forall\ x \in X$, $A_+ = \{\alpha \in R_+ \mid \alpha(x) e \succsim x \text{ and } \alpha(x) e \in Z\}$

     and $A_- = \{\alpha \in R_+ \mid \alpha(x) e \precsim x \text{ and } \alpha(x) e \in Z\}$ are nonempty and closed.

     Because $\succsim$ is complete, $R_+ \subseteq (A_+ \cup A_-)$

     (The nonemptyness and closeness of A_+, A_- and the connectness of R_+) $\rightarrow [(A_+ \cap A_-) \neq \emptyset]$

     Therefore, for $\forall\ x \in X$, $\exists\ \alpha(x) \in R_+$ s.t. $\alpha(x) e \sim x$.

     Furthermore, because $\succsim$ is monotone, $(\alpha_1 > \alpha_2) \rightarrow (\alpha_1 e \succ a_2 e)$.

     Therefore, for $\forall x \in X = R^L_+$, $\exists$ a unique $\alpha(x) \in R_+$ s.t. $\alpha(x) e \sim x$.

   Take $\alpha(x)$ as the utility function; that is, for $\forall x \in X = R^L_+$, let a utility value be $u(x) = \alpha(x)$

   - Claim 2
     (a) $[\alpha(x) \geq \alpha(x')] \leftrightarrow [x \succsim x']$
       i. $\rightarrow$:
          $\alpha(x) \geq \alpha(x')$
          By [(continuity and monotonicity) $\rightarrow$ weak monotonicity] of $\succsim$, $\alpha(x) e \succsim \alpha(x') e$
          By the proof of Claim 1, $\alpha(x) e \sim x \succsim x' \sim \alpha(x') e$
          By transitivity of $\succsim$, $x \succsim x'$
       ii. $\leftarrow$:
          $x \succsim x'$
          By the proof of Claim 1, $\alpha(x) e \sim x \succsim x' \sim \alpha(x') e$
          By transitivity of $\succsim$, $\alpha(x) e \succsim \alpha(x') e$
          By [(continuity and monotonicity) $\rightarrow$ weak monotonicity] of $\succsim$, $\alpha(x) \geq \alpha(x')$

(b) $u(x)$ is a continuous function
Note that the range of $u(x)$ is $[0,\infty)$ which is relatively closed.
WTS that for any $x \in X = R_+^L$ and $\forall \alpha \in [0,\infty)$, $u^{-1}([0,\alpha])$ and $u^{-1}([\alpha,\infty))$ are closed subsets of R_+^L.
$u^{-1}([0,\alpha]) = \{x \in X \mid u(x) \leq \alpha\} = \{x \in X \mid x \precsim \alpha(x)\, e\}$
$u^{-1}([\alpha,\infty)) = \{x \in X \mid u(x) \geq \alpha\} = \{x \in X \mid x \succsim \alpha(x)\, e\}$
These are closed by continuity of $\succsim$.

5. If a continuous utility function $u : R_+^L \to R$ represents $\succsim$, then $\succsim$ is continuous.

   By continuity of $u(\cdot)$, $u^{-1}((-\infty,\alpha])$ and $u^{-1}([\alpha,\infty))$ are closed for any $\alpha \in R$ and $\exists x' \in R_+^L$ s.t. $u(x') = \alpha$ for any $\alpha \in R$.

   $u^{-1}((-\infty,\alpha]) = \{x \in X \mid u(x) \leq u(x')\} = \{x \in X \mid x \precsim x'\}$

   $u^{-1}([\alpha,\infty)) = \{x \in X \mid u(x) \geq u(x')\} = \{x \in X \mid x \succsim x'\}$

   Thus $\{x \in X \mid x \precsim x'\}$ and $\{x \in X \mid x \succsim x'\}$ are closed.

   Therefore, $\succsim$ is continuous.

6. Even though $\succsim$ is a complete, continuous preorder on $X = R_+^L$ which a utility function $u : R_+^L \to R$ represents, u does not have to be continuous.

   $\Leftrightarrow$a discontinuous utility function can represent a complete, continuous preorder $\succsim$ on $X = R_+^L$

   Counter example | Sp 1999 III.2.(e) | ProS III-3

   In order for u to be continuous, we need monotonicity of $\succsim$.

   $[x_i + y_i]$ is equal to the integer part of $x_i + y_i$.

   $(x_1,y_1) \succ (x_2,y_2)$ if $[x_1 + y_1] > [x_2 + y_2]$

   $(x_1,y_1) \sim (x_2,y_2)$ if $[x_1 + y_1] = [x_2 + y_2]$

   $\succsim$ is complete because $(x_1,y_1) \succsim (x_2,y_2)$ or $(x_1,y_1) \precsim (x_2,y_2)$ or $(x_1,y_1) \sim (x_2,y_2)$ for all $(x,y) \in R_+^2$

   and is continuous because for all $(x,y) \in R_+^2$,

   $\{(x',y') \in R_+^2 \mid (x',y') \succsim (x,y)\}$and $\{(x',y') \in R_+^2 \mid (x',y') \precsim (x,y)\}$ are closed subsets.

   But $\succsim$ is not monotone obviously.

   We can represent this preference relation with $u_i(x_i,y_i) = [x_i + y_i]$

   But the utility function is not continuous.

   - This preference relation also show an example for nonsatiated but not l.n.s., not convex.\
   - A similar preference relation is $(x_1,y_1) \succsim (x_2,y_2)$ if $[x_1 y_1] \geq [x_2 y_2]$

7. Does a strong monotone preference have a utility function? Sp 1998 II.3

   Answer

   No, lexicographic preference is strict monotone, but not continuous so that there is no utility function representing $\succsim$.

8. If $\succsim$ is convex, and $u : R^L_+ \to R$ represents $\succsim$, is $u(\cdot)$ convex? Sp 1999 III.2.(h)

   Answer

   No, the utility function can be strictly concave.

   For all $x, y \in R_+$, $x \succ y \leftrightarrow \sqrt{x} > \sqrt{y}$ and $x \sim y \leftrightarrow \sqrt{x} = \sqrt{y}$

   Thus for all $\lambda \in (0, 1]$, $\left[\lambda\sqrt{x} + (1-\lambda)\sqrt{y} = \sqrt{y} + \lambda(x-y) > \sqrt{y}\right] \to [\lambda x + (1-\lambda)\, y \succ y]$ shows that it is convex.

   The utility function $u(x) = \sqrt{x}$ represents $\succsim$, but for all $x, y \in R_+$, $x \neq y$, and all $\lambda \in (0, 1)$, $\lambda u(x) + (1-\lambda)\, u(y) < u(\lambda x + (1-\lambda)\, y)$ which shows $u(\cdot)$ is strictly concave.

9. Other Properties of Utility Function

   (a) Additivity

   $u(x_1, x_2, ..., x_L) = u(x_1) + u(x_2) + ... + u(x_L)$

   (b) Differentiability

   What is required is that indifference sets should be smooth surfaces that fit together nicely so that the rates at which commodities substitute for each other depend differentiably on the consumption allocation.

10. If we can summarize the consumer's preference by means of a utility function, then mathematical programming problem(Utility Maximization problem) can be used to find maximal consumption allocations out of budget set. The utility function $u(\cdot)$ that represents a preference relation is not unique. Any strictly increasing transformation of $u(\cdot)$, $v(x) = f(u(x))$, where $f(\cdot)$ is a strictly increasing function, also represents the preference.

6 Endowments and Budget Set

1. From “complete market” and “given price” assumption, we have a price vector $p = (p_1, p_2, ..., p_L) \in R^L$ determined by auctioneer and it determines agents’ wealth(income) and the rate at which commodities can be exchanged.

2. We assume that all agents have something of some commodities, the initial resources of an individual(the means of exchange), given by a bundle in R^L_+. Given p, an agent can try to decide which bundle of commodities he will consume. Every agent faces an economic constraint: his consumption choice is limited to those commodity bundle within his possible wealth.

3. Correspondence

   A correspondence $\phi : X \to Y$ is a rule which associates to every elements $x \in X$ a nonempty subset $\phi(x) \subseteq Y$

   Graph of ϕ : given a correspondence ϕ of X into Y, the graph G_ϕ in $X \times Y$ defined by $G_\phi = \{(x, y) \in X \times Y \mid y \in \phi(x)\}$

4. Budget Correspondence

   Given $p \in R^L_+$ and initial endowments $e \in R^L_+$, an agent’s budget correspondence is $B : (p, e) \to X \left[\Longleftrightarrow \triangle \cdot R^L_+ \to R^L_+\right]$ s.t. the budget sets will be $B(p, e) = \{x \in X \mid p \cdot x \le p \cdot e\}$

5. Budget correspondence is nonempty and compact-valued if $p \in R^L_{++}$ and $e \in R^L_+$

   Later we will try to show that demand correspondence do behave continuously and to prove it, we need to know if the budget correspondence is continuous, i.e. the budget set changes continuously for some small changes in prices or initial endowments. The following simple counterexample shows that it is not always.

   - $e = (1, 0)$, $p_n = \left(\frac{1}{n}, 1\right) \to p = (0, 1)$
     For each n, the budget set is compact, but in the limit, $B(p, e) = [0, \infty)$ which is not bounded so that the budget correspondence is not l.h.c. If $p >> 0$, we can avoid this problem so that the correspondence will be continuous(u.h.c. and l.h.c) because his wealth $p \cdot e > 0 = \inf_{x \in X}(p \cdot x)$, the cheapest possible consumption bundle in R^L_+.

   $B(p, e) = \{x \in X \mid p \cdot x \le p \cdot e\}$ is bounded for all $p \in R^L_{++} (= \triangle)$ and closed for all $p \in R^L_+ (= \bar{\triangle})$ so that is compact for $p \in R^L_{++} (= \triangle)$

6. Budget correspondence is HD of 0.on p.

7. Continuous Correspondence(U.H.C. and L.H.C)

   How the set $\phi(x)$ depends on x when x is changed slightly

X, Y are subsets of Eunclidean space R^L.

- U.H.C.
  A correspondence $\phi : X \to Y$ is u.h.c. (at x) if the set $\phi(x)$ does not suddenly become much smaller with a small change in x
  A correspondence $\phi : X \to Y$ is u.h.c. (at x)
  (a) if for every open neighborhood $U \subseteq Y$ containing $\phi(x)$, $\exists$ a neighborhood V of x in X s.t. $\phi(x') \subseteq U$ for all x'
  (b) (in case $\phi(x)$ is compact-valued) if every sequence $x_n \to x$ and every sequence y_n with $y_n \in \phi(x_n)$ for all n, $\exists$ a convergent subsequence $\left\{y_{n_q}\right\} \to y$ s.t. $y \in \phi(x)$
  (c) (in case X is compact and Y is closed)if it has a closed graph and the image of compact sets are bounded.
- L.H.C.
  A correspondence $\phi : X \to Y$ is u.h.c. (at x) if the set $\phi(x)$ does not suddenly become much larger with a small change in x
  A correspondence $\phi : X \to Y$ is l.h.c. (at x)
  (a) if every sequence $x_n \to x$ and every $y \in \phi(x)$, $\exists$ a sequence $\{y_n\} \in Y$ with $y_n \in \phi(x_n)$ for all n s.t. $\{y_n\} \to y$
  (b) if for every open neighborhood $U \subseteq Y$ with $\phi(x) \cap U \neq \emptyset$, $\exists$ a neighborhood $V \subseteq X$ of x s.t. $\phi(x') \cap U \neq \emptyset$ for all $x' \in V$

The two concepts of continuity, u.h.c. and l.h.c, which are quite different for general correspondences, coincide and are equivalent to continuity in the case of functions. It is multivaluedness which plays the crucial role in making these concepts distinct from each other.

8. Continuity of Budget Correspondence $\boxed{\text{ProS IV-2}}$

   $\left[\Longrightarrow \left[p >> 0 \text{ and } e \in R^L_+\right] \to \text{budget correspondence is continuous}\right]$

   Let $p >> 0 (p \in \triangle)$ and $e \in R^L_+$. In the exchange economy, an element of X is price system and initial endowment.

   - U.H.C.
     From the environment, $B(\cdot, e)$ is compact on $\triangle$.
     Pick a sequence $\{p_n\}$ and $\{x_n\}$ with $x_n \in B(p_n, e)$ for $n \geq N$.
     Because $\triangle$ is bounded, $\exists$ a convergent subsequence $\left\{p_{n_q}\right\} \to \bar{p}$ with $x_{n_q} \in B\left(p_{n_q}, e\right)$ and
     since $B(\cdot, e)$ is compact, $\{x_n\}$ has a convergent subsequence $\left\{x_{n_q}\right\}$ s.t.$x_{n_q} \to \bar{x} \in B(p, e)$. for some p.

Now we need to show that $\bar{x} \in B\left(\bar{p}, e\right)$.
From the fact $x_{n_q} \in B\left(p_{n_q}, e\right)$, $p_{n_q} \cdot x_{n_q} \leq p_{n_q} \cdot e$.
By $\left\{p_{n_q}\right\} \to \bar{p}$ and $\left\{x_{n_q}\right\} \to \bar{x}$, and the continuity of dot product, $\bar{p} \cdot \bar{x} \leq \bar{p} \cdot e$.
Therefore, $\bar{x} \in B\left(\bar{p}, e\right)$

- L.H.C.
  Assume $\{p_n\} \to \bar{p} \in \triangle$ and $\bar{x} \in B\left(\bar{p}, e\right)$.
  Pick a sequence $\{x_n\}$ with $x_n \in B\left(p_n, e\right)$ for $n \geq N$.
  We need to show that $\{x_n\} \to \bar{x}$.
  (a) $p \cdot x = p \cdot e$
      From the assumption, $\bar{x} \in B\left(\bar{p}, e\right)$ and $x_n \in B\left(p_n, e\right)$.
      Thus, $\bar{p} \cdot \bar{x} = \bar{p} \cdot e$ and $p_n \cdot x_n = p_n \cdot e$.
      Let $x_n = \lambda x$.
      $[p_n \cdot x_n = p_n \cdot e] \Leftrightarrow [p_n \cdot \lambda x = p_n \cdot e]$
      $\lambda = \frac{p_n \cdot e}{p_n \cdot x}$ and $x_n = \frac{p_n \cdot e}{p_n \cdot x} x$.
      Then $\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{p_n \cdot e}{p_n \cdot x} x = \lim_{n\to\infty} e = e = x$.
  (b) $p \cdot x < p \cdot e$
      From the assumption, $\bar{x} \in B\left(\bar{p}, e\right)$ and $x_n \in B\left(p_n, e\right)$.
      Thus, $\bar{p} \cdot \bar{x} < \bar{p} \cdot e$ and $p_n \cdot x_n < p_n \cdot e$.
      Let $x_n = \begin{cases} a \text{ s.t. } a \in B\left(p_n, e\right) & \text{for } n < N \\ \bar{x} \text{ for } n \geq N & \end{cases}$
      Therefore, $\lim_{n\to\infty} x_n = x$.

We will use the continuity of Budget correspondence for the Maximum theorem.

7 Mathematical Stuffs

1. Cantor's Intersection Property

   $diam\ A_n = \sup_{x,x' \in A_n} ||x' - x||$

   Theorem

   Let X be a complete metric space and $\{A_n\}$ be a decreasing sequence of nonempty closed sets, i.e. $A_1 \supset A_2 \supset ... \supset A_n \supset ....$ with $diam\ A_n \rightarrow 0$.

   Then $\cap_{n=1}^{\infty} A_n = \{x\}$.

   Proof

   Choose a sequence $\{x_n\}$ s.t. $x_n \in A_n$.

   For any n, m with $n \geq m$, x_n, $x_m \in A_m$. Therefore, $||x_n - x_m|| \leq diam\ A_m$.

   Since $diam\ A_m \rightarrow 0$, this sequence is Cauchy and X is complete so that it converges to some limit x.

   Also, since $x_n \in A_m$ for all $n > m$ and since A_m is closed, $x \in A_m$ for all m.

   Thus, $x \in \cap_{n=1}^{\infty} A_n$.

   Suppose $\exists x'$ s.t.$x' \neq x$, $||x' - x|| = \varepsilon$, $x' \in \cap_{n=1}^{\infty} A_n$.

   But for n large enough, $diam\ A_n < \varepsilon$ so that it is impossible for $x', x \in A_n$ together.

   Therefore, $\cap_{n=1}^{\infty} A_n = \{x\}$.

2. Compactness and Finite Intersection Property

   Theorem

   A topological space X is compact

   iff

   every set of closed subsets of X with the finite intersection property has a property that the arbitrary intersection over the entire sets is nonempty.

   Proof

   - $\Rightarrow$
     Suppose X is compact. Then $\exists$an open cover of X, a set $\{A_\alpha\}_{\alpha \in I}$ of open subsets of X s.t. $X = \cup_{\alpha \in I} A_\alpha \Rightarrow X = \cup_{n=1}^{N} A_n$.
     It means $\{\cup_{\alpha \in I} A_\alpha\}^c = \emptyset \Rightarrow \left\{\cup_{n=1}^{N} A_n\right\}^c = \emptyset$.
     $\cap_{n=1}^{N} A_n^c \neq \emptyset \Rightarrow \cap_{\alpha \in I} A_\alpha^c \neq \emptyset$

- $\Leftarrow$
  Suppose a set of closed subset $\{B_\alpha\}_{\alpha\in I}$ satisfies the finite intersection property that is $\exists \cap_{n=1}^{N} B_n \neq \emptyset \Rightarrow \cap_{\alpha\in I} B_\alpha \neq \emptyset$
  By contraposition, $\cap_{\alpha\in I} B_\alpha = \emptyset \Rightarrow \exists \cap_{n=1}^{N} B_n = \emptyset$
  $\{\cap_{\alpha\in I} B_\alpha\}^c = \cup_{\alpha\in I} B_\alpha^c = X \Rightarrow \left\{\cap_{n=1}^{N} B_n\right\}^c = \cup_{n=1}^{N} B_\alpha^c = X$
  Therefore, X is compact.

3. Maximum Theorem

   Let $\phi : S \to T$ be a continuous correspondence where S is a metric space and T is a compact metric space.

   Let $f : S \times T \to R$ be a continuous function.

   Then the function $m : S \to R$ defined by $m(x) = \max\{f(x,y) \mid y \in \phi(x)\}$ is continuous and the correspondence $\mu : S \to T$ defined by $\mu(x) = \{y \in \phi(x) \mid f(x,y) = m(x)\}$ is nonempty, compact-valued, and u.h.c.

8 Demand Correspondence

1. The rule that assigns the consumer's set of optimal consumption allocations(the solution set of the utility maximization problem) to each price, endowment pair $(p, e) >> 0$ given consumers' preference, is denoted by $x\left(p, (\succsim, e)\right) \in R_{+}^{L}$ which is called as the Walrasian Demand Correspondence.

2. Continuity of Demand Correspondence

   If the budget set $B(p, e)$ is compact, then the demand set $x\left(p, (\succsim, e)\right)$ is nonempty and compact. Furthermore, if the budget correspondence $B(\cdot)$ is compact-valued, then the demand correspondence $x(\cdot)$ u.h.c. for $\forall\, p$ or e where $B(\cdot)$ is continuous.

   The continuity property says if in a particular environment p or e, a bundle x is preferred to another y then after a sufficiently small change of the environment from p or e to p' or e' and a sufficiently small change of the bundle x to x' and y to y', respectively, the agent will still prefer x' to y'.

3. Existence of Well Defined Demand Correspondence $\boxed{\text{Sp 1999 III.2.(d)}}$

   Theorem

   (a) Let $\succsim$ is a complete, continuous preorder on $X = R_{+}^{L}$ and $\boxed{e \in R_{+}^{L}}$.

   (b) And with $\boxed{p \in R_{++}^{L}}$, $B(p, e) = \{x \in X \mid p \cdot x \leq p \cdot e\}$ is a nonempty and compact set.

   Then $\exists \succsim$-maximal element in $B(p, e)$, i.e., $\exists$demand correspondence which is nonempty(well-defined) and compact-valued

   Proof

   N.T.S.: $\exists x^{*} \in \cap_{y \in B(p,e)} \{x \in B(p, e) \mid x \succsim y\}$ for $\forall y \in B(p, e)$.

   - For $\forall y \in B(p, e), \{x \in B(p, e) \mid x \succsim y\}$ is closed and bounded.
     Note that for $\forall y \in B(p, e), \{x \in B(p, e) \mid x \succsim y\} = \{x \in X \mid x \succsim y\} \cap B(p, e)$ which is an intersection of a closed set by continuity of preference(we just need u.s.c. of utility function) and the budget set which is compact by the assumption so that $.\{x \in B(p, e) \mid x \succsim y\}$ is closed and bounded.
   - Cantor's intersection property
     Using the cantor's intersection property, if we want to show $\cap_{y \in B(p,e)} \{x \in B(p, e) \mid x \succsim y\} \neq \emptyset$, we have to show for some finite set of y, $\{y_n\}_{n=1}^{N}, \cap_{y \in \{y_n\}_{n=1}^{N}} \{x \in B(p, e) \mid x \succsim y\} \neq \emptyset$.

     (a) $\left[\left[\succsim \text{ is } \underline{\text{complete}}\right] \rightarrow \left[\exists \text{ a function } \pi : \{1, 2, , , N\} \rightarrow \left\{y_{\pi(N)} \succsim y_{\pi(N-1)} \succsim \cdots \succsim y_{\pi(1)}\right\}\right]\right]$
     $\Rightarrow \left\{x \in B(p, e) \mid x \succsim y_{\pi(N)}\right\} \subset \left\{x \in B(p, e) \mid x \succsim y_{\pi(N-1)}\right\} \subset \ldots \subset \left\{x \in B(p, e) \mid x \succsim y_{\pi(1)}\right.$

$\Rightarrow \cap_{y\in\{y_n\}_{n=1}^N} \{x \in B(p,e) \mid x \succsim y\} = \left\{x \in B(p,e) \mid x \succsim y_{\pi(1)}\right\}$
$\Rightarrow \cap_{y\in\{y_n\}_{n=1}^N} \{x \in B(p,e) \mid x \succsim y\} \neq \emptyset$ because $\exists y_{\pi(1)} \in \cap_{y\in\{y_n\}_{n=1}^N} \{x \in B(p,e) \mid x \succsim y\}$ by reflexivity of $\succsim$.

(b) By the cantor's intersection property,
$\left[\cap_{y\in\{y_n\}_{n=1}^N} \{x \in B(p,e) \mid x \succsim y\} \neq \emptyset\right] \rightarrow \left[x(p,(\succsim,e)) = \cap_{y\in B(p,e)} \{x \in B(p,e) \mid x \succsim y\} \neq\right.$

- Furthermore, the infinite intersection of compact set is compact so $x(p,(\succsim,e))$ is compact.

4. Counter Example $\boxed{\text{ProS III-1}}$

   (a) $\succsim$ is not continuous.
   $\exists$ two goods #1, 2, $e = (0.5, 0.5)$, $p = (1,1) \in R^2_{++}$.
   Define $B(p,e) = \left\{(x_1,x_2) \in X = R^2_+ \mid x_1 + x_2 \leq 1\right\}$ which is nonempty and compact.
   Define $x \succ y$ iff $\begin{cases} x_1 \cdot x_2 > y_1 \cdot y_2 \text{ for } x_1,x_2,y_1,y_2 > 0 \text{ s.t.} x_1 \neq x_2 \text{ and } y_1 \neq y_2 \\ x_1 \neq x_2 \text{ and } y_1 = y_2 \text{ for } x_1,x_2,y_1,y_2 > 0 \\ x_1 \neq x_2 \text{ for } x_1,x_2 > 0 \text{ and } (y_1 = 0 \text{ or } y_2 = 0) \end{cases}$
   and $x \sim y$ iff $\begin{cases} x_1 \cdot x_2 = y_1 \cdot y_2 \text{ for } x_1,x_2,y_1,y_2 > 0 \text{ s.t.} x_1 \neq x_2 \text{ and } y_1 \neq y_2 \\ x_1 = x_2 \text{ and } y_1 = y_2 \text{ for } x_1,x_2,y_1,y_2 > 0 \\ x_1 = x_2 \text{ for } x_1,x_2 > 0 \text{ and } (y_1 = 0 \text{ or } y_2 = 0) \end{cases}$
   If $\succsim$ is not continuous, then utility function representing it is not continuous.
   We can represent $\succsim$ using the following utility function.
   $u(x_1,x_2) = \begin{cases} x_1 \cdot x_2 \text{ for } x_1,x_2 > 0 \text{ s.t.} x_1 \neq x_2 \\ 0 \text{ otherwise} \end{cases}$
   In the case, there is no $\succsim$ $-$maximal allocation on $B(p,e) \Rightarrow [x(p,(\succsim,e)) \neq \emptyset]$

   (b) $B(p,e)$ is not compact.

      i. Suppose $p \in R^2_+ \neq R^2_{++}$.
      Let $\succsim$ be a strict monotone, complete, continuous preorder on $X = R^2_+$ and $B(p,e) = \left\{(x_1,x_2) \in X = R^2_+ \mid x_1 \leq 1\right\}$ which is not bounded.
      Then for $\forall (x_1,x_2) \in X$, we can find (x_1,x_2'), i.e. $(x_1,x_2') = (x_1,x_2+1) \succ (x_1,x_2)$ s.t. $(x_1,x_2+1) \in B(p,e)$.
      Therefore, there is no $\succsim$ $-$maximal allocation on $B(p,e) \Rightarrow [x(p,(\succsim,e)) \neq \emptyset]$

      ii. Suppose $e \in R^2 \neq R^2_+$
      Let $\succsim$ be a strict monotone, complete, continuous preorder on $X = R^2_+$ and $B(p,e) = \emptyset$.
      Therefore, $x(p,(\succsim,e)) = \emptyset$, too.

5. Property of Demand Correspondence

Environment

$E = \left\{ I, (\succsim_i, e_i)_{i \in I} \right\}$

$X_i = R^L_+$ for $\forall i$ and $X = \{X_i\}_{i \in I}$

$\succsim_i$is a (strict monotone(l.n.s.)), (strict convex), complete, continuous preorder

$e_i \in R^L_+$ for $\forall i$ and $e = \{e_i\}_{i \in I}$

$B_i(p, e_i) = \{x_i \in X_i \mid p \cdot x_i \leq p \cdot e_i\}$ for $\forall i$

$\boxed{p \in R^L_{++} \text{ (or } \triangle)}$

- Individual Demand Correspondence
  $D_i(p, e_i) = x_i(p, (\succsim_i, e_i)) = \{x_i \in B_i(p, e_i) \mid x_i \succsim y_i \text{ for } \forall y_i \in B_i(p, e_i)\}$
  (a) nonempty, compacted-valued, u.h.c. for $\forall p \in R^L_{++}$
  Since $p \in R^L_{++}$, $e_i \in R^L_+$, and $\succsim_i$is a complete, continuous preorder,
  $B_i : (p, e_i) \to X_i$ is continuous where (p, e_i) is a metric space and $B_i(p, e_i) \in X_i$ is compact.
  $u : (p, e_i) \times X_i \to R$ be a continuous function.
  By the maximum theorem, the function $m : (p, e_i) \to R$ defined by $m(p, e_i) = \max\{u((p, e_i), x_i) \mid x_i \in B_i(p, e_i)\}$ is continuous and
  the correspondence $D_i : (p, e_i) \to X_i$ defined by
  $D_i(p, e_i) = \{x_i \in B_i(p, e_i) \mid u_i((p, e_i), x_i) = m(p, e_i)\}$ which is nonempty, compact-valued, and u.h.c.
  (b) HD 0 in $p \in R^L_{++}$
  Budget Correspondence $B_i(p, e_i)$ is HD 0 in $p \in R^L_{++}$ and $\succsim_i$maximization problem does not change, so $D_i(p, e_i) = D_i(\lambda p, e_i)$ for $\forall \lambda > 0$
  (c) Walras' law if $\succsim_i$ is l.n.s.
  For $\forall p \in R^L_{++}$ (or $\triangle$) $[x_i \in D_i(p, e_i)] \to [p \cdot x_i = p \cdot e_i]$
  Proof
  Suppose not. $\exists x'_i \in D_i(p, e_i)$ s.t. $p \cdot x'_i < p \cdot e_i$.
  By l.n.s. of $\succsim_i$, for $\varepsilon > 0$, $\exists x''_i \in N_\varepsilon(x'_i)$ with.$\|x''_i - x'_i\| < \varepsilon$ and $p \cdot x''_i < p \cdot e_i$ s.t.$x''_i \succ x'_i$.
  It contradicts $x'_i \in D_i(p, e_i)$.
  (d) Bounded from below
  $\exists s > 0$ s.t. $D^l_i(p, e_i) > -s + e^l_i$ for all $l \in L$ and all p
  Direct from $X_i = R^L_+$
  (e) Boundary condition
  If $\boxed{e_i \in R^L_{++}}$, $\succsim_i$is strictly monotone,
  then $[p_n \to p \in \partial\triangle] \to \left[\|D_i(p, e_i)\| \text{ is unbounded} \Leftrightarrow \sup_{z^l \in Z_i(p_n, e_i)} |Z_i(p_n, e_i)| = \infty\right]$

Proof
Suppose not. $e_i \in R^L_{++}$, $\succsim_i$is strictly monotone, $[p_n \to p \in \partial\triangle]$, but $||D_i(p,e_i)||$ is bounded.
For $\forall n$, because $D_i(p_n,e_i)$ is HD 0 in $p_n \in R^L_{++}$, we can represent the price system $p_n \in \triangle$ for $\forall p_n \in R^L_{++}$.
Since, for $p_n \in \triangle$, $D_i(p_n,e_i)$ is compact-valued, $\exists$ a convergent subsequence $\{x_{n_q}\} \to x$ with $p_{n_q}(\in \triangle) \to p \in \partial\triangle$ s.t. $x_{n_q} \in D_i(p_{n_q},e_i)$.
Then $p_{n_q} \cdot x_{n_q} \le p_{n_q} \cdot e_i$ for $\forall n_q$.
In the finite space, by the continuity of dot product, $[p_{n_q} \cdot x_{n_q} \le p_{n_q} \cdot e_i] \to [p \cdot x \le p \cdot e_i]$ with $x \in D_i(p,e_i)$
W.l.o.g., let $p^k = 0$, $k = 1$.
Let $x' = x + (1,0,0,...,0)$ and then $p \cdot x = p \cdot x' \le p \cdot e_i$
By strict monotonicity of $\succsim_i$, $x' \succ x$.
Because $\succsim_i$is continuous, $\exists N$ s.t.for $n \ge N$, $x_{n_q} \in N(x)$ and $x'_{n_q} \in N(x')$.
And .$p_n \cdot x_{n_q} = p_n \cdot x'_{n_q} \le p_n \cdot e_i$ with $x'_{n_q} \succ x_{n_q}$.
It contrdicts $x_{n_q} \in D_i(p_{n_q},e_i)$?!.

(f) HD 1 in $p \cdot e_i$ if $\succsim_i$ is homothetic

(g) Convex-valued if $\succsim_i$is weakly convex
$\succsim_i$is weakly convex $\to$for $\forall x_i, x'_i \in X_i$ with $x_i \ne x'_i$, $[x_i \succsim x'_i$ for $\forall\lambda \in [0,1]] \to [\lambda x_i + (1-\lambda)x'_i \succsim x'_i]$
Suppose not; $D_i(p,e_i)$ is not convex-valued.
Then $\exists p \in \triangle$ with $x_i, x'_i \in D_i(p,e_i)$ for $\lambda \in [0,1]$ s.t. $\lambda x_i + (1-\lambda)x'_i \notin D_i(p,e_i)$
By the definition of demand correspondence, $x_i \sim x'_i$.
By weak convexity of $\succsim_i$, $\lambda x_i + (1-\lambda)x'_i \succsim x_i \sim x'_i$
Therefore, $\lambda x_i + (1-\lambda)x'_i \notin D_i(p,e_i)$?!(contradiction)

(h) Singled-valued(function) if $\succsim_i$is strictly convex
Suppose not;$\exists x_i, x'_i \in D_i(p,e_i)$ and then $x_i \sim x'_i$.
By strict convexity of $\succsim_i$, $\lambda x_i + (1-\lambda)x'_i \succ x_i \sim x'_i$.
We have to make it sure that $\lambda x_i + (1-\lambda)x'_i$ is feasible.
From $x_i, x'_i \in D_i(p,e_i)$,
$p \cdot x_i \le p \cdot e_i \to \lambda \cdot p \cdot x_i \le \lambda \cdot p \cdot e_i$ for $\forall\lambda \in [0,1]$
$p \cdot x'_i \le p \cdot e_i \to (1-\lambda) \cdot p \cdot x'_i \le (1-\lambda) \cdot p \cdot e_i$ for $\forall\lambda \in [0,1]$
$p \cdot [\lambda \cdot x_i + (1-\lambda) \cdot x'_i] \le p \cdot [\lambda \cdot e_i + (1-\lambda) \cdot e_i]$ for $\forall\lambda \in [0,1]$
Therefore, $x_i, x'_i \notin D_i(p,e_i)$?!(contradiction)

- FP 1994 III-1
  Assumption on X_i, $\succsim_i$, e_i, p
  to satisfy that a excess demand function Z is continuous, HD of 0, bounded from below, Walras' law, Boundary condition
  $E = \{I, (\succsim_i, e_i)_{i \in I}\}$

$X_i = R^L_+$ for $\forall i$ and $X = \{X_i\}_{i \in I}$

$$\succsim_i \left\{ \begin{array}{ccc} \text{strict monotone(l.n.s)} & \rightarrow & \text{boundary condition(walras' law)} \\ \text{strict convex} & \rightarrow & \text{single-valued correspondence} \\ \text{continuous(+monotone)} & \rightarrow & \text{continuous utiity function} \rightarrow \text{existence of } D_i(p, e_i) \end{array} \right.$$

$$e_i \in \left\{ \begin{array}{ccc} R^L_+ & \rightarrow & \text{for all the other conditions} \\ R^L_{++} & \rightarrow & \text{Boundary condition} \end{array} \right. \rightarrow R^L_{++}$$

$\boxed{p \in R^L_{++}\ (\text{or } \triangle)}$→for every condition

- Aggregate Demand Correspondence

  $\Phi(p, e) = \sum_{i \in I} D_i(p, e_i)$

  (a) nonempty, compacted-valued, u.h.c. for $\forall p \in R^L_{++}$
  Sum of compact-valued and u.h.c. correspondences is compact-valued and u.h.c.

  (b) HD 0 in $p \in R^L_{++}$

  (c) Walras' law if $\succsim_i$ is l.n.s
  For $\forall p \in R^L_{++}$, $[x_i \in D_i(p, e_i)$ for $\forall i \in I] \rightarrow [p \cdot x = p \cdot e]$

  (d) Bounded from below

  (e) Boundary condition
  At least one agent satisfies it is sufficient

  (f) HD 1 in $p \cdot e_i$ if $\succsim_i$ is homothetic for $\forall i$

  (g) Convex-valued if $\succsim_i$is weakly convex for $\forall i$

  (h) Singled-valued(function) if $\succsim_i$is strictly convex for $\forall i$
  everyone satisfies it is necessary.

- Counter Example of Boundary condition

  Let consider two agents' pure exchange economy

  (a) $X_i = R^L_+$ for $\forall i$ and $X = \{X_i\}$

  (b) $\succsim_i$is strictly monotone, complete, continuous preorder
  $X_i = R^L_+$ for $\forall i, e_i \in R^L_{++} = ((1,1), (1,1))$.
  If two agents'preference are leontief which is not strict monotone, then $D_1((0,1), (1,1)) = D_2((0,1), (1,1)) = (1,1)$ which is bounded.

  (c) $e_i \in R^L_{++}$
  $X_i = R^L_+$ for $\forall i$,$\succsim_i$is strictly monotone, complete, continuous preorder→ $u_i(x_1, x_2) = x_1 + x_2$ for $\forall i$.
  If $e_1 = (1,0) = e_2 \in R^2_+ \neq R^2_{++}$, then $D_1((1,0), (1,0)) = D_2((1,0), (1,0)) = (1,0)$ which is bounded.

- $\boxed{\text{ProS V-1}}$

  Show that the boundary condition

If $e_i \in R_{++}^L$, $\underline{\succsim_i \text{is strictly monotone}}$,
then $[p_n \to p \in \partial\triangle] \to \left[||D_i\left(p, e_i\right)|| \text{ is unbounded} \Leftrightarrow \sup_{z^l \in Z_i(p_n, e_i)} \left|Z_i\left(p_n, e_i\right)\right| = \infty \right]$
is not equivalent to the statement "if $p_1 \to 0$, $Z^1\left(\left(p_1, p_{-1}\right), e\right) \to \infty$"

$\boxed{\text{Counter example}}$

$\exists 3$ goods. $e = (1, 1, 1)$
$\succsim$ is a strictly monotone, complete, continuous preorder$\to u\left(x_1, x_2, x_3\right) = \ln x_1 + x_2 + \ln x_3$ which is strictly increasing, continuous on R_+^3.
Let the three goods' price sequence $p_n = \left(\frac{1}{n}, \frac{1}{n}, \frac{n-2}{n}\right) \to p = (0, 0, 1) \in \partial\triangle$ as $n \to \infty$
From the maximization problem, we get $\left(x_{1,n}, x_{2,n}, x_{3,n}\right) = \left(1, n-2, \frac{1}{n-2}\right) \to x = (1, \infty, 0)$.
Therefore, even if $p_{1,n} \to 0$, $\left[Z^1\left(p_n, e\right) = x_{1,n} - 1\right] \to 0 \neq \infty$.

- $\boxed{\text{FP 1994 III-1}}$
  (a) Demand function is $\underline{\text{continuous, HD of 0, Walras' law}}$
    i. Continuous demand function
    Existence of Demand Correspondence
    $B\left(p, e\right)$ is nonempty, compact-valued, and continuous
    $u\left(\cdot\right)$ is continuous.
    $\to$ *Maximum Theorem*
    $\to$ Demand Correspondence exists and it is nonempty, compact-valued, u.h.c.
    In order to get demand function, we need singled valued, u.h.c. correpondence.
    $\succsim$ is stricyly convex or $u\left(\cdot\right)$ is strictly quasiconcave.
    ii. HD of 0
    iii. Walras' law
    $\succsim$ is l.n.s. or $u\left(\cdot\right)$ is weakly increasing($\Leftrightarrow \succsim$ is monotone).

9 Competitive Equilibrium

9.1 Definition

9.1.1 Static Pure Exchange Economy

An economy is a pure exchange economy if its only technological possibility is that of free disposal; that is, if for $\forall j \in J$, $Y_j = -R_+^L$

1. Environment

   Commodity Space=R^L

   $X_i = R_+^L$ for $\forall i \in I$

   $\succsim_i$is a complete continuous preorder defined on $X_i = R_+^L$ for $\forall i \in I$

   $e_i \in R_+^L$

2. Competitive Equilibrium

   In the pure exchange economy E, a Competitive Equilibrium consists of $\{x_i^*\}_{i\in I}$ s.t. $x_i^* \in D_i(p^*, e_i)$ for $\forall i \in I$ and $p^* \in R_+^L \backslash \{0\}$ s.t.

   (a) for $\forall i \in I$, x_i^* is $\succsim_i$ − maximal in $B_i(p^*, e_i) = \{x_i \in X_i \mid p^* \cdot x_i \leq p^* \cdot e_i\}$
   $\Leftrightarrow x_i^* \in \arg\max_{x_i \in B_i(p^*, e_i)} u_i(x_i)$ s.t. $B_i(p^*, e_i) = \{x_i \in X_i \mid p^* \cdot x_i \leq p^* \cdot e_i\}$

   (b) $\sum_{i\in I} x_i^* \leq \sum_{i \in I} e_i$

3. C.E. and l.n.s.$\succsim$

   If $\succsim_i$is l.n.s. and $\exists (p^*, x^*)$ given e, then $p^* \cdot x_i^* = p^* \cdot e_i$ for $\forall i \in I$ and $[x_i \succsim x_i^*] \rightarrow [p^* \cdot x_i \geq p^* \cdot x_i^* = p^* \cdot e_i]$

4. Free disposal and nonnegative price vector $p^* \geq 0$

   - Free disposal
     $X = R_+^L$
     For $\forall x \in X$, $x_l \geq x_l'$ for $\forall l \in L \rightarrow x' \in X$

   Theorem

   With l.n.s. preference and free disposal technology in the economy, then $p^* \geq 0$

   Proof

   Since (p^*, x^*) is a C.E., $p^* \cdot x_i^* = p^* \cdot e_i$ for $\forall i \in I \rightarrow p^* \cdot x^* = p^* \cdot e$

   From free disposality, for $\forall \varepsilon > 0$, $x' = (x_1 - \varepsilon, x_2, x_3, ..., x_L) \in X = R_+^L$ and $p^* \cdot x' \leq p^* \cdot e$

Suppose $p^* \ngeq 0$;w.l.o.g. $p_1^* < 0$
Given $\varepsilon > 0$, $p^* \cdot x' > p^* \cdot e$ which contradicts $p^* \cdot x' \leq p^* \cdot e$.
Therefore, $p^* \geq 0$.

5. $p^* \geq 0$ and monotone $\succsim$
   If at least one consumer has monotone preference, then $p^* \geq 0$
6. $p^* >> 0$ and stricy monotone $\succsim$
   If at least one consumer has strict monotone preference, then $p^* > 0$

Remark 1. *An economy is a pure exchange economy if its only technological possibility is that of free disposal; that is, if for* $\forall j \in J$, $Y_j = -R_+^L$

9.1.2 Production Economy

1. Environment
   Commodity Space=R^L
   $X_i = R_+^L$ for $\forall i \in I$
   $\succsim_i$is a complete continuous preorder defined on $X_i = R_+^L$ for $\forall i \in I$
   $e_i \in R_+^L$
   $Y_j = R^L$ for $\forall j \in J$
   $\theta_{ij} \in [0,1]$ for $\forall i, j$ s.t. $\sum_{i\in I} \theta_{ij} = 1$ for $\forall j$
2. Competitive Equilibrium
   (a) In the production economy E_p, a Competitive Equilibrium consists of $\left\{ \{x_i^*\}_{i\in I}, \left\{y_j^*\right\}_{j\in J} \right\}$ and $p^* \in R_+^L \backslash \{0\}$ s.t.
      i. for $\forall i \in I, x_i^*$ is $\succsim_i -$ maximal in $B_i(p^*, e_i) = \left\{ x_i \in X_i \mid p^* \cdot x_i \leq p^* \cdot e_i + p^* \sum_{j\in J} \theta_{ij} \cdot y_j^* \right\}$
      $\Leftrightarrow x_i^* \in \arg\max_{x_i \in B_i(p^*, e_i)} u_i(x_i)$ s.t. $B_i(p^*, e_i) = \left\{ x_i \in X_i \mid p^* \cdot x_i \leq p^* \cdot e_i + p^* \sum_{j\in J} \theta_{ij} \cdot \right.$
      ii. for $\forall j \in J$, $y_j^* \in Y_j$, $p^* \cdot y_j \leq p^* \cdot y_j^*$ for $\forall y_j \in Y_j$
      iii. $\sum_{i\in I} x_i^* \leq \sum_{i\in I} e_i + \sum_{j\in J} y_j^*$
   (b) In the production economy E_p, a price equilibrium with transfers consists of $\left\{ \{x_i^*\}_{i\in I}, \left\{y_j^*\right\}_{j\in J} \right\}$ and $p^* \in R_+^L \backslash \{0\}$ s.t.
      i. $\exists (w_1, w_2, ..., w_I)$ s.t. $\sum_{i\in I} w_i = \sum_{i\in I} p^* \cdot e_i + \sum_{j\in J} p^* \cdot y_j^*$

   ii. for $\forall i \in I$, x_i^* is $\succsim_i -$ maximal in $B_i(p^*, e_i) = \{x_i \in X_i \mid p^* \cdot x_i \leq w_i\}$
   iii. for $\forall j \in J$, $y_j^* \in Y_j$, $p^* \cdot y_j \leq p^* \cdot y_j^*$ for $\forall y_j \in Y_j$
   iv. $\sum_{i \in I} x_i^* \leq \sum_{i \in I} e_i + \sum_{j \in J} y_j^*$

(c) In the production economy E_p, a price quasiequilibrium with transfers consists of $\left\{\{x_i^*\}_{i\in I}, \left\{y_j^*\right\}\right\}$ and $p^* \in R_+^L \backslash \{0\}$ s.t.
   i. $\exists (w_1, w_2, ..., w_I)$ s.t. $\sum_{i \in I} w_i = \sum_{i \in I} p^* \cdot e_i + \sum_{j \in J} p^* \cdot y_j^*$
   ii. for $\forall i \in I$, $x_i^* \in X_i = R_+^L$ s.t. $p^* \cdot x_i^* \leq w_i$,
   and $x_i \in X_i = R_+^L$ and $x_i \succ_i x_i^*$ s.t. $p^* \cdot x_i \geq w_i$ for $\forall x_i \in X_i, x_i \neq x_i^*$
   iii. for $\forall j \in J$, $y_j^* \in Y_j$, $p^* \cdot y_j \leq p^* \cdot y_j^*$ for $\forall y_j \in Y_j$
   iv. $\sum_{i \in I} x_i^* \leq \sum_{i \in I} e_i + \sum_{j \in J} y_j^*$

(d) Relationship among concepts
   i. In order to relate the idea of pareto optimality to supportability by means of price taking behavior, it is useful to introduce a notion of equilibrium that allows for a more general determination of consumer's wealth levels than that in a private ownership economy. By way of motivation, we can imagine a situation where a social planner is able to carry out (lump-sum) redistributions of wealth, and where society's aggregate wealth can therefore be redistributed among agents in any desired manner.

9.2 Excess Demand Correspondence

- We charaterize C.E. as solutions to a system of aggregate excess demand equations

  Given $e_i \in R^L_+$ for $\forall i$ and $e = \{e_i\}_{i \in I} >> 0$ and $p \in R^L_+$, $Z_i(p, e_i) = x_i(p, e_i) - e_i$ s.t. $x_i(p, e_i) \in D_i(p, e_i)$

  Given $e_i \in R^L_+$ for $\forall i$ and $e = \{e_i\}_{i \in I} >> 0$ and $p \in R^L_+$, $Z(p, e) = \sum_{i \in I} Z_i(p, e_i) = \sum_{i \in I} x_i(p, e_i) - \sum_{i \in I} e_i$ s.t. $x_i(p, e_i) \in D_i(p, e_i)$

  Therefore, $p \in R^L_+$ is an equilibrium price system iff $Z(p, e) = \sum_{i \in I} x_i(p, e_i) - \sum_{i \in I} e_i \leq 0$.

  Note that if p is an equilibrium price system in pure exchange economy with locally non-satiated preferences, then $p \geq 0$, $Z(p, e) \leq 0$, and $p \cdot Z(p, e) = 0$. Thus for every $l \in L$, $Z^l(p, e) \leq 0$, but also $Z^l(p, e) = 0$ iff $p > 0$. It means that at an equilibrium, a good l can be in an excess supply($Z(p, e) < 0$) iff it is free($p_l = 0$)

- Furthermore, with strong monotonicity assumption, $p >> 0$; otherwise agents would demand an unboundedly large amount of all the free goods which is not feasible. We can conclude that with strong monotone preferences, p an equilibrium price system iff it clears all markets; that is, iff it solves the system of L equations in L unknowns.

$$\left[Z^l(p, e) = 0 \text{ for } \forall l \in L\right] \rightarrow Z(p, e) = 0$$

- Aggregate Excess Demand Correspondence

1. nonempty, compacted-valued, u.h.c. for $\forall p \in R^L_{++}$

   Sum of compact-valued and u.h.c. correspondences is compact-valued and u.h.c.

2. HD 0 in $p \in R^L_{++}$
3. Walras' law if $\succsim_i$ is l.n.s

   For $\forall p \in R^L_{++}, [x_i \in D_i(p, e_i) \text{ for } \forall i \in I] \rightarrow [p \cdot x = p \cdot e]$

4. Bounded from below
5. Boundary condition

   At least one agent satisfies it is sufficient

6. HD 1 in $p \cdot e_i$ if $\succsim_i$ is homothetic for $\forall i$
7. Convex-valued if $\succsim_i$is weakly convex for $\forall i$
8. Singled-valued(function) if $\succsim_i$is strictly convex for $\forall i$

   everyone satisfies it is necessary.

- Assumption on X_i, $\succsim_i$, e_i, p

  to satisfy that a excess demand function Z is continuous, HD of 0, bounded from below, Walras' law, Boundary condition

  $E = \left\{I, (\succsim_i, e_i)_{i\in I}\right\}$

  $X_i = R^L_+$ for $\forall i$ and $X = \{X_i\}_{i\in I}$

$$\succsim_i \left\{ \begin{array}{ccc} \text{strict monotone(l.n.s)} & \rightarrow & \text{boundary condition(walras' law)} \\ \text{strict convex} & \rightarrow & \text{single-valued correspondence} \\ \text{continuous(+monotone)} & \rightarrow & \text{continuous utiity function} \rightarrow \text{existence of } D_i\left(p, e_i\right) \end{array} \right.$$

$$e_i \in \left\{ \begin{array}{lll} R^L_+ & \rightarrow & \text{for all the other conditions} \\ R^L_{++} & \rightarrow & \text{Boundary condition} \end{array} \right. \rightarrow R^L_{++}$$

$\boxed{p \in R^L_{++} \text{ (or } \triangle)}$ →for every condition

- 1997 Final

  Define $Z\left(p, e\right) = \sum_{i\in I} x_i\left(p, e_i\right) - \sum_{i\in I} e_i$ s.t. $x_i\left(p, e_i\right) \in D_i\left(p, e_i\right)$

  1. Let $Z : \triangle \cdot R^{IL}_{++} \rightarrow R^L$ be a continuous function.

     Answer

     Prove $E = \left\{(p, e) \in \triangle \cdot R^{IL}_{++} \mid Z\left(p, e\right) = 0\right\}$ is closed.

     Under a continuous function $Z\left(\cdot\right)$, the inverse image of a closed set through a continuous function is closed.

     Because $Z\left(\cdot\right)$ is a continuous function, the inverse image of $\{0\}$ which is closed as a singleton set, $Z^{-1}\left(\{0\}\right)$, is closed on $\triangle \cdot R^{IL}_{++}$.
  2. If $Z\left(p, e\right)$ is the aggregate excess demand at price vector $p \in \triangle$ and endowment $e \in R^{IL}_{++}$, then provide economic interpretation of E.

     Answer

     If $Z\left(p, e\right)$ is the aggregate excess demand at price vector $p \in \triangle$ and endowment $e \in R^{IL}_{++}$, $Z\left(p, e\right) = 0$ means that x^* is the equilibrium allocation and it is possible that there are more than one equilibrium so that for each equilibrium allocation, we can find a equilibrium price system p^* given $e \in R^{IL}_{++}$.

     Therefore, $E =$ the set of pairs of equilibrium prices and endowments which support equilibrium allocations.
  3. Suppose that the aggregate excess demand in a pure exchange economy in which each consumer $i \in I$ has a continuous utility function $u_i : R^L_+ \rightarrow R$.

     Are there any additional assumptions needed to guarantee that $Z\left(\cdot\right)$ is a continuous function?

Under the assumption from C, prove that $Z(\cdot)$ is a continuous function.

Answer

Given (p, e),

First, for $Z(\cdot)$ to be a function, the demand correspondence $D_i(p, e_i)$ for $\forall i \in I$ has to be a function, which means that a single-valued correspondence. For this, we need $\succsim_i$to be strictly convex for $\forall i \in I$ and the equivalent property of utility function is that u_i is strictly quasiconcave for $\forall i \in I$.

Second, we know from mathematical fact that a upper hemi continuous single-valued function is continuous so that we need $D_i(p, e_i)$ for $\forall i \in I$ to be upper hemi continuous. In order to show it, we need to use the maximum theorem so that we have to show the assumptions for the theorem to be satisfied.

$D_i : \triangle \cdot R^{IL}_{++} \rightarrow R^L_+$ s.t. $D_i(p, e_i) \subseteq B_i(p, e_i)$ which is nonempty-valued, compact-valued, and continuous by the condition .$p \in \triangle$ and $e \in R^{IL}_{++}$.

(a) $\triangle \cdot R^{IL}_{++}$:a metric space, $D_i(p, e_i) \subseteq B_i(p, e_i)$ is a compact.metric space.

(b) $B_i : \triangle \cdot R^{IL}_{++} \rightarrow R^L_+$ is continuous

(c) $u_i : R^L_+ \rightarrow R$ is a continuous function for $\forall i \in I$

(d) By the maximum theorem, $D_i : \triangle \cdot R^{IL}_{++} \rightarrow R^L_+$ is nonempty-valued, compact-valued, and u.h.c.

Therefore, the only additional assumption we needed is that u_i is strictly quasiconcave for $\forall i \in I$

9.3 Existence of C.E.

9.3.1 A pure exchange economies with finite commodity spaces

Question

- Given the fact that the behavioral assumption of price taking and the institutional assumptions of complete markets hold,

  under what conditions does the L system of equations have solutions?

  Now, we will argue that under the assumption above for the excess demand function to have the five properties, ∃an equilibrium price p s.t. $Z(p,e)=0$

  Conceptually, the assurance of existence of an equilibrium means that our equilibrium notion passes the logical test of consistency. It tells us that the mathematical model is well suited to the purposes it has been designed for.

9.3.2 $L=2$(Two commodities case)

1. For this case, we do not need any complex mathematical theory to show ∃an equilibrium.

   $E=\left\{I,(\succsim_i,e_i)_{i\in I}\right\}$

   $X_i=R^2_+$ for $\forall i$ and $X=\{X_i\}_{i\in I}$

   $$\succsim_i \left\{\begin{array}{ccc} \text{strict monotone(l.n.s)} & \rightarrow & \text{boundary condition(walras' law)} \\ \text{strict convex} & \rightarrow & \text{single-valued correspondence} \\ \text{continuous(+monotone)} & \rightarrow & \text{continuous utiity function} \rightarrow \text{existence of } D_i(p,e_i) \end{array}\right.$$

   $$e_i \in \left\{\begin{array}{ccl} R^2_+ & \rightarrow & \text{for all the other conditions} \\ R^2_{++} & \rightarrow & \text{Boundary condition} \end{array}\right. \rightarrow R^2_{++}$$

   $\boxed{p\in R^2_{++}\text{ (or }\triangle)}$→for every condition

   Define $Z_i(p,e_i)=x_i(p,e_i)-e_i$ s.t. $x_i(p,e_i)\in D_i(p,e_i)$ and $Z(p,e)=\sum_{i\in I}x_i(p,e_i)-\sum_{i\in I}e_i$ s.t. $Z(p,e)=\left\{Z^1(\cdot),Z^2(\cdot)\right\}$

   (a) By $\boxed{\text{homogeneity of degree 0}}$ of $Z(\cdot)$, we can normalize $p_1=1$ and look for equilibrium price vectors of the form $(1,p_2)$

   (b) By $\boxed{\text{Walras' law}}$, an equilibrium can be obtained as a solution to the single equation $Z^2(1,p_2)=0$

   This one variable problem can be represented by Figure.

   (c) When p_2^o is very small, $Z^2(1,p_2^o)>0$; if p_2'' is very large, then $Z^2(1,p_2'')<0$

   In particular, $\boxed{\text{boundary condition}}$, $\boxed{\text{bounded from below}}$, and $\boxed{\text{Walras' law}}$ imply that the value of intended supply(sale) and demand(purchase) are bounded. Because by

boundary condition, intended demand become unbounded for good 2 as $p_2 \to 0$. Hence $Z^2\left(1, p_2^0\right) > 0$ for p_2^0 sufficiently small. By symmetry, as $p_2 \to \infty$, for large enough p_2'', $Z^2\left(1, p_2''\right) < 0$
Because the $\boxed{\text{function } Z^2\left(1, p_2\right) \text{ is continuous(u.h.c. and single-valued)}}$, by the intermediate value theorem there must be an intermediate value p_2^* with $Z^2\left(1, p_2^*\right) = 0$ and hence, an equilibrium price vector must exist.

Of course, from the figure, we know it is possible that there are more than one equilibrium; that is, it is possible that there are finite(odd) numbers of equilibria in $L = 2$ case.

9.3.3 L commodities space

In the general case of more than two commodities, the proof that a solution exists is more complicated, and involves the use of some powerful mathematical tools.

Fixed point theorems

1. Fixed point of weakly increasing function
   Theorem
   If f is a weakly increasing function from $[0,1]$ to $[0,1]$, then f has a fixed point.
   Proof

   - If $f(0) = 0$, then $f(\cdot)$ has a fixed point on $[0,1]$
   - If $f(1) = 1$, then $f(\cdot)$ has a fixed point on $[0,1]$
   - $f(0) > 0$ and $f(1) < 1$
     Define a sequence $[a_n, b_n]$ s.t.
     Given $a_0 = 0$ and $b_0 = 1$,
     $a_n = a_{n-1}, b_n = \frac{a_{n-1}+b_{n-1}}{2}$ if $f\left(\frac{a_{n-1}+b_{n-1}}{2}\right) \leq \frac{a_{n-1}+b_{n-1}}{2}$
     $a_n = \frac{a_{n-1}+b_{n-1}}{2}, b_n = b_{n-1}$ if $f\left(\frac{a_{n-1}+b_{n-1}}{2}\right) \geq \frac{a_{n-1}+b_{n-1}}{2}$
     Then we have shrinking sequences of closed sets s.t. $[a_0, b_0] \supset [a_1, b_1] \supset ... \supset [a_n, b_n] \supset ...$ s.t. $\{a_n\}_{n=1}^{\infty}$ is an increasing sequence and $\{b_n\}_{n=1}^{\infty}$ is a decreasing sequence, so that $b_n - a_n \to 0$ as $n \to \infty$
     Note that $[0,1]$ is a compact set in a metric space and every convergent subsequence has a limit point in this set so that it is complete.
     By cantor's intersection property, in $[0,1]$ which is a complete metric space and $\{[a_n, b_n]\}$ be a decreasing sequence of nonempty closed sets, i.e. $[a_0, b_0] \supset [a_1, b_1] \supset ... \supset [a_n, b_n] \supset ...$ with $|b_n - a_n| \to 0$, $\exists x^*$ s.t. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = x$;i.e.,, $\cap_{n=1}^{\infty} [a_n, b_n] = x^*$.
     Therefore, $[\ f(x^*) \leq x^*$ and $f(x^*) \geq x^*] \to f(x^*) = x^*$

2. Brouwer's fixed point theorem

   If $S \subset R^L$ is a nonempty, convex, and compact set, and $f : S \rightarrow S$ is a continuous function, then $f(\cdot)$ has a fixed point; i.e. $\exists x^* \in S$ s.t. $x^* = f(x^*)$

3. Intermediate Value Theorem

   - $L = 1$ and $S = [0, 1]$ so that the graph of f is in R^2
     Then Brouwer's fixed point theorem is a consequence of Intermediate Value Theorem.
   - Let f be a $\boxed{\text{continuous}}$ function s.t. $f : [a, b] \rightarrow R$. If $f(a) < f(b)$ and $c \in R$ s.t. $f(a) < c < f(b)$, then $\exists x \in [a, b]$ s.t. $f(x) = c$

4. Kakutani's fixed point theorem

   If $S, T \subset R^L$ is a nonempty, convex, and compact sets with $S = T$, and $\phi : S \rightarrow T$ is a nonempty-valued, convex-valued, u.h.c(=has a closed graph because S is compact) correspondence for $\forall x \in S$, then $f(\cdot)$ has a fixed point; i.e. $\exists x^* \in S$ s.t. $x^* \in f(x^*)$

Existence proof

- There are two ways to follow. The proof of the second case has to deal with the fact that excess demand is not defined when the prices of some commodities are 0. The first one contains a very easy proof for the case of excess demand functions defined for all nonzero, nonnegative prices.

1. (Very) Easy existence theorem

   To facilitate a clear understanding of the nature of the fixed point argument, it is helpful to consider the easy existence proof first in which boundary conditions are eliminated by studying continuous, HD of 0 functions $Z(p, e)$ satisfying Walras' law and defined for all nonnegative, nonzero price vectors.

   Within the framework of $\boxed{\text{continuous and strictly convex preferences}}$, this type of excess demand function is not compatible with monotone preferences but can arise with $\boxed{\text{l.n.s}}$ preferences. Recall also that the equilibrium condition when zero prices are allowed is $Z(p, e) \leq 0$

   Theorem

   If a function $Z : R^L_+(\bar{\triangle}) \rightarrow R^L$, $Z(p, e) = \sum_{i \in I} x_i(p, (\succsim_i, e_i)) - \sum_{i \in I} e_i = \{Z_1(\cdot), Z_2(\cdot), ..., Z_L(\cdot)\}$, satisfying continuity, HD of zero in p, and Walras' law for $\forall p$,
   then $\exists p^* \in R^L_+(\bar{\triangle})$ s.t. $Z(p^*, e) \leq 0$

   Proof

   (a) Because $Z(\cdot)$ is HD of zero in p, we can restrict $p \in \bar{\triangle} = \left\{ p \in R^L_+ \mid \sum_{l=1}^{L} p_l = 1 \right\}$

(b) Define $f : \bar{\triangle} \to \bar{\triangle}$ s.t. $f(p) = \{f_1(\cdot), f_2(\cdot), ..., f_L(\cdot)\}$ s.t. $f_l(p) = \frac{p_L + \max(Z_L(p,e),0)}{\sum_{l=1}^{L}(p_l + \max(Z_l(p,e),0))}$ for $\forall l \in L$

Corresponding to intuition, this fixed point function tends to increase the price of commodities in excess demand.

(c) To use Brouwer's fixed point theorem, we need to show all assumptions satisfied.
$\bar{\triangle} \subset R^L$: nonempty, convex, and compact set
$f : \bar{\triangle} \to \bar{\triangle}$ is a continuous function because $Z(p,e)$ is continuous.
therefore, $f(\cdot)$ has a fixed point; i.e. $\exists p^* \in \bar{\triangle}$ s.t. $p^* = f(p^*)$

(d) We need to show $Z(p,e) \leq 0$ for the $p^* \in \bar{\triangle}$ with. $p^* = f(p^*)$.
Let $\lambda = \frac{1}{\sum_{l=1}^{L}\left(p_l^* + \max Z_l(p^*,e),0\right)}$.

i. $\lambda = 1$
$\sum_{l=1}^{L}(p_l^* + \max(Z_l(p^*,e),0)) = 1$
$\to$For $\forall l \in L$, $p_l^* = f_l(p^*) = p_l^* + \max(Z_l(p^*,e),0)$
$\to$For $\forall l \in L$, $\max(Z_l(p^*,e),0) = 0$
$\to$For $\forall l \in L$, $Z_l(p^*,e) \leq 0$
$\to Z(p^*,e) \leq 0$

ii. $\lambda < 1$
$\sum_{l=1}^{L}(p_l^* + \max(Z_l(p^*,e),0)) > 1$
$\to \sum_{l=1}^{L}\max(Z_l(p^*,e),0) > 0$
Furthermore for $\forall\ l \in L$, $p_l^* = f_l(p^*) = \lambda(p_l^* + \max(Z_l(p^*,e),0))$
$\to$For $\forall l \in L$, $p_l^* < (p_l^* + \max(Z_l(p^*,e),0))$
$\to$For $\forall l \in L, \max(Z_l(p^*,e),0) > 0$
$\to$For $\forall l \in L, Z_l(p^*,e) > 0$
Because $p^* \neq 0$, $p^* \cdot Z(p^*,e) = \sum_{l \in L} p_l^* \cdot Z_l(p^*,e) > 0$ which contradicts to Walras' law.

- $\boxed{\text{Fall 98 III-4}}$

  Let $\bar{\triangle}$ denote the unit simplex in R^L, $\left\{p \in R_+^L \mid \sum_{l=1}^{L} p_l = 1\right\}$.
  Let $f : \bar{\triangle} \to R^l$ be a continuous function satisfying $p \cdot f(p) = 0$ for every $p \in \bar{\triangle}$

  (a) Prove that $\exists p^* \in \bar{\triangle}$ s.t.$f(p^*) \leq 0$
  Very easy existence theorem

  (b) Does there necessarily exist some $p^* \in \bar{\triangle}$ s.t. $f(p^*) = 0$?
  No!!!
  Counter-example)
  Let $\exists$two goods in this economy, $L = 2$ and $p^* \in \bar{\triangle} = \left\{p \in R_+^2 \mid \sum_{l=1}^{2} p_l = 1\right\}$.

Let $f(p) : \bar{\triangle} \to R^2$ s.t.$f(p_1) = p_2$ and $f(p_2) = -p_1$which is linear and continuous satisfying $p_1 \cdot f(p_1) + p_2 \cdot f(p_2) = 0$ for $\forall p \in \bar{\triangle}$.
In this case, $f(p^*) = (p_2, -p_1) \neq 0$ for $\forall p^* \in \bar{\triangle}$
Suppose not; $f(p^*) = (p_2, -p_1) = 0$.
Then $(p_2^*, -p_1^*) = (0, 0) \to p_1^* + p_2^* = 0 \notin \bar{\triangle}$ which is a contradiction.
Therefore, it is $\boxed{\text{not}}$ necessary that $\exists p^* \in \bar{\triangle}$ s.t. $f(p^*) = 0$

(c) Frequently, existence theorem differ from very easy existence theorem in that they assume

i. Free disposability(the only feasible technology is free disposal)
ii. The function f is defined on $\triangle$
iii. The function f is continuous on $\triangle$
iv. A boundary condition such as if p^* belongs to the boundary of $\bar{\triangle}$, then for any sequence of $p^k \in \triangle$ with $p^k \to p^*$, $||Z(p^*)|| = \sum_{l=1}^{L} Z_l(p^*) = \infty$

State conditions on consumer preferences, endowments, and consumption sets that in the preference of assumption $(i), (ii), (iii)$ which imply a boundary condition such as (iv)

$\boxed{\text{Answer}}$

To satisfy that a excess demand $\boxed{\text{function}}$ Z is $\underline{\text{continuous}}$ and $\underline{\text{Boundary condition}}$
$E = \left\{I, (\succsim_i, e_i)_{i \in I}\right\}$
$X_i = R_+^L$ for $\forall i$ and $X = \{X_i\}_{i \in I}$

$$\succsim_i \begin{cases} \text{strict monotone(l.n.s)} & \to & \text{boundary condition(walras' law)} \\ \text{strict convex} & \to & \text{single-valued correspondence} \to \text{function} \\ \text{continuous(+monotone)} & \to & \text{continuous utiity function} \to \text{existence of } D_i\,(p, \end{cases}$$

$$e_i \in \begin{cases} R_+^L & \to & \text{for all the other conditions} \\ R_{++}^L & \to & \text{Boundary condition} \end{cases} \to R_{++}^L$$

$\boxed{p \in R_{++}^L \text{ (or } \triangle)}$$\to$for every condition

$\boxed{\text{Boundary condition}}$

$\underline{\text{Proof}}$

Suppose not. $e_i \in R_{++}^L$, $\underline{\succsim_i \text{is strictly monotone}}$, $[p_n \to p \in \partial\triangle]$, but $||D_i(p, e_i)||$ is bounded.
For $\forall n$, because $D_i(p_n, e_i)$ is HD 0 in $p_n \in R_{++}^L$, we can represent the price system $p_n \in \triangle$ for $\forall p_n \in R_{++}^L$.
Since, for $p_n \in \triangle$, $D_i(p_n, e_i)$ is compact-valued, $\exists$ a convergent subsequence $\left\{x_{n_q}\right\} \to x$ with $p_{n_q} (\in \triangle) \to p \in \partial\triangle$ s.t. $x_{n_q} \in D_i\left(p_{n_q}, e_i\right)$.
Then $p_{n_q} \cdot x_{n_q} \leq p_{n_q} \cdot e_i$ for $\forall n_q$.
In the finite space, by the continuity of dot product, $\left[p_{n_q} \cdot x_{n_q} \leq p_{n_q} \cdot e_i\right] \to [p \cdot x \leq p \cdot e_i]$ with $x \in D_i(p, e_i)$
W.l.o.g., let $p^k = 0$, $k = 1$.

Let $x' = x + (1, 0, 0, ..., 0)$ and then $p \cdot x = p \cdot x' \leq p \cdot e_i$
By strict monotonicity of $\succsim_i$, $x' \succ x$.
Because $\succsim_i$is continuous, $\exists N$ s.t.for $n \geq N$, $x_{n_q} \in N(x)$ and $x'_{n_q} \in N(x')$.
And $.p_n \cdot x_{n_q} = p_n \cdot x'_{n_q} \leq p_n \cdot e_i$ with $x'_{n_q} \succ x_{n_q}$.
It contrdicts $x_{n_q} \in D_i\left(p_{n_q}, e_i\right)$?!.

- $\boxed{\text{Spring 1997}}$

(a) Sketch a proof showing that $\exists p^* \in \bar{\triangle}$ s.t.$f_l(p^*) < 0$ only if $p_l^* = 0$ and give a brief economic interpretation.
$E = \left\{I, (\succsim_i, e_i)_{i \in I}\right\}$
$X_i = R_+^L$ for $\forall i$ and $X = \{X_i\}_{i \in I}$

$$\succsim_i \left\{ \begin{array}{ccc} \text{l.n.s} & \rightarrow & \text{walras' law} \\ \text{strict convex} & \rightarrow & \text{single-valued correspondence} \rightarrow \text{function} \\ \text{continuous(+monotone)} & \rightarrow & \text{continuous utiity function} \rightarrow \text{existence of } D_i(p, \end{array} \right.$$

$e_i \in R_+^L$
$\boxed{p \in R_+^L \backslash \{0\}}$
By very easy existence theorem, we know that under certain assumptions like above, $\exists p^* \in R_+^L \backslash \{0\}$ s.t.$f(p^*) \leq 0$
It implies not only for $\forall l \in L$, $.f_l(p^*) \leq 0$ but also $f_l(p^*) = 0$ if $p_l^* > 0$
Thus we can see that at the equilibrium, a good l can be in excess supply($f_l(p^*) < 0$) but only if the good is free, $p_l^* = 0$
As a simple example, a good l might be a "bad." Then we would expect that the good l's price $p_l = 0$ because consumer's demand is 0 and there will be $f_l(p) = \sum_{i \in I}(D_i^l(p^*) - e_i^l) = \sum_{i \in I}(-e_i^l) < 0$ and it will be dumped by free disposal technology.

- $\boxed{\text{Prob V-2}}$

(a) In a pure exchange economy $E = \left\{I, (\succsim_i, e_i)_{i \in I}\right\}$ s.t.
$X_i = R_+^L$ for $\forall i$ and $X = \{X_i\}_{i \in I}$

$$\succsim_i \left\{ \begin{array}{ccc} \boxed{\text{strict monotone}} & \rightarrow & \text{boundary condition(walras' law)} \\ \boxed{\text{strict convex}} & \rightarrow & \text{convex-valued correspondence} \\ \text{continuous(+monotone)} & \rightarrow & \text{continuous utiity function} \rightarrow \text{existence of } D_i(p, \\ \text{complete} & & \end{array} \right.$$

$e_i \in R_{++}^L \Rightarrow \sum_{i \in I} e_i >> 0$
Preference represented by $u_i : R_+^L \rightarrow R$
$\boxed{u_i(x_i) = \sum_{l=1}^L \alpha_i^l \cdot \ln x_i^l}$
for $\forall i$ and $\forall l \in L$, $\alpha_i^l > 0$ and for $\forall i$, $\sum_{l=1}^L \alpha_i^l = 1$
In this environment, $VEET$ is not satisfied because
$\boxed{\text{demand function can not be well-defined on } p^* \in \bar{\triangle}}$

$\boxed{\text{Proof}}$

We need to show $Z(\cdot)$ is a continuous function and to show it, we need a well-defined continuous function on $p^* \in \bar{\triangle}$.

But with the utility function defined above,

$\boxed{\text{demand function can not be well-defined on } p^* \in \bar{\triangle}}$.

Let $p_1^* = 0$ w.l.o.g.

Suppose demand is well-defined and $\exists \bar{x}_i \in D_i(p^*, e_i)$ s.t.

$D_i(p, e_i) = \{x_i \in B_i(p, e_i) \mid x_i \succsim_i x_i' \text{ for } \forall x_i' \in B_i(p, e_i)\}$, that is

$\bar{x}_i$ solves

$\max u_i(x_i)$

$s.t.\ B_i(p, e_i) = \{x_i \in X_i \mid p^* \cdot x_i \leq p^* \cdot e_i\}$

Consider $\tilde{x}_i = \bar{x}_i + (1, 0, 0, ..., 0)$.

Then $p^* \cdot \bar{x}_i = p^* \cdot \tilde{x}_i \leq p^* \cdot e_i$ but by strict increasing u_i, $u_i(\tilde{x}_i) > u_i(\bar{x}_i)$.

Therefore, $\bar{x}_i \notin D_i(p^*, e_i)$ which is a contradiction.

(b) In a pure exchange economy $E = \left\{I, (\succsim_i, e_i)_{i \in I}\right\}$ s.t.

$X_i = R_+^L$ for $\forall i$ and $X = \{X_i\}_{i \in I}$

$$\succsim_i \left\{ \begin{array}{c} \boxed{\text{strict monotone}} \\ \boxed{\text{convex}} \\ \text{continuous(+monotone)} \\ \text{complete} \end{array} \right. \quad \begin{array}{c} \to \\ \to \\ \to \end{array} \quad \begin{array}{c} \text{boundary condition(walras' law)} \\ \text{convex-valued correspondence} \\ \text{continuous utiity function} \to \text{existence of } D_i(p, \end{array}$$

$e_i \in R_{++}^L \Rightarrow \sum_{i \in I} e_i >> 0$

Preference represented by $u_i : R_+^L \to R$

$\boxed{u_i(x_i) = \sum_{l=1}^{L} \alpha_i^l \cdot x_i^l}$

for $\forall i$ and $\forall l \in L$, $\alpha_i^l > 0$ and for $\forall i$, $\sum_{l=1}^{L} \alpha_i^l = 1$

In this environment, $VEET$ is not satisfied.

because $\boxed{\text{demand function can not be well-defined on } p^* \in \bar{\triangle}}$

$\boxed{\text{Proof}}$

We need to show $Z(\cdot)$ is a continuous function and to show it, we need a well-defined continuous function on $p^* \in \bar{\triangle}$.

But with the utility function defined above,

$\boxed{\text{demand function can not be well-defined on } p^* \in \bar{\triangle}}$.

Let $p_1^* = 0$ w.l.o.g.

Suppose demand is well-defined and $\exists \bar{x}_i \in D_i(p^*, e_i)$ s.t.

$D_i(p, e_i) = \{x_i \in B_i(p, e_i) \mid x_i \succsim_i x_i' \text{ for } \forall x_i' \in B_i(p, e_i)\}$, that is

$\bar{x}_i$ solves

$\max u_i(x_i)$

$s.t.\ B_i(p, e_i) = \{x_i \in X_i \mid p^* \cdot x_i \leq p^* \cdot e_i\}$

Consider $\tilde{x}_i = \bar{x}_i + (1, 0, 0, ..., 0)$.

Then $p^* \cdot \bar{x}_i = p^* \cdot \tilde{x}_i \leq p^* \cdot e_i$ but by strict increasing u_i, $u_i\left(\tilde{x}_i\right) > u_i\left(\bar{x}_i\right)$.
Therefore, $\bar{x}_i \notin D_i\left(p^*, e_i\right)$ which is a contradiction.

(c) In a pure exchange economy $E = \left\{I, \left(\succsim_i, e_i\right)_{i\in I}\right\}$ s.t.
$X_i = R_+^L$ for $\forall i$ and $X = \left\{X_i\right\}_{i\in I}$

$$\succsim_i \left\{ \begin{array}{ccc} \boxed{\text{monotone}} & \rightarrow & \text{boundary condition(walras' law)} \\ \boxed{\text{convex}} & \rightarrow & \text{convex-valued correspondence} \\ \text{continuous(+monotone)} & \rightarrow & \text{continuous utiity function} \rightarrow \text{existence of } D_i\left(p, \right. \\ \text{complete} & & \end{array} \right.$$

$e_i \in R_{++}^L \Rightarrow \sum_{i\in I} e_i >> 0$
Preference represented by $u_i : R_+^L \rightarrow R$
$u_i\left(x_i\right) = \min\left(x_i^l\right)_{l=1}^L$
In this environment, $VEET$ is not satisfied $\boxed{\text{because demand mapping is not a function}}$.

$\boxed{\text{Proof}}$
Counter-example
Let $L = 2$.
$X = R_+^2$.
$e = (1, 4)$
$p_1^* = 0$, $p_2^* = 1$.
Then $x = \left\{x \in B(p, e) \mid \min\left(x^1, x^2\right)\right\} = \left\{x \in B(p, e) \mid x_1 \in [1, \infty], x_2 = 1\right\}$
So $D : p \rightarrow R_+^2$ is a correspondence in $p^* = (0, 1)$

2. General existence theorem(Proposition 17.c.1.)

$\boxed{p \in R^L_{++}\text{ (or }\triangle)}$→for every condition

$\boxed{\text{Theorem}}$

In an economy $E = \left\{I, (\succsim_i, e_i)_{i\in I}\right\}$ s.t.
$X_i = R^L_+$ for $\forall i$ and $X = \{X_i\}_{i\in I}$

$$\succsim_i \left\{\begin{array}{ccc} \text{strict monotone(l.n.s)} & \rightarrow & \text{boundary condition(walras' law)} \\ \text{strict convex} & \rightarrow & \text{single-valued correspondence} \\ \text{continuous(+monotone)} & \rightarrow & \text{continuous utiity function} \rightarrow \text{existence of } D_i\left(p, e_i\right) \end{array}\right.$$

$$e_i \in \left\{\begin{array}{ccc} R^L_+ & \rightarrow & \text{for all the other conditions} \\ R^L_{++} & \rightarrow & \text{Boundary condition} \end{array}\right. \rightarrow R^L_{++} \Rightarrow \textstyle\sum_{i\in I} e_i >> 0,$$

a function $Z : R^L_{++}(\triangle) \rightarrow R^L$, $Z\left(p,e\right) = \sum_{i\in I} x_i\left(p, (\succsim_i, e_i)\right) - \sum_{i\in I} e_i = \left\{Z^1\left(\cdot\right), Z^2\left(\cdot\right), ..., Z^L\left(\cdot\right)\right\}$ satisfies

(a) continuity,

(b) HD of zero in p,

(c) Walras' law,

(d) Bounded from below
$\exists s > 0$ s.t. $D^l_i\left(p, e_i\right) > -s + e^l_i$ for all $l \in L$ and all p

(e) Boundary condition
$[p_n \rightarrow p \in \partial\triangle] \rightarrow \left[||D_i\left(p, e_i\right)|| \text{ is unbounded} \Leftrightarrow \sup_{z^l \in Z_i(p_n, e_i)} \left|Z_i\left(p_n, e_i\right)\right| = \infty\right]$

and then $\exists p^* \in R^L_{++}(\triangle)$ s.t. $Z\left(p^*, e\right) = 0$

$\boxed{\text{Proof}}$

For notational clarity, in defining the set $\mu\left(p\right) \subset \bar{\triangle}$, we denote the vectors that are elements of $\mu\left(p\right)$ by q.

(a) Construct a fixed point correspondence for $p \in \bar{\triangle}$
Define $\mu : \bar{\triangle} \rightarrow \bar{\triangle} = (\triangle \cup \partial\triangle)$ s.t.

  i. For $p \in \triangle$, $\mu\left(p\right) = \left\{q \in \bar{\triangle} \mid q \cdot Z\left(p,e\right) = \max_{q' \in \bar{\triangle}} q' \cdot Z\left(p,e\right)\right\}$
  $= \left\{q \in \bar{\triangle} \mid q \cdot Z\left(p,e\right) \geq q' \cdot Z\left(p,e\right) \text{ for } \forall q' \in \bar{\triangle}\right\}$

  ii. For $p \in \partial\triangle$, $\mu\left(p\right) = \left\{q \in \bar{\triangle} \mid q \cdot p = 0\right\}$

(b) Check assumptions for Kakutani's fixed point theorem

  i. $\bar{\triangle}$ is nonempty, convex, and compact in R^L_+

ii. $\mu(\cdot)$ is nonempty-valued, convex-valued, and u.h.c. in R_{+}^{L}

- nonempty-valued
  $\underline{p \in \triangle}$
  $q \cdot Z(p,e)$ is a linear transformation of a continuous function $Z(p,e)$ so that it is continuous, and a continuous function attains a maximum on a compact set; i.e. $\exists q \in \bar{\triangle}$ s.t.$q \cdot Z(p,e) = \max_{q' \in \bar{\triangle}} q' \cdot Z(p,e)$
  $\underline{p \in \partial\triangle}$
  $\exists q \in \partial\triangle \subset \bar{\triangle}$ s.t.$q \cdot p = 0$
- convex-valued
  $\underline{p \in \triangle}$
  let $q_1, q_2 \in \bar{\triangle}$ s.t. $q_1 \cdot Z(p,e) = \max_{q' \in \bar{\triangle}} q' \cdot Z(p,e) = q_2 \cdot Z(p,e)$.
  For $\forall \lambda \in [0,1]$, $\lambda \cdot q_1 \cdot Z(p,e) + (1-\lambda) \cdot q_2 \cdot Z(p,e) = \max_{q' \in \bar{\triangle}} q' \cdot Z(p,e)$.
  Hence, $(\lambda \cdot q_1 + (1-\lambda) \cdot q_2) \in \mu(p)$
  $\underline{p \in \partial\triangle}$
  let $q_1, q_2 \in \bar{\triangle}$ s.t. $q_1 \cdot Z(p,e) = 0 = q_2 \cdot Z(p,e)$.
  For $\forall \lambda \in [0,1]$, $\lambda \cdot q_1 \cdot Z(p,e) + (1-\lambda) \cdot q_2 \cdot Z(p,e) = 0$
  Hence, $(\lambda \cdot q_1 + (1-\lambda) \cdot q_2) \in \mu(p)$
- u.h.c.(=a correspondence $\mu : \bar{\triangle} \to \bar{\triangle}$ has a closed graph)
  Given $p_n \to p$ and $q_n \to q$ with $q_n \in \mu(p_n)$ for all n, we need to show $q \in \mu(p)$
  $\underline{p \in \triangle}$
  $p_n >> 0$ for sufficiently large n. From $q_n \in \mu(p_n)$ for all n, $q_n \cdot Z(p_n, e) \geq q' \cdot Z(p_n, e)$ for $\forall q' \in \bar{\triangle}$.
  By the continuity of $Z(\cdot)$, $q \cdot Z(p,e) \geq q' \cdot Z(p,e)$ for $\forall q' \in \bar{\triangle}$
  Hence, $q \in \mu(p)$.
  $\underline{p \in \partial\triangle}$
  case 1) $p_n \in \partial\triangle$ infinitely often
    Take a subsequence $\{p_{n_i}\}$ s.t. $p_{n_i} \in \partial\triangle$ for $\forall n_i$.
    Along the subsequence, $\exists q_{n_i} \in \mu(p_{n_i})$ so that $q_{n_i} \cdot p_{n_i} = 0$ for $\forall n_i$.
    By the continuity of dot product, $q \cdot p = 0$ as $n_i \to \infty$.
    Hence, $q \in \mu(p)$
  case 2) $p_n \in \partial\triangle$ only finite times
    $\exists N$ s.t. for $n \geq N$, $p_n \in \triangle$.
    Suppose $q \notin \mu(p)$. It means for $p \in \partial\triangle$, $q \cdot p \neq 0$
    Then w.l.o.g. $\exists$ at least one $l \in L$ s.t. $q^l \cdot p^l > 0$.
    Because $p_l > 0$, $\exists \varepsilon > 0$ s.t. $p_n^l > \varepsilon > 0$ for $\forall n \geq N$.
    Because $Z(\cdot)$ is bounded from below, $\exists s > 0$ s.t. $Z^l(p_n, e) > -s$ for all $l \in L$ and all p_n.
    Because $Z(\cdot)$ satisfies Walras' law for $\forall p \in \triangle$, $p_n^l \cdot Z^l(p_n, e) \leq s$ for suffi-

ciently large n.

$$Z^l(p_n, e) \le \frac{s}{p_n^l} < \frac{s}{\varepsilon}$$

Let $\exists k \in L$ s.t. in $Z(p_n, e)$, $Z^k(p_n, e) = \max_{l \in L} Z^l(p_n, e)$.

$$-s < Z^l(p_n, e) \le Z^k(p_n, e) < \frac{s}{\varepsilon}$$

It implies that when $p_n \to p \in \partial\triangle$, $||Z(p_n, e)||$ is bounded which is a contradiction.

Hence, $q \in \mu(p)$.

(c) Apply Kakutani's fixed point theorem
A correspondence $\mu : \bar{\triangle} \to \bar{\triangle}$ in which
   i. $\bar{\triangle}$ is nonempty, convex, and compact in R_+^L
   ii. $\mu(\cdot)$ is nonempty-valued, convex-valued, and u.h.c. in R_+^L
then $\mu(\cdot)$ has a fixed point; i.e. $\exists p^* \in \bar{\triangle}$ s.t. $p^* \in \mu(p^*)$

(d) Restriction on p^*
Recall that we define the correspondence $\mu(p)$ at $p \in \partial\triangle$ s.t $\mu(p) = \left\{q \in \bar{\triangle} \mid q \cdot p = 0\right\}$
Because $p \cdot p > 0$ when $p \in \partial\triangle$, $p^* \notin \partial\triangle$.
Therefore, we can restrict $p^* \in \triangle$

(e) $Z(p^*, e) = 0$
For $p^* \in \triangle$, if $Z(p^*, e) \neq 0$, by Walras' law we have $Z^l(p^*, e) < 0$ for some l and $Z^{l'}(p^*, e) > 0$ for some $l' \neq l$.
Thus, for such p^*, any $q^* \in \mu(p^*)$ has$(q^*)^l = 0$ for l and then $q^* \in \partial\triangle$. which is incompatible with $p^* \in \mu(p^*)$ and $p^* \in \triangle$.
Hence, if $p^* \in \mu(p^*)$ and $p^* \in \triangle, Z(p^*, e) = 0$

3. General existence theorem with excess demand correspondence
   If a correspondence $Z : R_{++}^L(\triangle) \to R^L$, $Z(p, e) = \sum_{i \in I} x_i(p, (\succsim_i, e_i)) - \sum_{i \in I} e_i = \{Z_1(\cdot), Z_2(\cdot), ..., Z$
   satisfying

   (a) u.h.c. and convex-valued,
   (b) HD of zero in p,
   (c) Walras' law,
   (d) Bounded from below($\exists s > 0$ s.t. $D_i^l(p, e_i) > -s + e_i^l$ for all $l \in L$ and all p)
   (e) Boundary condition(If $\boxed{e_i \in R_{++}^L}$, $\underline{\succsim_i \text{is strictly monotone}}$)
   (f) $[p_n \to p \in \partial\triangle] \to \left[||D_i(p, e_i)|| \text{ is unbounded} \Leftrightarrow \sup_{z^l \in Z_i(p_n, e_i)} |Z_i(p_n, e_i)| = \infty\right]$

   then $\exists p^* \in R_{++}^L(\triangle)$ s.t. $0 \in Z(p^*, e)$

Previous Problem

- $\boxed{\text{Spring 98 III-3}}$

1. In a pure exchange economy $E = \left\{I, (\succsim_i, e_i)_{i\in I}\right\}$ s.t.

   Commodity space$\left(R^L\right)$

   $X_i = R^L_+$ for $\forall i$ and $X = \{X_i\}_{i\in I}$

   $$\succsim_i \left\{\begin{array}{c}\boxed{\text{monotone}}\\ \boxed{\text{convex}}\\ \text{continuous(+monotone)}\\ \text{complete}\end{array}\right. \quad \begin{array}{cc} \rightarrow & \text{boundary condition(walras' law)}\\ \rightarrow & \text{convex-valued correspondence}\\ \rightarrow & \text{continuous utiity function} \rightarrow \text{existence of } D_i\left(p, e_i\right)\end{array}$$

   $e_i \in R^L_{++} \Rightarrow \sum_{i\in I} e_i >> 0$

   Excess demand function $Z : \triangle \rightarrow R$ satisfying

   Walras' law

   bounded from below

   boundary condition

   (a) Walras' law

   For $\forall p \in \triangle$, $p \cdot Z\left(p\right) = 0$

   (b) N utility maximizing consumers with consumption sets R^L_+, $e_i \in R^L_{++}$, and utility function $u_i : R^L_+ \rightarrow R$

   Which conditions do we need for Z to be continuous?

   First, we need demand correspondence is well-defined.

   For it, the utility function has to be continuous and it comes from that $\succsim_i$is continuous and monotone.

   Once demand correspondence is well-defined, we need it to be u.h.c. and single-valued.

   Given the conditions, we have a continuous budget correspondence and a continuous utility function. Using maximum theorem, we have demand correspondence is u.h.c., convex-valued, nonempty-valued

   Last for single-valued correspondence, we need $u_i\left(\cdot\right)$ is strictly quasiconcave which is equivalent to that $\succsim_i$is strictly convex.

   Therefore, for Z to be continuous, we need $\succsim_i$is monotone, strictly convex, and continuous.

   $\boxed{\text{Maximum Theorem}}$

   Let $\phi : S \rightarrow T$ be a continuous correspondence where S is a metric space and T is a compact metric space.

Let $f : S \times T \rightarrow R$ be a continuous function.

Then the function $m : S \rightarrow R$ defined by $m(x) = \max \{f(x, y) \mid y \in \phi(x)\}$ is continuous and

the correspondence $\mu : S \rightarrow T$ defined by $\mu(x) = \{y \in \phi(x) \mid f(x, y) = m(x)\}$ is non-empty, compact-valued, and u.h.c.

(c) Easy Existence Theorem

9.3.4 Production Economy

- The applicability of the existence proof is not limited to exchange economies. If we allow for production sets that are closed, strictly convex, and bounded above(and if a positive aggregate consumption buncle is producible from the initial aggregate endowments), then the production inclusive convex-valued, upper-hemi continuous excess demand correspondence $Z(\cdot)$(or a continuous function) satisfies the properties of excess demand correspondence. Hence, $\exists$a C.E.

1. Easy Market equilibrium lemma
   (a) If a function $Z : R^L_{++}(\triangle) \to R^L$, $Z(p) == \left\{Z^1(\cdot), Z^2(\cdot), ..., Z^L(\cdot)\right\}$, satisfying
      i. continuity,
      ii. HD of zero in $p \in \triangle$,
      iii. Walras' law in $p \in \triangle$,
      iv. Bounded from below($\exists s > 0$ s.t. $Z^l > -s$ for all $l \in L$ and all p)
      v. Boundary condition(If $\boxed{e_i \in R^L_{++}}$, $\underline{\succsim_i \text{is strictly monotone}}$)
      $[p_n \to p \in \partial\triangle] \to \left[\left\|D_i\left(p, e_i + \sum_{j\in J}\theta_{ij}\cdot y_j\right)\right\| \text{ is unbounded} \Leftrightarrow \sup_{z^l \in Z_i(p_n)}|Z_i(p_n)| = \infty\right]$
      then $\exists p^* \in R^L_{++}(\triangle)$ s.t. $Z(p^*) = 0$

2. Extended Market equilibrium lemma

   Lemma

   - If a correspondence $Z : R^L_{++}(\triangle) \to R^L$, $Z(p) = \left\{Z^1(\cdot), Z^2(\cdot), ..., Z^L(\cdot)\right\}$ s.t..$Z^l(p) = \sum_{i\in I} x^l_i(p) - \sum_{i\in I} e^l_i - \sum_{j\in J} y^l_j$ satisfying

   (a) i. u.h.c. and convex-valued,
      ii. HD of zero in p,
      iii. Walras' law,
      iv. Bounded from below($\exists s > 0$ s.t. $D^l_i\left(p, e_i + \sum_{j\in J}\theta_{ij}\cdot y_j\right) > -s + e^l_i$ for all $l \in L$ and all p)
      v. Boundary condition(If $\boxed{e_i \in R^L_{++}}$, $\underline{\succsim_i \text{is strictly monotone}}$)
      $[p_n \to p \in \partial\triangle] \to \left[\left\|D_i\left(p, e_i + \sum_{j\in J}\theta_{ij}\cdot y_j\right)\right\| \text{ is unbounded} \Leftrightarrow \sup_{z^l \in Z_i(p_n)}|Z_i(p_n)| = \infty\right]$
      then $\exists p^* \in R^L_{++}(\triangle)$ s.t. $0 \in Z(p^*)$

3. Existence Theorem with Production

Theorem

In $E = \left[\left\{I, (\succsim_i, e_i)_{i\in I}\right\}, \left\{J, (Y_j)_{j\in J}\right\}\right]$ s.t.

$X_i = R_+^L$ for $\forall i$ and $X = \{X_i\}_{i\in I}$

$$\succsim_i \left\{\begin{array}{ccc} \text{strict monotone(l.n.s)} & \rightarrow & \text{boundary condition(walras' law)} \\ \text{(weakly)convex} & \rightarrow & \text{convex-valued correspondence} \\ \text{continuous(+monotone)} & \rightarrow & \text{continuous utiity function} \rightarrow \text{existence of } D_i\left(p, e_i\right) \end{array}\right.$$

$$e_i \in \left\{\begin{array}{ccc} R_+^L & \rightarrow & \text{for all the other conditions} \\ R_{++}^L & \rightarrow & \text{Boundary condition} \end{array}\right. \rightarrow R_{++}^L \Rightarrow \sum_{i\in I} e_i >> 0$$

$Y_j = R^L$ for $\forall j$ and $Y = \{Y_j\}_{j\in J}$ satisfying

$0 \in Y_j$
Y_j is closed and convex
$Y \cap R_+^L = \emptyset$,
$Y \cap -Y = \emptyset$

the Market Equilibrium Lemma is applicable and

$\exists$an equilibrium with production.

Proof

(a) The set of feasible allocations is compact
Define $A = \Big\{(x, y) \in \Pi_{i=1}^I X_i \times \Pi_{j=1}^J Y_j \mid$ for $\forall i, j, x_i \in X_i$ and $y_j \in Y_j, \sum_{i\in I} x_i^* \leq \sum_{i\in I} e_i + \sum_{j\in}$

(b) Truncate the economy
Let $\hat{Y}_j$ be the projection of A on producer j's coordinate; the compact set of feasible production plans for j. Choose a compact convex set $K \subset R^L$ s.t. $\hat{Y}_j \subset intK$ for $\forall j$
Define $Y_j^K = Y_j \cap K$ which is closed, convex, and $0 \in Y_j^K$.

(c) Existence of equilibrium in Y_j^K
   i. Define $Z_K(p) = \left\{Z_K^1(\cdot), Z_K^2(\cdot), ..., Z_K^L(\cdot)\right\}$ s.t..$Z_K^l(p) = \sum_{i\in I} x_{iK}^l(p) - \sum_{i\in I} e_i^l - \sum_{j\in J} y_{jK}^l$ for $\forall l$
   ii. Check assumptions for Market Equilibrium Lemma
      - $\sum_{i\in I} x_{iK}^l(p) \in D\left(p^*, e + \sum_{j\in J} y_j^*\right)$
        is nonempty-valued, convex-valued, u.h.c.,
        satisfies HD of 0 in p,
        satisfies Walras' law,
        is bounded from below,
        satisfies boundary condition

    - $\sum_{j \in J} y^{l}_{jK} \in Y^K$
      is nonempty-valued, convex-valued, compacted-valued

  iii. Therefore, $Z_K(p)$ satisfies all assumptions for Extended Market Equilibrium Lemma
  then $\exists p^* \in R^L_{++}(\triangle)$ s.t. $0 \in Z(p^*)$ and
  for $\forall i \in I, x_i^* \in D_i\left(p^*, e_i + \sum_{j \in J} \theta_{ij} \cdot y_j^*\right)$
  for $\forall j \in J, y_j^* \in Y_j^K$

(d) Equilibrium for truncated economy Y^K is also al equilibrium for the original economy Y
We need to show $y^* \in Y^K$ is also profit-maximizing with p^* in Y.
→for $\forall j \in J, y_j^* \in Y_j$ s.t. $p^* \cdot y_j \leq p^* \cdot y_j^*$ for $\forall y_j \in Y_j$
Suppose not. Then $\exists y_j' \in Y_j$ s.t. $p^* \cdot y_j' \geq p^* \cdot y_j^*$
Consider $\lambda y_j^* + (1-\lambda)\, y_j' = y_j''$ for $\lambda \in [0,1]$
For $\lambda < 1$, $p^* \cdot y_j'' > p^* \cdot y_j^*$ and λ close enough to 1, $y_j'' \in K$ and $y_j'' \in Y_j$ so that $y_j'' \in Y_j \cap K = Y_j^K$
It implies $y_j^* \notin Y_j^K$ which is a contradiction.

9.3.5 $L = \infty$

9.3.6 Conclusion

- Although the following tells us that ∃ an equilibrium, it does not give us the equilibrium price system and equilibrium allocation explicitly. The issue of how to actually find equilibria was first considered by Scarf(1973). By now, a variety of useful techniques are available.

10 Properties of C.E.

10.1 Local Uniqueness

1. The Competitive Equilibrium Theory is not completely deterministic. The uniqueness of equilibria is assured only under special conditions. That is, it is possible there are multiple equilibria. From the theoretical point of view, if uniqueness is not achievable, the next-best property is local uniqueness. We say that an equilibrium price vector is locally unique or locally isolated, if we can not find another price vector arbitrary close to it. The local uniqueness property is of interest because it may not be difficult to complete the theory in any particular application. For example, we may determine the region where equilibrium lies.

- Assumption

1. $E = \left\{ I, (\succsim_i, e_i)_{i \in I} \right\}$

   $X_i = R^L_+$ for $\forall i$ and $X = \{X_i\}_{i \in I}$

   $$\succsim_i \left\{ \begin{array}{ccc} \text{strict monotone(l.n.s)} & \rightarrow & \text{boundary condition(walras' law)} \\ \text{strict convex} & \rightarrow & \text{single-valued correspondence} \\ \text{continuous(+monotone)} & \rightarrow & \text{continuous utiity function} \rightarrow \text{existence of } D_i(p, e_i) \end{array} \right.$$

   $$e_i \in \left\{ \begin{array}{lll} R^L_+ & \rightarrow & \text{for all the other conditions} \\ R^L_{++} & \rightarrow & \text{Boundary condition} \end{array} \right. \rightarrow R^L_{++}$$

   $\boxed{p \in R^L_{++} \text{ (or } \triangle)}$ →for every condition

   Define $Z_i(p, e_i) = x_i(p, e_i) - e_i$ s.t. $x_i(p, e_i) \in D_i(p, e_i)$ and $Z(p, e) = \sum_{i \in I} x_i(p, e_i) - \sum_{i \in I} e_i$ s.t. $Z(p, e) = \left\{ Z^1(\cdot), Z^2(\cdot), ..., Z^L(\cdot) \right\}$

   Then the aggregate excess demand correspondence satisfies

   i) continuity

   ii) HD of 0 in price

   iii) Walras' law

   iv) Bounded from below

   v) Boundary Condition

   Additionally, we assume that $Z(\cdot)$ is continuously differentiable.

- We hope to determine relative prices, we normalize $p_L = 1$ and denote $Z(p) = \left\{ 1, Z^2(\cdot), Z^3(\cdot), \ldots, Z^L \right.$

  A normalized price vector $p = (1, p_2, ..., p_L)$ constitutes a CE equilibrium iff it solves the system of $L-1$ equations in $L-1$ unknowns; $Z(p) = 0$

1. An equilibrium price vector $p = (1, p_2, ..., p_L)$ is regular if the $(L-1) \cdot (L-1)$ matrix of price effects $DZ(p)$ is nonsingular, that is, has rank $L-1$. If every normalized equilibrium price vector is regular, we say that the economy is regular.

1. Every equilibrium in the graph is regular because the slope of excess demand $\frac{\partial Z^2(1,p_2)}{\partial p_2}$ is nonzero at every solution.

- The significance of the technical concept of regularity derives from the fact that a regular equilibrium price vector is isolated, and a regular economy can only have a finite number of price equilibria. Typically(generically) there is a finite number of equilibria, and each equilibrium is therefore locally isolated. Even more, this number is odd, and the equilibria fall naturally into two categories according to the sign of their index.

   1. Any regular equilibrium price vector $p^* \in \triangle$ is locally unique; that is $\exists \varepsilon > 0$ s.t. if $p^* \neq p$ and $||p^* - p|| < \varepsilon$, then $Z(p) \neq 0$.
      Moreover, if the economy is regular, then the number of equilibrium price vector is finite.

- In the above figure, the boundary conditions on the excess demand function $Z^2(\cdot)$ necessarily imply that for a regular economy, there is an odd number of equilibria and the slopes of the excess demand function at the equilibrium must alternate between being negaitve and being positive, starting with negative. If we say that an equilibrium with an associated negative slope of excess demand has an index of +1 and that one with a positive slope has an index of -1, then no matter how many equilibria there are, the sum of the indices of the equilibria of a regular economy is always +1. With appropriate definitions, it turns out that this invariance of index property also holds in the general case with any number of commodities, where it has some important implications for comparative statics and uniqueness questions.

   1. Suppose that $p = (1, p_2, ..., p_L)$ is a regular equilibrium of the economy. Then we denote

      $$\textit{index } p = (-1)^{L-1} \operatorname{sign} |DZ(p)|$$

      where $|DZ(p)|$ is the determinant of the $(L-1) \cdot (L-1)$ matrix $DZ(p)$.
   2. For any regular economy, we have

      $$\sum_{\{p; Z(p)=0, p_1=1\}} \textit{index } p = +1$$

      (a) It implies that the number of equilibria of a regular economy is odd. In particular, the number cannot be zero; so the existence of at least one equilibrium is a particular case of the proposition.

(b) The index concept provides a classification of equilibria into two types. The type with positive index is more fundamental because the presence of at least one equilibrium of positive type is unavoidable. In fact, it is typically the case that any search for well-behaved equilibria can be confined to the positive index equilibria.

(c) The index result has implications for the uniqueness and the multiplicity of equilibria.

(d) Without imposing additional strong assumption, the index theorem is all we can hope.

- Typically(generically) economies are regular. The solution to the excess demand equations are locally isolated and finite in number, and the index formula holds.

  Essence of genericity analysis rests on counting equations and unknowns.

  Suppose $\exists M$ equations and N unknowns.

  1. Implict Function Theorem
     Suppose that every equation $Z^l(\cdot)$ is continuously differentiable with respect to its $L+M$ variables and that we consider a solution $x^* = (x_1^*, ..., x_L^*)$ at exogenous variables $p^* = (p_1^*, ..., p_M^*)$, that is, satisfying $Z^l(x^*, p^*) = 0$ for $\forall l$. If the jacobian matrix of the system

     $$\begin{aligned} Z^1(x_1, ..., x_L; p_1, ..., p_M) &= 0 \\ Z^2(x_1, ..., x_L; p_1, ..., p_M) &= 0 \\ &\vdots \\ Z^L(x_1, ..., x_L; p_1, ..., p_M) &= 0 \end{aligned}$$

     with respect to the endogenous variables, evaluated at (x^*, p^*), is nonsingular, that is, if

     $$\begin{vmatrix} \frac{\partial Z^1(x^*,p^*)}{\partial x_1} & \cdots & \frac{\partial Z^1(x^*,p^*)}{\partial x_L} \\ \vdots & \ddots & \cdots \\ \frac{\partial Z^L(x^*,p^*)}{\partial x_1} & \cdots & \frac{\partial Z^L(x^*,p^*)}{\partial x_L} \end{vmatrix} \neq 0$$

     then the system can be locally solved at (x^*, p^*) by implicitly defined functions $x_l : B' \to A'$ that are continuously differentiable. Moreover, the first-order effects of p on x at (x^*, p^*) are given by $\frac{\partial x(p^*)}{\partial p} = -\left[\frac{\partial Z(x^*,p^*)}{\partial x}\right]^{-1} \cdot \frac{\partial Z(x^*,p^*)}{\partial p}$.

  2. Inverse Function Theorem
     When $L = M$ and every equation has the form $Z^l(x, p) = f^l(x) - p_l = 0$ is known as the inverse function theorem

3. The normal situation should be one in which, with N unknowns and M equations, we have N-M degree of freedom available for the description of the solution set.

   If M>N, the system should be overdetermined and have no solution

   If M=N, the system should exactly determined with the solutions locally unique.

   If M<N, the system should be underdetermined and the solution not locally unique.

   Clearly, all these statements are not always true.

   The implicit function theorem provides an answer; one needs the equations which we assume are differetiable to be independent at the solutions.

4. The system of M equations in N unknowns $Z(p) = 0$ is regular if rank $DZ(p) = M$ whenever $Z(p) = 0$.

   For a regular system, the implicit function theorem yields the existence of the right number of degrees of freedom.

   If M>N, then rank $DZ(p) \leq N < M$ for $\forall p$. In this case, $Z(p) = 0$ is regular iff the system admits no solution.

   If M=N, equilibria must be locally unique.

   If M<N, we can choose M variables corresponding to M linearly independent columns of DZ and we can express the values of these M variables that solve the M equations $Z(p) = 0$ as a function of N-M remaining variables.

5. Robustness

   Suppose there are some parameters $p = (p_1, ..., p_M)$ s.t. for $\forall p$, we have a system of equations $Z(x, p) = 0$. The set of possible values is R^M. We can then justifiably say that $Z(., p')$ is a pertubation of $Z(., p)$ if p' is close to p. Hence, the notion that the regularity of a system $Z(., p) = 0$ is generic could be captured by demanding that for almost every p, $Z(., p) = 0$ is regular; in other words, that nonregular systems have probability zero of occuring.

11 Pareto Optimality

- An allocation is Pareto optimal if there is no waste: it is impossible to make any agent strictly better without making some other agent worse off.

11.1 Pure Exchange Economy

11.1.1 Definition

1. Strict Pareto Optimal

   A feasible allocation $\{\hat{x}_i\} \in X = R_+^{LI}$ is strictly pareto optimal if there is no other feasible allocation $\{\bar{x}_i\} \in X$ s.t.

   for $\forall i \in I$, $\bar{x}_i \succsim_i \hat{x}_i$ and for at least one $i' \in I$, $\bar{x}_{i'} \succ_{i'} \hat{x}_{i'}$

2. Weakly Pareto Optimal

   A feasible allocation $\{\hat{x}_i\} \in X = R_+^{LI}$ is weakly pareto optimal if there is no other feasible allocation $\{\bar{x}_i\} \in X$ s.t.

   for $\forall i \in I$, $\bar{x}_i \succ_i \hat{x}_i$

3. Equivalence

   (a) Strict Pareto Optimal→Weak Pareto Optimal
   If a feasible allocation $\{\hat{x}_i\} \in X = R_+^{LI}$ is strictly pareto optimal, then it is automatically weakly pareto optimal by the definitions.

   (b) Weak Pareto Optimal→Strict Pareto Optimal
   Suppose for $\forall i \in I$, $\succsim_i$is a strictly monotone, continuous, complete preorder on X_i.
   Then Weak Pareto Optimal→Strict Pareto Optimal
   $\boxed{\text{Proof}}$
   Suppose a feasible allocation $\{\hat{x}_i\}_{i\in I} \in X = R_+^{LI}$ is not strictly pareto optimal; i.e. $\exists$another feasible allocation $\{\bar{x}_i\}_{i\in I} \in X$ s.t.
   for $\forall i \in I$, $\bar{x}_i \succsim_i \hat{x}_i$ and for at least one $i' \in I, \bar{x}_{i'} \succ_{i'} \hat{x}_{i'}$
   By strict monotonicity of $\succsim_i$, for at least one l, $\bar{x}_{i'}^l > \hat{x}_{i'}^l$.
   Then from continuity of $\succsim_i$, we can find another feasible allocation $\{x_i'\}_{i\approx I}$ s.t. $\exists\varepsilon > 0$,
   for $\forall i \in I$, $x_i' = \bar{x}_i + \left(0, ..., 0, \frac{\varepsilon}{I-1}, 0, ..., 0\right) \succ_i \bar{x}_i \succsim_i \hat{x}_i \rightarrow x_i' \succ_i \hat{x}_i$
   for i', $x_{i'}' = \bar{x}_{i'} - (0, ..., 0, \varepsilon, 0, ..., 0) \succ_i \hat{x}_i \rightarrow x_{i'}' \succ_i \hat{x}_i$
   Moreover, $\sum_{i\in I} x_i' = \sum_i \bar{x}_i + (I-1) \cdot \left(0, ..., 0, \frac{\varepsilon}{I-1}, 0, ..., 0\right) + \bar{x}_{i'} - (0, ..., 0, \varepsilon, 0, ..., 0) = \sum_{i\in I} \bar{x}_i$
   Therefore, $\{\hat{x}_i\}_{i\in I}$ is not weakly pareto optimal, which is a contradiction.

4. Conclusion

- Note that the Pareto optimality concept does not concern itself with distributional issues. For example, in a pure exchange economy, an allocation that gives all of society's endowments to one consumer who has strongly monotone preferences is necessarily Pareto optimal.

11.2 Production Economy

1. Strict Pareto Optimal

   A feasible allocation $\{\hat{x}, \hat{y}\} \in X \times Y = R_{+}^{L(I+J)}$ is strictly pareto optimal if there is no other feasible allocation $\{\bar{x}, \bar{y}\} \in X \times Y = R_{+}^{L(I+J)}$ s.t.
   with $\sum_{i \in I} \hat{x}_i \leq \sum_{i \in I} e_i + \sum_{j \in J} \hat{y}_j$,
   for $\forall i \in I$, $\bar{x}_i \succsim_i \hat{x}_i$ and for at least one $i' \in I$, $\bar{x}_{i'} \succ_{i'} \hat{x}_{i'}$

2. Weakly Pareto Optimal

   A feasible allocation $\{\hat{x}, \hat{y}\} \in X \times Y = R_{+}^{L(I+J)}$ is weakly pareto optimal if there is no other feasible allocation $\{\bar{x}, \bar{y}\} \in X \times Y = R_{+}^{L(I+J)}$ s.t.
   with $\sum_{i \in I} \hat{x}_i \leq \sum_{i \in I} e_i + \sum_{j \in J} \hat{y}_j$,for $\forall i \in I$, $\bar{x}_i \succ_i \hat{x}_i$

12 Welfare Theorem

12.1 Pure Exchange Economy

12.1.1 First Welfare Theorem

It provides a formal and very general confirmation of Adam Smith's asserted "invisible hand" property of the market.

1. local nonsatiation

   On a set $X = R_+^L$, a preorder $\succsim$ is locally nonsatiated on X,

   if for $\forall\ x \in X, \forall \varepsilon > 0$, $\exists x'$ s.t. $||x' - x|| < \varepsilon$ and $x' \succ x$

   It will be satisfied if there are some desirable commodities.

2. First welfare theorem

   If preferences are l.n.s., and if (x^*, p^*) is a C.E. in a pure exchange economy, then the allocation x^* is pareto optimal.

   $\boxed{\text{Proof}}$

   Suppose that (x^*, p^*) is a C.E. in a pure exchange economy, but x^* is not pareto optimal; i.e., $\exists$another feasible allocation $x \in X$ s.t. for $\forall i \in I$, $x_i \succsim_i x_i^*$ and for at least one i', $x_{i'} \succ_{i'} x_{i'}^*$.

   Individual agents' preference maximization of the definition of C.E. implies that if $x_i \succ_i x_i^*$, then $p^* \cdot x_i > p^* \cdot e_i$ for $\forall i \in I$.

   l.n.s and continuity of preference implies that if $x_i \succsim_i x_i^*$, then $p^* \cdot x_i \geq p^* \cdot e_i$ for $\forall i \in I$.

   $$\begin{array}{ll} & \text{for } \forall i \in I,\ x_i \succsim_i x_i^* \rightarrow p^* \cdot x_i \geq p^* \cdot e_i \\ \text{Then} & \text{for at least one } i',\ x_{i'} \succ_{i'} x_{i'}^* \rightarrow p^* \cdot x_{i'} > p^* \cdot e_i \quad \Rightarrow \text{a contradiction?!} \\ & \rightarrow \quad \sum_{i \in I} p^* \cdot x_i > \sum_{i \in I} p^* \cdot e_i \end{array}$$

   But from the definition of feasibility of allocation, $\sum_{i \in I} x_i \leq \sum_{i \in I} e_i$ and it implies that $\sum_{i \in I} p^* x_i \leq \sum_{i \in I} p^* e_i$.

   Therefore, if (x^*, p^*) is a C.E. in a pure exchange economy, then the allocation x^* is pareto optimal.

12.1.2 Second Welfare Theorem

This theorem gives us conditions under which any desired distributional aims can be achieved through the use of competitive markets. The second welfare theorem gives conditions under which a pareto optimum allocation can be supported as a price equilibrium with transfers. It means that we can achieve any desired pareto optimal allocation as a market based equilibrium using an appropriate lump-sum distribution plan.

1. Separating Hyperplane Theorem

   Suppose that convex sets A and $B \subset R^L$ are disjoint$(A \cap B = \emptyset)$.

   Then $\exists p^* \in R^L \backslash \{0\}$ and $\exists r \in R$ s.t. $p^* \cdot a \geq r$ for $\forall a \in A$ and $p^* \cdot b \leq r$ for $\forall b \in B$

   It means that $\exists$a hyperplane that separate A and B.

2. $\boxed{\text{Theorem}}$

   In $E = \left[\left\{I, (\succsim_i, e_i)_{i \in I}\right\}\right]$ s.t.

   $X_i = R^L_+$, $0 \in X_i$, and convex for $\forall i$

   For $\forall i \in I$, $\succsim_i$is a complete preorder s.t. $\begin{cases} \text{strictly monotone} \\ \text{(weakly)convex:} \{x_i \in X_i \mid x_i \succsim_i x'_i\} \text{ is convex for } \forall x'_i \\ \text{continuous} \end{cases}$

   For $\forall i \in I$, $e_i \in R^L_+$

   for every pareto optimal allocation (x^*), $\exists p^* \in R^L$, $p^* \neq 0$ s.t. (x^*, p^*) is a C.E. with $\bar{e}_i = x^*_i$ for $\forall i \in I$

   $\boxed{\text{Proof}}$→application of the separating hyperplane theorem for convex sets

   (a) Find an equilibrium candidate price system p^*

      i. Define, for $\forall i \in I, V_i = \{x_i \in X_i \mid x_i \succ_i x^*_i\}$ and
      $V = \sum_{i \in I} V_i = \left\{\sum_{i \in I} x_i \in X = R^L \mid x_1 \in V_1, ..., x_I \in V_I\right\}$.
      $\{e\} = \sum_{i \in I} x^*_i$

      ii. Convexity

      - For $\forall i \in I, V_i$ is convex because $\succsim_i$is weakly convex and transitive.
        And then V is convex because the sum of convex sets is convex.
        $\{e\}$ is convex as a singleton set

      iii. $V \cap \{e\} = \emptyset \leftarrow$Pareto optimality of (x^*, y^*)
      Suppose not. $\exists\,(x) \in V \cap \{e\}$.
      Then for $\forall i \in I$, $x_i \succ_i x^*_i$ because $(x) \in V$ and (x) is feasible because $(x) \in Y + \{e\}$.
      Therefore, (x^*) is not pareto optimal which is a contradiction.

      iv. Separating Hyperplane Theorem
      By separating hyperplane theorem, $\exists p^* \in R^L, p^* \neq 0$ s.t.$\exists r$ s.t $p^* \cdot x \geq r \geq p^* \cdot x'$ for $\forall x \in V$ and $\forall x' \in \{e\}$

      v. $p^* \in R^L_+$
      Let o_k be the k th unit vector s.t $k \in L$
      By strict monotonicity of $\succsim_i, \{e\} + o_k \in V$.
      Then $p^* \cdot [\{e\} + o_k] \geq r \geq p^* \cdot \{e\} \rightarrow p^* \cdot o_k \geq 0 \rightarrow p^*_k \geq 0$ for $\forall k \in L$.
      Therefore, $p^* \in R^L_+$

(b) Show (x^*, y^*) with p^* consists of a price quasiequilibrium

i. Show [For $\forall i, x_i \succ_i x_i^* \rightarrow p^* \cdot x_i \geq p^* \cdot x_i^*$]

Suppose $x_i \succ_i x_i^*$.

By strict monotonicity and continuity of $\succsim_i$, for $\forall \varepsilon > 0$,

$\bar{x}_i = x_i - (0, ..., 0, \varepsilon, 0, ...0) \succ_i x_i^*$

$\bar{x}_{i'} = x_i^* + \left(0, ..., 0, \frac{\varepsilon}{I-1}, 0, ...0\right) \succ_i x_{i'}^*$ for $\forall i' \neq i$

$\bar{x}_i \in V_i \rightarrow \sum_{i \in I} \bar{x}_i \in V$

$\sum_{i \in I} p^* \cdot \bar{x}_i \geq r \geq \sum_{i \in I} p^* \cdot x_i^*$ because $\{x_i^*\} \in \{e\}$

$\rightarrow p^* \cdot x_i - p^* \cdot (0, ..., 0, \varepsilon, 0, ...0) + \sum_{i' \neq i} p^* \cdot x_i^* + p^* \cdot \sum_{i' \neq i} \left(0, ..., 0, \frac{\varepsilon}{I-1}, 0, ...0\right)$

$= p^* \cdot x_i + \sum_{i' \neq i} p^* \cdot x_i^* \geq \sum_{i \in I} p^* \cdot x_i^*$

$\rightarrow p^* \cdot x_i \geq p^* \cdot x_i^*$

ii. Show [For $\forall i, x_i \succ_i x_i^* \rightarrow p^* \cdot x_i > p^* \cdot x_i^*$]

Suppose not. $\exists x_i \in X_i$ s.t. $x_i \succ_i x_i^*$ and $p^* \cdot x_i = p^* \cdot x_i^*$

By convexity of X_i and continuity of $\succsim_i$, for $\lambda < 1$ close enough to 1, $\exists \lambda x_i \in X_i$ s.t. $\lambda x_i \succ_i x_i^*$.

$\lambda x_i \succ_i x_i^* \rightarrow p^* \cdot \lambda x_i \geq p^* \cdot x_i^*$

On the other hand, $[x_i \succ_i x_i^*$ and $p^* \cdot x_i = p^* \cdot x_i^*]$

$\rightarrow$ [for $\lambda < 1, \lambda x_i \succ_i x_i^*$ and $p^* \cdot \lambda x_i < p^* \cdot x_i^*$] which is a contradiction?!

12.2 Production Economy

12.2.1 First Welfare Theorem

1. First welfare theorem

   If preferences are l.n.s., and if (x^*, y^*, p^*) is a price equilibrium with transfers, then the allocation (x^*, y^*) is pareto optimal.

   $\boxed{\text{Proof}}$

   Suppose that (x^*, y^*, p^*) is a price equilibrium with transfers and the associated wealthes $(w_1, w_2, ..., w_I)$ s.t. $\sum_{i\in I} w_i = \sum_{i\in I} p^* \cdot e_i + \sum_{j\in J} p^* \cdot y_i^*$, but it is not pareto optimal; i.e. $\exists$a feasible allocation $(x, y) \in X \times Y$ s.t. for $\forall i \in I$, $x_i \succsim_i x_i^*$ and for at least one i', $x_{i'} \succ_{i'} x_{i'}^*$.

   Individual agents' preference maximization of the definition of price equilibrium with transfer implies that if $x_i \succ_i x_i^*$, then $p^* \cdot x_i > p^* \cdot x_i^* = w_i$ for $\forall i \in I$. With this fact and l.n.s. of preference implies that if $x_i \succsim_i x_i^*$, then $p^* \cdot x_i \geq p^* \cdot x_i^* = w_i$ for $\forall i \in I$.

   $$\begin{array}{ll} & x_i \succsim_i x_i^* \rightarrow p^* \cdot x_i \geq w_i \text{ for } \forall i \in I \\ \text{Then} & x_{i'} \succ_{i'} x_{i'}^* \rightarrow p^* \cdot x_{i'} > w_i \text{ for at least one } i' \\ & \rightarrow \ \sum_{i\in I} p^* \cdot x_i > \sum_{i\in I} w_i = \sum_{i\in I} p^* \cdot e_i + \sum_{j\in J} p^* \cdot y_i^* \end{array}$$

   And from firms' profit maximization of the definition of equilibrium implies that $p^* \cdot y_j \leq p^* \cdot y_j^*$ for $\forall j \in J$.

   $$\begin{array}{ll} \text{Therefore,} & \sum_{i\in I} p^* \cdot x_i > \sum_{i\in I} w_i = \sum_{i\in I} p^* \cdot e_i + \sum_{j\in J} p^* \cdot y_i^* \geq \sum_{i\in I} p^* \cdot e_i + \sum_{j\in J} p^* \cdot y_j \\ & \Leftrightarrow \sum_{i\in I} p^* \cdot x_i > \sum_{i\in I} p^* \cdot e_i + \sum_{j\in J} p^* \cdot y_j \end{array}$$

   It is a contradiction because from the definition of feasible allocation,

   we have $\sum_{i\in I} x_i \leq \sum_{i\in I} e_i + \sum_{j\in J} y_j \rightarrow \sum_{i\in I} p^* \cdot x_i \leq \sum_{i\in I} p^* \cdot e_i + \sum_{j\in J} p^* \cdot y_j$.

2. Interpretation

   (a) Although the result appear to follow from very simple hypothetheses, note that we are already assuming universal price quoting of commodities(complete market) and price taking by agents. Under a certain circumstances like externality, market power, and asymmetric information, the first welfare theorem fail to be satisfied.

   (b) The first welfare theorem is entirely silent about the desirability of the equilibrium allocation from a distributional standpoint.

12.2.2 Second Welfare Theorem

There are two ways to prove this theorem.

A price quasiequilibrium with transfers

1. $\boxed{\text{Theorem}}$

   In $E = \left[\left\{I, (\succsim_i, e_i)_{i\in I}\right\}, \left\{J, (Y_j)_{j\in J}\right\}\right]$ s.t.

   $X_i = R^L_+$, $0 \in X_i$, and $\boxed{\text{convex}}$ for $\forall i$

   $$\text{For } \forall i \in I, \succsim_i \text{is a preorder s.t.} \begin{cases} \text{l.n.s.} \\ \text{(weakly)convex:} \{x_i \in X_i \mid x_i \succsim_i x_i'\} \text{ is convex for } \forall x_i' \\ \text{continuous} \end{cases}$$

   $(w_1, w_2, ..., w_I) \in R^L_{++}$

   $Y_j = R^L$ for $\forall j$ and $Y = \{Y_j\}_{j\in J}$ satisfying

   For $\forall j \in J$, Y_j is $\boxed{\text{convex}}$,

   for every pareto optimal allocation (x^*, y^*), $\exists p^* \in R^L$, $p^* \neq 0$ s.t. (x^*, y^*, p^*) is a price equilibrium with transfers.

   $\boxed{\text{Proof}}$→application of the separating hyperplane theorem for convex sets

   (a) Find an equilibrium candidate price system p^*

      i. Define, for $\forall i \in I, V_i = \{x_i \in X_i \mid x_i \succ_i x_i^*\}$
      and $V = \sum_{i\in I} V_i = \left\{\sum_{i\in I} x_i \in X = R^L \mid x_1 \in V_1, ..., x_I \in V_I\right\}$.
      For $\forall j \in J, Y = \sum_{j\in J} Y_j = \left\{\sum_{j\in J} y_i \in Y \mid y_1 \in Y_1, ..., y_J \in Y_J\right\}$ and $Y + \{e\}$ is the aggregate production set shifted to $\{e\}$

      ii. Convexity

         - For $\forall i \in I, V_i$ is convex because $\succsim_i$is weakly convex and transitive.
           And then V is convex because the sum of convex sets is convex.
           For $\forall j \in J, Y + \{e\}$ is convex because the sum of convex sets is convex

      iii. $V \cap Y + \{e\} = \emptyset$ ←Pareto optimality of (x^*, y^*)
      Suppose not. $\exists (x, y) \in V \cap Y + \{e\}$.
      Then for $\forall i \in I$, $x_i \succ_i x_i^*$ because $(x, y) \in V$ and (x, y) is feasible because $(x, y) \in Y + \{e\}$.
      Therefore, (x^*, y^*) is not pareto optimal which is a contradiction.

      iv. Separating Hyperplane Theorem
      By separating hyperplane theorem, $\exists p^* \in R^L, p^* \neq 0$ s.t.$\exists r$ s.t $p^* \cdot x \geq r$ for $\forall (x, y) \in V$ and $p^* \cdot x \leq r$ for $\forall (x, y) \in Y + \{e\}$

   (b) Show (x^*, y^*) with p^* consists of a price quasiequilibrium

      i. Show $\left[\text{For } \forall i \in I, x_i \succsim_i x_i^* \to \sum_{i\in I} p^* \cdot x_i \geq r\right]$
      Suppose for $\forall i \in I, x_i \succsim_i x_i^*$.

By l.n.s. of preferences for $\forall i$, $\exists \tilde{x}_i$ which is arbitrary close to x_i with $\tilde{x}_i \succ_i x_i$. Hence $\tilde{x}_i \in V$ and it implies $\sum_{i\in I}\tilde{x}_i \in V$ and $p^* \cdot \sum_{i\in I}\tilde{x}_i \geq r$
Take the limit as $\tilde{x}_i \to x_i$, and then $\sum_{i\in I} p^* \cdot x_i \geq r$

ii. Show $\left[\sum_{i\in I} p^* \cdot x_i^* = \sum_{i\in I} p^* \cdot e_i + \sum_{j\in J} p^* \cdot y_j^* = r\right]$
$\left[\text{For } \forall i \in I, x_i^* \succsim_i x_i^* \to \sum_{i\in I} p^* \cdot x_i^* \geq r\right]$
$\sum_{i\in I} x_i^* = \sum_{i\in I} e_i + \sum_{j\in J} y_j^* \in Y + \{e\} \to \sum_{i\in I} p^* \cdot x_i^* \leq r$
Therefore, $\sum_{i\in I} p^* \cdot x_i^* = \sum_{i\in I} p^* \cdot e_i + \sum_{j\in J} p^* \cdot y_j^* = r$

iii. Show $\left[\text{For } \forall j \in J, p^* \cdot y_j \leq p^* \cdot y_j^* \text{ for } \forall y_j \in Y_j\right]$
Consider for any $j \in J$, $\sum_{h\neq j} y_h^* + y_j + \{e\} \in Y + \{e\}$
$p^* \cdot \left[\sum_{h\neq j} y_h^* + y_j + \{e\}\right] \leq r = p^* \cdot \left[\sum_{i\in I} e_i + \sum_{j\in J} y_j^*\right]$
$p^* \cdot y_j \leq p^* \cdot y_j^*$

iv. Show [For $\forall i, x_i \succ_i x_i^* \to p^* \cdot x_i \geq p^* \cdot x_i^*$]
Suppose $x_i \succ_i x_i^*$.
By $\sum_{i\in I} p^* \cdot x_i \geq r$ and $\sum_{i\in I} p^* \cdot x_i^* = r$,
$p^* \cdot \left(x_i + \sum_{i'\neq i} x_{i'}^*\right) \geq r = p^* \cdot \left(x_i^* + \sum_{i'\neq i} x_{i'}^*\right) \to p^* \cdot x_i \geq p^* \cdot x_i^*$

v. Define $w_i = p^* \cdot x_i^*$ for $\forall i \in I$.
Then $\sum_{i\in I} w_i = \sum_{i\in I} p^* \cdot x_i^* = p^* \cdot \left(\sum_{i\in I} e_i + \sum_{j\in J} y_j^*\right)$

Therefore, for every pareto optimal allocation (x^*, y^*), $\exists p^* \in R^L$, $p^* \neq 0$ s.t. (x^*, y^*, p^*) is a price quasiequilibrium with transfers

(c) Any price quasiequilibrium (x^*, y^*, p^*) with $(w_1, w_2, ..., w_I) \in R^L_{++}$ is a price equilibrium (x^*, y^*, p^*) with $(w_1, w_2, ..., w_I) \in R^L_{++}$
NTS: $x_i \succ_i x_i^* \to p^* \cdot x_i > w_i$
Suppose not. $\exists x_i \in X_i$ s.t. $x_i \succ_i x_i^*$ and $p^* \cdot x_i = w_i$
By convexity of X_i, $\exists x_i' \in X_i$ s.t. $p^* \cdot x_i' < w_i$ and $\exists \bar{x}_i = \lambda x_i + (1-\lambda) x_i'$ s.t. $p^* \cdot \bar{x}_i < w_i$
By continuity of $\succsim_i, \exists \varepsilon > 0\,(\text{for } \lambda \text{ close enough to } 1), \exists \bar{x}_i = \lambda x_i + (1-\lambda) x_i' \in B(x_i)$ s.t. $p^* \cdot \bar{x}_i < w_i$ and $\bar{x}_i \succ_i x_i^*$.
It contradicts (x^*, y^*, p^*) is a price quasiequilibrium?!

2. Conclusion

(a) The second welfare theorem identifies conditions under any pareto optimal allocation can be implemented through competitive markets and offers a strong conceptual affirmation of the use of competitive markets, even for dealing with distributional concerns.

(b) Some limitations on the use of welfare theorem

(c) A planning authority wishing to implement a particular pareto optimal allocation must be able to insure the supporting prices will be taken as given by consumers and firms.

(d) The authority must have good information to identify the pareto optimal allocation to be implemented and to compute the right supporting price vector. For this purpose, the authority must know at least the statistical joint distribution of preferences, endowments, and other relevant characteristics of consumers and firms. And moreover, in order to implement the correct transfer levels for each consumer, the authority must know who is who, which is unlikely in practice. As a result, most common transfer plan fails to be lump-sum transfers.

(e) Even if the authority observes all the required information, it must actually have the power to enforce the necessary transfers through some tax and transfer mechanism that individual can not evade.

(f) Because of informational and enforceability limitations, it is practically unlikely that extensive lump-sum taxation will be possible. If these transfer plan is not possible, then the second welfare theorem fails in the sense that for a typical economy, only a limited range of pareto optima are supportable by means of prices supplemented by the usual sort of taxation plan. For the typical economy, redistribution plans are distortionary; that is , they trade off distributional aims against pareto optimality.

(g) In summary, the second welfare theorem is a very useful theoretical reference point. But it is far from a direct prescription for policy practice. On the contrary, by pointing out what is necessary to achieve any desired pareto optimal allocation, it serves a cautionary purposes.

12.2.3 Generalized second welfare theorem

1. Nonconvex Production Technologies and Marginal Cost Pricing

   The second welfare theorem runs into difficulties in the presence of nonconvex production sets. In the first place, large nonconvexities caused by the presence of fixed costs or extensive increasing returns lead to a world of a small number of large firms(in the limit, natural monopoly), making price taking assumption unlikely.

   Although nonconvexities may prevent us from supporting the pareto optimal production allocation as a profit maximizing choice, under the differentiability assumptions, we can use the first order necessary conditions derived there to formulate a weaker result that parallels the second welfare theorem.

2. Marginal cost price equilibrium with transfers

   In the production economy E_p, a marginal cost price equilibrium with transfers consists of $\left\{\{x_i^*\}_{i\in I}, \left\{y_j^*\right\}_{j\in J}\right\}$ and $p^* \in R_+^L \backslash \{0\}$ s.t.

   (a) $\exists\,(w_1, w_2, ..., w_I)$ s.t. $\sum_{i\in I} w_i = \sum_{i\in I} p^* \cdot e_i + \sum_{j\in J} p^* \cdot y_j^*$

(b) for $\forall i \in I$, x_i^* is $\succsim_i -$ maximal in $B_i(p^*, e_i) = \{x_i \in X_i \mid p^* \cdot x_i \leq w_i\}$

(c) for $\forall j \in J$, $p^* = \gamma_j F_j'\left(y_j^*\right)$ for some $\gamma_j > 0$

(d) $\sum_{i \in I} x_i^* \leq \sum_{i \in I} e_i + \sum_{j \in J} y_j^*$

For two goods case, $p^* = \gamma_j F_j'\left(y_j^*\right)$ means that the price of input must equal the price of the output multiplied by the marginal productivity of the input.

The price of output equals the marginal cost.

3. First welfare theorem fails.

   because marginal cost pricing neglects second order conditions and it may therefore happen that the second order conditions for the social utility maximization are not satisfied.

4. Generalized second welfare theorem

   Suppose $u_i(x_i)$ is C^2 and quasiconcave, $u_i'(x_i) >> 0$, and $u_i(0) = 0$

   and $F_j(y_j) = 0$ at transformation frontier, $F_j(y_j)$ is C^2, $F_j(0) \leq 0$, $F_j'(y_j) >> 0$

   Then if $(\hat{x}, \hat{y})$ is pareto optimal, then $\exists$a price vector p^* and wealth levels $(w_i)_{i \in I}$ with $\sum_{i \in I} w_i = \sum_{i \in I} p^* \cdot e_i + \sum_{j \in J} p^* \cdot \hat{y}_j$ s.t. $(\hat{x}, \hat{y})$ is a marginal cost price equilibrium allocation.

   →The firm incurs a loss at the prices that locally support the pareto optimal allocation.

13 Pareto Optimality and Social Welfare Function

- Given a family of $u_i(\cdot)$ of continuous utility functions representing the preferences $\succsim_i$of the I consumers, we can capture the attainable vectors of utility levels for an economy specified by $E = \left[\left\{I, (\succsim_i, e_i)_{i\in I}\right\}, \left\{J, (Y_j)_{j\in J}\right\}\right]$ by means of the utility possibility set s.t.
$$U = \left\{(u_1, u_2, ..., u_I) \in R^L \mid \exists (x, y) \text{ s.t.} u_i(x_i) \geq u_i \text{ for } \forall i \in I\right\}$$
- In two consumers' economy, we can depict the utility possibiliity set which is closed under sufficient conditions.
- By the definition of pareto optimality, the utility levels of a pareto optimal allocation must belong to the boundary of the utility possibility set; the pareto frontier UP
$$UP = \left\{(u_1, u_2, ..., u_I) \in U \mid \nexists \left(u_1', ..., u_I'\right) \in U \text{ s.t.} u_i' \geq u_i \text{ for } \forall i \in I \text{ and } u_{i'}' > u_{i'} \text{ for at least one } i'\right\}$$
- A feasible allocation (x, y) is a pareto optimum iff $(u_1(x_1), ..., u_I(x_I)) \in UP$.
- Note that if every X_i and Y_j is convex and the utility functions $u_i(\cdot)$ are concave, then the utility possibility set U is convex.

1. Social Planner's Problem
   (a) Suppose that society's distributional principles can be summarized in a social welfare function $W(u_1, u_2, ..., u_I) = \sum_{i\in I} \lambda_i u_i$ for some constant $\lambda = (\lambda_1, \lambda_2, ..., \lambda_I)$ s.t. $\lambda \geq 0$ because social welfare should be nondecreasing in the consumers' utility levels.

   (b) For economies with $\boxed{\text{convex}}$ utility possibility sets, there is a close relation between pareto optima and linear social welfare optima;
   every linear social welfare optimum with weights $\lambda >> 0$ is pareto optimal and
   every pareto optimal allocation is a social welfare optimum for some welfare weights $\lambda \geq 0$

   (c) Let (x^*, y^*) be pareto optimal.
   With $\lambda >> 0$, define a utility function $u_\lambda(\bar{x})$ on aggregate consumption vectors in X by
$$u_\lambda(\bar{x}) = \max_x \sum_{i\in I} \lambda_i \cdot u_i(x_i)$$
$$\text{s.t. } x_i \in X_i \text{ for all } i \text{ and } \sum_i x_i = \bar{x}$$
   Then (x^*, y^*) is a solution to the problem
$$\max u_\lambda(\bar{x})$$
$$\text{s.t. } \bar{x} = \bar{e} + \bar{y},\ \bar{x} \in X \text{ and } \bar{y} \in Y$$

2. First order conditions for pareto optimality

   With differentiability assumptions, we show how prices and the optimality properties of price taking behavior emerge from an examination of the first order condition of pareto optimality problems.

   (a) Assumptions

      i. $u_i(x_i)$ is twice differentiable and $u_i'(x_i) >> 0 \leftarrow$ preference strictly monotone, and $u_i(0) = 0$
      ii. $Y_j = \left\{y \in R^L \mid F_j(y_j) \leq 0\right\}$
      $F_j(y_j) = 0$ at transformation frontier, $F_j(y_j)$ is C^2, $F_j(0) \leq 0$, $F_j'(y_j) >> 0$

   (b) Problem

   The problem of identifying the pareto optimal allocations for this economy can be reduced to the selection of allocations that solve the following problem

   $$\begin{array}{ll} \max u_1(x_1) & \\ \text{s.t. } u_i(x_i) \geq \bar{u}_i \text{ for } \forall i \neq 1 & \delta_i \\ \sum_{i \in i} x_i^l \leq \sum_{i \in I} e_i^l + \sum_{j \in J} y_j^l \text{ for } \forall l \in L & \mu_l \\ F_j(y_j) \leq 0 \text{ for } \forall j \in J & \gamma_j \end{array}$$

   By solving it for varing required levels of utility for these other consumers $(\bar{u}_i)_{i=2}^{I}$, we can identify all the pareto optimal allocations for this economy.

   (c) First order conditions

   Under $\bar{u}_i \geq 0$ for $\forall i \neq 1$, all the constraints of problem will be binding at a solution.

   $$x_i^l;\ \delta_i \frac{\partial u_i}{\partial x_i^l} - \mu_l \left\{ \begin{array}{l} \leq 0 \\ = 0 \text{ if } x_i^l > 0 \end{array} \right. \text{ for } \forall i, l$$

   $$y_j^l;\ \mu_l - \gamma_j \cdot F_j'\left(y_j^l\right) = 0 \text{ for } \forall j, l$$

   At the interior solution,

   $$MRS_i(l, l') = MRS_{i'}(l, l')$$
   $$MRT_j(l, l') = MRT_{j'}(l, l')$$
   $$MRS_i(l, l') = MRT_{j'}(l, l')$$

   (d) Interpretations

      i. The multiplier μ_l at an optimal solution is exactly equal to the increase in consumer 1' utility derived from a relaxation of the corresponding constraint, that is, from a marginal increase in the available social endowment $\sum_{i \in I} e_i^l$; that is, the multiplier μ_l can be interpreted as the marginal value or shadow price of good l in terms of consumer 1's utility
      ii. The multiplier δ_i equals the marginal change in consumer 1's utility if we decrease the utility requirement $(\bar{u}_i)_{i=2}^{I}$.

At the optimal interior allocation, weighted by the amount that relaxing consumer i's utility constraint is worth in terms of raising consumer 1's utility, the increase in the utility of any consumer i from receiving an additional unit of good l should be equal to the marginal value μ_l of good l.

iii. The multiplier γ_j is the marginal benefit from relaxing the jth production constraint(the marginal cost from tighting the jth production constraint)
At an optimum, the marginal cost is equated to the marginal benefit μ_l of good l for $\forall j \in J$

3. Welfare theorem and social planner's problem

   (a) Assumptions

      i. $\succsim_i$is $\boxed{\text{strictly monotone}}$, $\boxed{\text{convex}}$, continuous, complete preorder on $R^L \rightarrow u_i(x_i)$ is C^2 and quasiconcave, $u_i'(x_i) >> 0$, and $u_i(0) = 0$

      ii. $F_j(y_j) = 0$ at transformation frontier, $F_j(y_j)$ is C^2, $F_j(0) \leq 0$, $F_j'(y_j) >> 0$, and a $\boxed{\text{convex function}}$(production sets Y_j is convex for $\forall j$)
      It means we do not have to check the second order condition for having maximum.

   (b) Problem

      Let (x^*, y^*, p^*) be a price equilibrium with transfers s.t $w_i = p^* \cdot e_i + p^* \cdot \sum_{j\in J} \theta_{ij} \cdot y_j^*$ for $\forall i \in I$ iff

      i. $\begin{array}{l} \max u_i(x_i) \\ \text{s.t. } p^* \cdot x_i \leq w_i \text{ for } \forall i \in I \end{array} \rightarrow \alpha$

      ii. $\begin{array}{l} \max p^* \cdot y_j \\ \text{s.t. } F_j(y_j) \leq 0 \text{ for } \forall j \in J \end{array} \rightarrow \beta$

      iii. $\sum_{i\in I} x_i^* = \sum_{i\in I} e_i + \sum_{j\in J} y_j^*$

   (c) First order conditions

$$\begin{array}{l} x_i^l; u_i'\left(x_i^l\right) - \alpha_i \cdot p_l \begin{cases} \leq 0 \\ = 0 \text{ if } x_i^l > 0 \end{cases} \text{ for } \forall i, l \\ y_j^l; p_l - \beta_j \cdot F_j'\left(y_j^l\right) = 0 \text{ for } \forall j, l \end{array}$$

$$\Leftrightarrow \begin{array}{l} x_i^l; \delta_i \frac{\partial u_i}{\partial x_i^l} - \mu_l \begin{cases} \leq 0 \\ = 0 \text{ if } x_i^l > 0 \end{cases} \text{ for } \forall i, l \\ y_j^l; \mu_l - \gamma_j \cdot F_j'\left(y_j^l\right) = 0 \text{ for } \forall j, l \end{array}$$

   (d) Social welfare function

$$\begin{array}{ll} \max_x \sum_{i\in I} \lambda_i \cdot u_i(x_i) & \\ \text{s.t. } \sum_{i\in i} x_i^l \leq \sum_{i\in I} e_i^l + \sum_{j\in J} y_j^l \text{ for } \forall l \in L & \rightarrow \Psi \\ F_j(y_j) \leq 0 \text{ for } \forall j \in J & \rightarrow \Phi \end{array}$$

$$\Leftrightarrow \begin{array}{l} x_i^l; \lambda_i u_i' \left(x_i^l\right) - \Psi_l \begin{cases} \leq 0 \\ = 0 \text{ if } x_i^l > 0 \end{cases} \text{for } \forall i, l \\ y_j^l; \Psi_l - \Phi_j \cdot F_j' \left(y_j^l\right) = 0 \text{ for } \forall j, l \end{array}$$

if $\mu_l = p_l = \Psi_l$, $\frac{1}{\delta_i} = \alpha_i \cdot = \frac{1}{\lambda_i}$ and $\gamma_j = \beta_j = \Phi_j$

(e) Interpretations

i. Since both sets of conditions are necessary and sufficient for their respective problems, it means that the allocation (x^*, y^*) is pareto optimal iff it is a price equilibrium with transfers with respect to some price vector p^*. Note that the equilibrium price p_l exactly equal to μ_l, the marginal value of good l in the pareto optimal problem.

(f) Conclusion

Under the assumptions made about the economy E,
in particular $u_i(\cdot)$ is concave and $F_j(\cdot)$ is convex for $\forall i, j$,
every pareto optimal allocation (and, hence, every price equilibrium with transfers) maximizes a weighted sum of utilities subject to the resource and technological constraints. Moreover, the weight λ_i of the utility of the ith consumer equals the reciprocal of consumer i's marginal utility of wealth evaluated at the supporting prices and imputed wealth.

14 Core

14.1 Definition

- Given an exchange economy $E = \left\{I, (\succsim_i, e_i)_{i\in I}\right\}$ s.t. $X_i = R^L_+$ for $\forall i$ and $e \in R^L_+$ for $\forall i$, the core of this economy E is the set of allocation $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, ..., \tilde{x}_I)$ satisfying

  i) feasibility: $\sum_{i\in I} \tilde{x}_i \leq \sum_{i\in I} e_i$

  ii) there is no other feasible allocation $\bar{x} = (\bar{x}_1, \bar{x}_2, ..., \bar{x}_I)$ s.t $\exists$a coalition $S \subseteq I\,(S \neq \emptyset)$ s.t. $\bar{x}_i \succ_i \tilde{x}_i$ for $\forall i \in S$ and $\sum_{i\in S} \bar{x}_i \leq \sum_{i\in S} e_i$

- Given a production economy $E_p = \left[\left\{I, (\succsim_i, e_i)_{i\in I}\right\}, 1\right]$ s.t. $X_i = R^L_+$ for $\forall i$, $e \in R^L_+$ for $\forall i$, and a publicly available CRS convex technology $Y \subset R^L$

  the core of this economy E_p is the set of allocation $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, ..., \tilde{x}_I)$ satisfying

  i) feasibility: $\sum_{i\in I} \tilde{x}_i \leq Y + \sum_{i\in I} e_i$

  ii) there is no other feasible allocation $\bar{x} = (\bar{x}_1, \bar{x}_2, ..., \bar{x}_I)$ s.t $\exists$a coalition $S \subseteq I\,(S \neq \emptyset)$ s.t. $\bar{x}_i \succ_i \tilde{x}_i$ for $\forall i \in S$ and $\sum_{i\in S} \bar{x}_i \leq Y + \sum_{i\in S} e_i$

14.2 Existence

- $\boxed{\text{Claim}}$

  Given an exchange economy $E = \left\{I, (\succsim_i, e_i)_{i\in I}\right\}$ s.t. $X_i = R^L_+$ and $\succsim_i$is a convex, continuous, and complete preorder on X_i for $\forall i$, and $e \in R^L_+ \backslash \{0\}$ for $\forall i$, Core(E) is nonempty and compact.

  $\boxed{\text{Proof}}$

  1. Nonemptiness
  2. Compactness
     (a) boundedness
     By the definition of Core(E) , every allocation $\tilde{x} \in R^{LI}_+$ in it has to satisfy feasibility condition.
     $0 \leq \sum_{i\in I} \tilde{x}_i \leq \sum_{i\in I} e_i$
     Thus, $\forall \tilde{x} \in$Core(E) is bounded and Core(E) is bounded.
     (b) closedness
     We have to show that if a sequence of core allocations $\tilde{x}^n \in R^{LI}_+$ has a limit point $\tilde{x} \in R^{LI}_+$, $\tilde{x} \in$Core(E) .
     Suppose not: $\tilde{x} \notin$Core(E) .
     Then $\exists$nonempty coalition S and another allocation $\bar{x} \in R^{LI}_+$ s.t.
     i) $\bar{x}_i \succ_i \tilde{x}_i$ for $\forall i \in S$

ii)$\sum_{i\in S}\bar{x}_i = \sum_{i\in S} e_i$
By the continuity of preferences, $\exists N$ s.t. for $n > N$, $\bar{x}_i \succ_i \tilde{x}_i^n$ for $\forall i \in S$ which is a contradiction.
Thus, Core(E) is closed.

(c) Therefore, Core(E) is compact.

14.3 Relationships

14.3.1 ($C.E. \subseteq Core \subseteq P.O.$)

1. $Core \subseteq (weakly)\, P.O.$

   We do not need any assumption.

   $\boxed{\text{Proof}}$

   For Core, there is no subset of I(coalition S) improving upon the core allocation. It implies that obviously there is no allocation pareto dominates Therefore, if $\tilde{x} \in Core$, then $\tilde{x} \in (weakly)\, P.O.$

   Furthermore, with strictly monotone, continuous, complete preorder of preference, we have $Core \subseteq P.O.$

2. $C.E. \subseteq Core$

   We do not need any assumption.

   $\boxed{\text{Proof}}$

   Suppose not: $\tilde{x} \in C.E.$ but $\tilde{x} \notin Core$

   Then $\exists$nonempty coalition S and another allocation $\bar{x} \in R_+^{LI}$ s.t.

   i) $\bar{x}_i \succ_i \tilde{x}_i$ for $\forall i \in S$

   ii)$\sum_{i\in S}\bar{x}_i = \sum_{i\in S} e_i$
   $[\sum_{i\in S}\bar{x}_i = \sum_{i\in S} e_i] \to [p \cdot \sum_{i\in S}\bar{x}_i = p \cdot \sum_{i\in S} e_i]$
   On the other hand, because $\tilde{x} \in C.E$ and i), for $\forall i \in S$, $p \cdot \bar{x}_i > p \cdot \tilde{x}_i$ and then $p \cdot \sum_{i\in S}\bar{x}_i > p \cdot \sum_{i\in S} e_i$ which is a contradiction.
   Furthermore, with l.n.s. of preference, $C.E. \subseteq P.O.$

3. Therefore, without l.n.s. of preference, $C.E. \subseteq Core$ but $C.E. \nsubseteq P.O(?)$

4. $Core \nsubseteq C.E.$

   It is only true if there are large number of agents.

14.3.2 Large Economy

1. R^{th}-fold replica economy E^R

   # of agents: RI

   # of types of agents: R

   $E^R = \left\{RI, \left((\succsim_{ir}, e_{ir})_{i\in I}\right)_{r\in R}\right\}$

   each R agent in one type has the same preference and endowment.

   An allocation in E^R, $x \in R_+^{LRI}$

2. CE in E^R

   Given a replicated economy $E^R = \left\{RI, \left((\succsim_i, e_i)_{i\in I}\right)_R\right\}$, a CE consists of a CE allocation $x^* = \{(x^*_{11}, ..., x^*_{1R}), ..., (x^*_{I1}, ..., x^*_{IR})\}$ and a CE price system $p^* \in R^L_+ \backslash \{0\}$ s.t.

   i) for $\forall i \in I$ and $\forall r \le R, x^*_{ir}$ is maximal for $\succsim_{ir}$in $B_{ir}(p^*, e_{ir}) = \left\{x_{ir} \in R^L_+ \mid p^* \cdot x_{ir} \le p^* \cdot e_{ir}\right\}$

   ii) $\sum_{r\le R}\sum_{i\in I} x^*_{ir} \le \sum_{r\le R}\sum_{i\in I} e_{ir}$

   (a) CE allocation satisfies ETP

   Suppose $x^* \in R^{LRI}_+$ is a CE allocation associated with $p^* \in R^L_+ \backslash \{0\}$, but it does not satisfy ETP; that is, $\exists i' \in I$ s.t. $\exists r' \in R$ s.t. $x^*_{ir} \ne x^*_{ir'}$ for all other $r \in R$.

   By the definition of CE and strict monotonicity of preferences, we have $p^* \cdot x^*_{ir} = p^* \cdot e_{ir}$ and $p^* \cdot x^*_{ir'} == p^* \cdot e_{ir'}$ for $\forall i \in I$ and $\forall r \le R$

   Because $e_{ir} = e_{ir'}$, for $\forall \lambda \in (0, 1)$, we have $p^* \cdot \left[\lambda x^*_{ir} + (1-\lambda)\, x^*_{ir'}\right] = p^* \cdot x^*_{ir} = p^* \cdot e_{ir}$. Therefore, $\lambda x^*_{ir} + (1-\lambda)\, x^*_{ir'}$ is feasible.

   W.l.o.g. $x^*_{ir} \succsim_i x^*_{ir'}$ and strict convexity of preferences, $\lambda x^*_{ir} + (1-\lambda)\, x^*_{ir'} \succ_i x^*_{ir'}$

   It contradicts $x^*_{ir'}$ is an element of CE allocation x^*.

3. Equal Treatment Property

   Given $E^R = \left\{RI, \left((\succsim_{ir}, e_{ir})_{i\in I}\right)_{r\in R}\right\}$ in which, for each $i \in I$ and $\forall r, r' \in R$, $\succsim_{ir} = \succsim_{ir'}$which is a strictly convex, strictly monotone, continuous, and complete preorder and $e_{ir} = e_{ir'} \in R^L_+ \backslash \{0\}$,

   if .$\tilde{x} = \{(\tilde{x}_{11}, \tilde{x}_{12}, ..., \tilde{x}_{1R}), (\tilde{x}_{21}, \tilde{x}_{22}, ..., \tilde{x}_{2R}), ..., (\tilde{x}_{I1}, \tilde{x}_{I2}, ..., \tilde{x}_{IR})\} \in Core\left(E^R\right)$, then $\tilde{x}_{ir} = \tilde{x}_{ir'}$ for each $i \in I$ and $\forall r, r' \in R$.

   $\boxed{\text{Proof}}$

   Suppose not: $\exists i' \in I$ s.t. $\exists r' \in R$ s.t. $\tilde{x}_{ir} \ne \tilde{x}_{ir'}$ for all other $r \in R$.

   W.l.o.g. assume that the first agent is the worst-off agent in each type($r' = 1$) and among types, the first type does not satisfy equal treatment($i' = 1$).

Define the average consumption for each type s.t. $\hat{x}_i = \frac{1}{R}\sum_{r=1}^{R} \tilde{x}_{ir}$.

By strict convexity of preferences, $\forall i \in I$, $\hat{x}_i \succsim_i \tilde{x}_{i1}$ and for $i' = 1 \in I$, $\hat{x}_1 \succ_1 \tilde{x}_{11}$ s.t.

$$\begin{aligned} \sum_{i\in I} \hat{x}_i &= \sum_{i\in I} \frac{1}{R} \sum_{r=1}^{R} \tilde{x}_{ir} = \frac{1}{R} \sum_{i\in I} \sum_{r=1}^{R} \tilde{x}_{ir} = \frac{1}{R} \sum_{r=1}^{R} \sum_{i\in I} \tilde{x}_{ir} \\ &= \frac{1}{R} \sum_{r=1}^{R} \sum_{i\in I} e_{ir} = \sum_{i\in I} \frac{1}{R} \sum_{r=1}^{R} e_{ir} = \sum_{i\in I} e_{ir} = \sum_{i\in I} e_{i1} \end{aligned}$$

In this situation, we will show that a coalition $S = \{11, 21, 31, ..., I1\}$ formed by I members can attain x so that it blocks the nonequal treatment allocation $\tilde{x}$.

By strict monotonicity and continuity of preferences, $\hat{x}_1 \succ_1 \tilde{x}_{11}$ implies that $\exists l^{th}$ goods and for $\forall \varepsilon > 0$, $x_1 = \hat{x}_1 - (0, ..., 0, \varepsilon, 0, ..., 0) \succ_1 \tilde{x}_{11}$

By strict monotonicity and continuity of preferences, for $\forall i \in I\,(i \neq 1)\,, x_i = \hat{x}_i + \left(0, ..., 0, \frac{\varepsilon}{I-1}, 0, ..., 0\right)$ $\tilde{x}_{i1}$.

$$\begin{aligned} \sum_{i\in I} x_i &= \sum_{i\neq 1} \left[\hat{x}_i + \left(0, ..., 0, \frac{\varepsilon}{I-1}, 0, ..., 0\right)\right] + \hat{x}_1 - (0, ..., 0, \varepsilon, 0, ..., 0) \\ &= \sum_{i\in I} \hat{x}_i = \sum_{i\in I} e_{i1} \end{aligned}$$

Therefore, it contradicts that $\tilde{x} \in Core\left(E^R\right)$.

4. If we replicate the original economy in biased way, then Equal Treatment Property does not have to be satisfied.

   We have a counterexample.

   Two type $I = 2$

   Two agents for type 1, one agent for type 2.

   $L = 2$

   $e_1 = e_2 = (1, 14)\,, e_3 = (27, 1)$

   $u_i(x) = x_1 x_2$ for $\forall agents$

   Then $x^1 = (6, 6) \neq x^2 = (7, 7)\,, x^3 = (16, 16)$ is in the core.

5. Equivalent Treatment Property

   Given $E^R = \left\{RI, \left((\succsim_{ir}, e_{ir})_{i\in I}\right)_{r\in R}\right\}$ in which, for $\forall i \in I$ and $\forall r, r' \in R$, $\succsim_{ir} = \succsim_{ir'}$ which is a convex, strictly monotone, continuous, and complete preorder and $e_{ir} = e_{ir'} \in R_+^L$, if $.\tilde{x} = \{(\tilde{x}_{11}, \tilde{x}_{12}, ..., \tilde{x}_{1R}), (\tilde{x}_{21}, \tilde{x}_{22}, ..., \tilde{x}_{2R}), ..., (\tilde{x}_{I1}, \tilde{x}_{I2}, ..., \tilde{x}_{IR})\} \in Core$, then $\tilde{x}_{ir} \sim \tilde{x}_{ir'}$ for $\forall i \in I$ and $\forall r, r' \in R$.

$\boxed{\text{Proof}}$

14.3.3 $\cap_{R=1}^{\infty} Core\left(E^R\right) \subseteq CE\left(E\right) : Debrew - Scarf\ theorem$

- The role of Equal Treatment Property

  The importance of Equal Treatment Property is that the core of any R^{th} replica is fully described by the core allocation consisting one member of each type.

  It allows us to regard the core allocations as vectors of fixed size LI no matter which size of replica we are concerned with. As a matter of terminology, we call a vector $(x_1, ..., x_I) \in R_+^{LI}$ a type allocation and for any replica R, interpret it as the equal treatment allocation to consumers where each consumer of type i get x_i. Note that for any replica R the corresponding equal-treatment allocatioin is feasible because $\sum_i Rx_i = R\sum_i e_i$. Therefore, we call $(x_1, ..., x_I) \in R_+^{LI}$ as a feasible type allocation for any replica.

- $\boxed{\text{Theorem}}$

  Consider a pure exchange economy E s.t.

  Let for $\forall i \in I$, $\succsim_i$be a strictly convex, strictly monotone, continuous, complete preorder and $e_i \in R_+^L \backslash \{0\}$

  and there is its R^{th}-fold replica economy E^R in which all allocations in the core satisfy Equal Treatment Property.

  Let $C.E.(E)$ be the set of C.E. allocation in the original economy E, $Core\left(E^R\right)$ be the set of core allocation in E^R, and C_R be the set of allocations consisting one member of each type from $Core\left(E^R\right)$:feasible type allocation.

  If $.\tilde{x}^R = \{(\tilde{x}_{11}, \tilde{x}_{12}, ..., \tilde{x}_{1R}), (\tilde{x}_{21}, \tilde{x}_{22}, ..., \tilde{x}_{2R}), ..., (\tilde{x}_{I1}, \tilde{x}_{I2}, ..., \tilde{x}_{IR})\} \in Core\left(E^R\right)$, then as $R \to \infty$, $Core\left(E^R\right)$ shrinks to $\tilde{x} = \{\tilde{x}_1, \tilde{x}_2, ..., \tilde{x}_I\}$ which is the feasible type allocation of $R^{th}-$replica and $\exists \tilde{p}$ s.t. $(\tilde{p}, \tilde{x})$ is CE in E.

  That is, $\cap_{R=1}^{\infty} Core\left(E^R\right) = CE\left(E\right)$

- Separating Hyperplane Theorem

  Suppose that convex sets A and $B \subset R^L$ are disjoint$(A \cap B = \emptyset)$.

  Then $\exists p^* \in R^L \backslash \{0\}$ and $\exists r \in R$ s.t. $p^* \cdot a \geq r$ for $\forall a \in A$ and $p^* \cdot b \leq r$ for $\forall b \in B$

  It means that $\exists$a hyperplane that separate A and B.

- $\boxed{\text{Proof}}$

1. $CE\left(E\right) \to \cap_{R=1}^{\infty} Core\left(E^R\right)$
   Suppose $\tilde{x} \in CE\left(E\right).$ Because $CE\left(E\right) \subseteq Core\left(E\right),$ we have $\tilde{x} \in Core\left(E\right)$ and using Equal Treatment Property, we can construct allocations $\tilde{x}^R = \left\{\left(\tilde{x}_{11,}..., \tilde{x}_{1R}\right), ..., \left(\tilde{x}_{I1,}..., \tilde{x}_{IR}\right)\right\} \in Core\left(E^R\right)$ s.t. $\tilde{x}_{i1} = \tilde{x}_{iR}$ for $\forall i$ and $\forall R > 2$.
2. $\cap_{R=1}^{\infty} Core\left(E^R\right) \to CE\left(E\right)$
   Consider $\tilde{x}^R = \left\{\left(\tilde{x}_{11,}..., \tilde{x}_{1R}\right), ..., \left(\tilde{x}_{I1,}..., \tilde{x}_{IR}\right)\right\} \in Core\left(E^R\right).$
   By Equal Treatment Property, we just have to focus on a feasible type allocation $\tilde{x} = \left\{\tilde{x}_1, \tilde{x}_2, ..., \tilde{x}_I\right\}$ from $\tilde{x}^R$.
   We need to show that $\exists \tilde{p}$ s.t. $(\tilde{p}, \tilde{x})$ is CE; that is,
   i) $\tilde{p} \cdot \tilde{x}_i = \tilde{p} \cdot e_i$ for $\forall i \in I$
   ii) $x_i \succ_i \tilde{x}_i$ implies $\tilde{p} \cdot x_i > \tilde{p} \cdot e_i$ for $\forall i \in I$
   (a) Find an equilibrium candidate price system $\tilde{p}$
      Define the set of net trade prefered to $\tilde{x}$,
      for $\forall i \in I, V_i = \left\{x_i \in X_i \mid x_i + e_i \succ_i \tilde{x}_i\right\}$ and
      $V = co\left\{\cup_{i \in I} V_i\right\}.$
      i. $V \cap \{0\} = \emptyset$
         Suppose not. $0 \in V$
         By the definition of Covex hull V, $\exists \alpha_i \in [0, 1]$ s.t. $\sum_{i \in I} \alpha_i = 1$ and $\sum_{i \in I} \alpha_i V_i = 0$.
         case 1) $\alpha_i \in Q$
         Then $\alpha_i = \frac{\beta_i}{n}$ s.t. $\beta_i, n\,(fixed) \in N$ for $\forall i \in I$.
         In order to obtain an expression of a form $\sum_{i \in I} \beta_i V_i = 0$, among $\{\alpha_1, \alpha_2, ..., \alpha_I\}$ find the least common mulitiplier of denominator of α_i.
         Let $R = \max\{\beta_i\}$ and replicate the original economy E to E^R.
         Then we can find a coalition $S \subset RI$ with β_i agents of type i (worst-off agents) blocks $\tilde{x}$ with $V_i + e_i$.

$$\sum_{i \in I} \beta_i \left(V_i + e_i\right) = \sum_{i \in I} \beta_i V_i + \sum_{i \in I} \beta_i e_i = \sum_{i \in I} \beta_i e_i$$

         Then for $\forall i \in I$, $x_i \succ_i \tilde{x}_i$ because $x \in V$ and x is feasible
         Therefore, $\tilde{x}$ does not belong to the core which is a contradiction.
         case 2) $\alpha_i \in Irrational\ number$
         we can find an rational approximation of α_i s.t. $\exists q_i^n \in Q$, $\lim_{n \to \infty} q_i^n = \alpha_i$
         Then $\sum_{i \in I} \alpha_i V_i = \sum_{i \in i} q_i^n \left(\frac{\alpha_i}{q_i^n}\right) V_i = 0$
         By continuity of preferences, for $\forall \varepsilon > 0$, $\left|\left|\left(\frac{\alpha_i}{q_i^n}\right) V_i - V_i\right|\right| < \varepsilon$
         Furthermore, we can use the same step as case 1.

ii. Convexity

For $\forall i \in I, V_i$ is convex because $\succsim_i$is strictly convex and transitive.

And then V is convex because it is a convex hull of union of convex set and $\{0\}$ is convex as a singleton set.

iii. Separating Hyperplane Theorem

By separating hyperplane theorem, $\exists \tilde{p} \in R^L, \tilde{p} \neq 0$ s.t.$\exists 0$ s.t $\tilde{p} \cdot x \geq 0$ for $\forall x \in V$

iv. $\tilde{p} \in R^L_+$

Let o_k be the k th unit vector s.t $k \in L$

By strict monotonicity of $\succsim_i, \tilde{x}_i - e_i + o_k \succ_i \tilde{x}_i - e_i$.

Then $\tilde{p} \cdot [\tilde{x}_i - e_i + o_k] \geq \tilde{p} \cdot [\tilde{x}_i - e_i] \to \tilde{p} \cdot o_k \geq 0 \to \tilde{p} \geq 0$ for $\forall k \in L$.

Therefore, $\tilde{p} \in R^L_+ \setminus \{0\}$

(b) Show $\tilde{x}$ with $\tilde{p}$ consists of CE

i. Show [For $\forall i, x_i \succ_i \tilde{x} \to \tilde{p} \cdot x_i \geq \tilde{p} \cdot e_i$]

Suppose $x_i \succ_i \tilde{x}$.

$x_i \succ_i \tilde{x} \to x_i - e_i + e_i \succ_i \tilde{x} \to x_i - e_i \in V_i$

$\tilde{p} \cdot [x_i - e_i] \geq 0 \to \tilde{p} \cdot x_i \geq \tilde{p} \cdot e_i$ for $\forall i \in I$.

ii. Show [For $\forall i, x_i \succ_i \tilde{x} \to \tilde{p} \cdot x_i > \tilde{p} \cdot e_i$]

Suppose not. $\exists x_i \in X_i$ s.t. $x_i \succ_i \tilde{x}$ and $\tilde{p} \cdot x_i = \tilde{p} \cdot e_i$

By convexity of X_i and continuity of $\succsim_i$, for $\lambda < 1$ close enough to 1, $\exists \lambda x_i \in X_i$ s.t. $\lambda x_i \succ_i \tilde{x}$.

$\lambda x_i \succ_i \tilde{x} \to \tilde{p} \cdot \lambda x_i \geq \tilde{p} \cdot e_i$ for $\forall i \in I$

On the other hand, $[x_i \succ_i \tilde{x}$ and $\tilde{p} \cdot x_i = \tilde{p} \cdot e_i]$

$\to$ [for $\lambda < 1, \lambda x_i \succ_i \tilde{x}$ and $\tilde{p} \cdot \lambda x_i < \tilde{p} \cdot e_i$ for $\forall i \in I$] which is a contradiction?!

- Key difference from the Second welfare theorem

  V does not need to be convex and therefore a nonzero $\tilde{p}$ supporting V at $\sum_{i \in I} e_i$ may not exist. The reason for the lack of convexity is that individual set V_i need not be convex. If the sets V_i being added are numerous, then the sum $\sum_i V_i$ is almost convex. Thus, the existence of almost supporting prices for core allocations can be seen as another instance of the convexifying effects of aggregation.

- Intuition

  Define $C_R \subset R^{LI}_+$ be a feasible type allocation for $R^{th}-$replica of the original economy with equal treatment property. Note that $C_{R+1} \subseteq C_R$ because a type allocation blocked in the R^{th}-replica will be blocked also in $(R+1)^{th}$ $-$replica by a coalition having exactly the same composition as the one that blocked in the $R^{th}-$replica. Thus as a subset of R^{LI}, the core can only get smaller when $R \to \infty$.

  At the same time, we know that the core cannot vanish because CE allocations belong to C_R for $\forall R$. The set of CE type allocations is independent of R and contained in all C_R.

Debrew-Scarf core convergence theorem asserts that CE allocations are the only surviving allocations in the core when $R \to \infty$.

- Claim in Mas-colell

  If the feasible type allocation $\tilde{x} = \{\tilde{x}_1, \tilde{x}_2, ..., \tilde{x}_I\}$ satisfies Equal Treatment Property for $\forall R$, that is, $\tilde{x} \in C_R$ for $\forall R$, then $\tilde{x}$ is CE allocation.

  Proof

  We need to show that $C_R \to CE(E) \subset Core(E)$; $\exists \tilde{x} \in C_R$ for $\forall R$ s.t. $\tilde{x} \in CE(E)$

  Let $\{\tilde{x}_R\}_{R=1}^{\infty}$ be a sequence with $\tilde{x}_R \in C_R$. We know that $Core(E)$ is nonempty and compact so that $Core\left(E^R\right)$ is also nonempty and compact, and so C_R is.

  Moreover C_R is a shrinking sequence so that $\{\tilde{x}_R\}_{R=1}^{\infty} \subset C_1$.

  Since C_1 is compact, $\exists$a convergent subsequence $\{\tilde{x}_{R_k}\}_{k=1}^{\infty}$ converging $\tilde{x} \in C_1$.

  We want to see whether $\tilde{x} \in CE(E)$.

  Let $k \geq 1$ be any natural number. Then $C_{R_j} \subseteq C_{R_k}$ for $j \geq k$.

  Then we have $x_{R_j} \in C_{R_k}$.

  Since C_{R_k} is compact, $x = \lim_{j \to \infty} x_{R_j} \in C_{R_k}$

  Therefore, $x \in \cap_{k=1}^{\infty} C_{R_k} = \cap_{R=1}^{\infty} C_R$

  From Theorem, $x \in CE(E)$

Microeconomics II

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1 Summary

Commodity Space $\left(R^L\right)$
$\downarrow$
Consumption Space(X_i)
$X_i = R^L_+ \subset R^L$ ←Physical constraints
and Production Set(Y_j) ←Technological constraints
$Y_j \subset R^L$ defined $Y_j = \left\{ y_j \in R^L \mid F\left(y_j\right) \leq 0 \right\}$
or $Y_j = \left\{ \left(-z_1, -z_2, ..., -z_{L-1}, q\right) \in R^L \mid q - f\left(z_1, z_2, ..., z_{L-1}\right) \leq 0 \right\}$ ←one output case
Firm maximization problem⇒ y_j^* maximizes Π_j
$\downarrow$ ←Economic Constraints
Budget Set$(B_i\left(p, e_i, y^*\right))$ = Set of feasible consumption bundle
$B_i\left(p, e_i, y^*\right) = \left\{ x_i \in X_i \mid p \cdot x_i \leq p \cdot e_i + p \cdot \sum_j \theta_{ij} \cdot y_j^* \right\}$
$\downarrow$
Consumer's maximization programming problem: ←we can summarize information
choosing $\succsim$ −maximal consumption bundles from $B_i\left(p, e_i, y^*\right)$ of preference using utility function
given p, $e_i \in R^L_+$ for $\forall i$, and y^*
$\downarrow$ ←Finite intersection property
Demand Correspondence$(x_i \in D_i\left(p, e_i, y^*\right)$ for $\forall i)$ ←Maximum theorem
Consumer's maximal utility value
Value function(indirect utiliity function)

2 Production

2.1 Production Set

2.1.1 Definition

identifying production vector y that is technologically possible
→Production Set Y
the set of all production vector y technologically possible for the firm

2.1.2 Properties

1. Nonempty
2. closed
3. No free lunch: $Y \cap R_+^L = \{0\}$
4. inactivity: $0 \in Y$
5. Free disposal: $Y - R_+^L \subset Y$
6. Irreversibility: $y \in Y \rightarrow -y \notin Y$
7. nonincreasing return: $y \in Y \rightarrow \alpha y \in Y$ for $\forall \alpha \in [0,1]$
   nondecreasing: $y \in Y \rightarrow \alpha y \in Y$ for $\forall \alpha > 1$
   constant return: $y \in Y \rightarrow \alpha y \in Y$ for $\forall \alpha \geq 0$
8. Convex: $y, y' \in Y \rightarrow \alpha y + (1-\alpha) y' \in Y$ for $\forall \alpha \in [0,1]$
9. Additivity: $y, y' \in Y \rightarrow y + y' \in Y$
10. Convex cone: $y, y' \in Y \rightarrow \alpha y + \beta y' \in Y$ for $\forall \alpha \geq 0$ and $\forall \beta \geq 0$
11. Y is convex cone$\Leftrightarrow$ Y is nonincreasing return to scale and additive
    - →Immediate
    - ←Let $k > \max\{\alpha, \beta\}$ and $\min\{\alpha, \beta\} \geq 0$.
      By nonincreasing return and $\alpha y = \frac{\alpha}{k} ky$ and $\beta y = \frac{\beta}{k} ky$, αy, $\beta y \in Y$
      By additivity, $\alpha y + \beta y \in Y$ for $\forall \alpha, \beta \geq 0$.

2.1.3 Description of Production Set

1. Transformation Function

   Y decribed by $F(y)$: transformation function
   $Y = \left\{ y \in R^L \mid F(y) \leq 0 \right\}$

   So far we need not distinguish output from input

   $MRT_{12}(\bar{y}) = \frac{\frac{\partial F(\bar{y})}{\partial y_1}}{\frac{\partial F(\bar{y})}{\partial y_2}} = -\frac{\partial y_2}{\partial y_1}$

   at profit maximizing production level, $MRT_{12}(\bar{y}) = \frac{\frac{\partial F(\bar{y})}{\partial y_1}}{\frac{\partial F(\bar{y})}{\partial y_2}} = -\frac{\partial y_2}{\partial y_1} = \frac{p_1}{p_2}$

2. Production Function

   In case that we can distinguish output from input and # of output is 1, we can describe Y by $f(z)$

   $Y = \left\{ (-z, q) \in R^L \mid q - f(z) \leq 0, z \geq 0 \right\}$

   $MRST_{12}(\bar{q}) = \frac{\frac{\partial f(\bar{q})}{\partial z_1}}{\frac{\partial f(\bar{q})}{\partial z_2}} = -\frac{\partial z_2}{\partial z_1} = \frac{p_1}{p_2}$

   $MP_1 = \frac{p_1}{p_q}$

   Therefore, $MRST_{12}$ is a renaming of MRT_{12} in the special case of a single output and multiple input.

   For single output case, properties of production set can be translated into properties of production function.

   (a) constant return: $y \in Y \rightarrow \alpha y \in Y$ for $\forall \alpha \geq 0 \Leftrightarrow f(\cdot)$ is HD of 1

   (b) Convex: $y', y'' \in Y \rightarrow \alpha y' + (1-\alpha) y'' \in Y$ for $\forall \alpha \in [0,1] \Leftrightarrow f(\cdot)$ is concave
   $f(\cdot)$ is concave$\Leftrightarrow f(\alpha z' + (1-\alpha) z'') \geq \alpha f(z') + (1-\alpha) f(z'')$
   $y' \in Y \Leftrightarrow q' \leq f(z')$
   $y'' \in Y \Leftrightarrow q'' \leq f(z'')$
   $\alpha y' + (1-\alpha) y'' \in Y \Leftrightarrow \alpha q' + (1-\alpha) q'' \leq f(\alpha z' + (1-\alpha) z'')$ for $\forall \alpha \in [0,1]$

2.2 Profit Maximization

2.2.1 Firm's Problem

$\max_y \ p \cdot y$
s.t. $y \in Y \ (F(y) \leq 0)$ $\quad MRT_{12}(\bar{y}) = \frac{\frac{\partial F(\bar{y})}{\partial y_1}}{\frac{\partial F(\bar{y})}{\partial y_2}} = -\frac{\partial y_2}{\partial y_1} = \frac{p_1}{p_2}$
when $\exists$a single output

$$\rightarrow \begin{array}{l} \max_y \, p_q \cdot f(z) - p_z \cdot z \\ \text{s.t. } q \leq f(z) \end{array} \quad MRST_{12}(\bar{q}) = \frac{\frac{\partial f(\bar{q})}{\partial z_1}}{\frac{\partial f(\bar{q})}{\partial z_2}} = -\frac{\partial z_2}{\partial z_1} = \frac{p_1}{p_2}$$

If Y is convex($f(\cdot)$ is HD of 1), then these conditions are necessary and sufficient conditions for profit maximizing.

2.2.2 Prifit function & Supply Correspondence

$\pi(p) = \max\{p \cdot y \mid F(y) \leq 0\}$
$y(p) = \{y \in Y \mid \pi(p) = p \cdot y\}$

properties(PSet #1)

1. $\pi(p)$ is HD of 1 in p
2. $\pi(p)$ is a convex function.
3. If Y is convex, then $Y = \left\{y \in R^L \mid p^* \cdot y \leq p^* \cdot y^* \text{ for } \forall p^* >> 0\right\}$
4. $y(\cdot)$ is HD of 0 in p
5. If Y is convex, then $y(p)$ is a convex set for all p
   Moreover, if Y is strictly convex, then $y(p)$ is single-valued
6. (Hotelling's Lemma)If $y(\bar{p})$ is single-valued, then $\pi(p)$ is differetiable at $\bar{p}$ and $D\pi(\bar{p}) = y(\bar{p})$
   The fact that $y(\bar{p})$ is single-valued is from that Y is strictly convex.
   Then $Y = \left\{y \in R^L \mid \bar{p} \cdot y \leq \bar{p} \cdot y(\bar{p}) \text{ for } \forall \bar{p} >> 0\right\}$
   Assume $y >> 0$.
   Define $g(p) = \pi(p) - p \cdot y(\bar{p})$. Then $g(p) > 0$ from $Y = \left\{y \in R^L \mid \bar{p} \cdot y \leq \bar{p} \cdot y(\bar{p}) \text{ for } \forall \bar{p} >> 0\right\}$
   And $g(\bar{p}) = 0$.
   Therefore, $g(p) > 0 = g(\bar{p})$ so that $g(p)$ takes its minimum at $\bar{p}$.
   $0 = \frac{\partial g(\bar{p})}{\partial p} = \frac{\partial \pi(\bar{p})}{\partial p} - y(\bar{p}) \rightarrow \frac{\partial \pi(\bar{p})}{\partial p} = y(\bar{p})$
7. If $y(\cdot)$ is a differetiable function at $\bar{p}$,
   then $Dy(\bar{p}) = D^2\pi(\bar{p})$ is a symmetric and positive semidefinite matrix with $Dy(\bar{p}) \cdot \bar{p} = 0$
   (a) (Young's Theorem)$Dy(\bar{p}) = D^2\pi(\bar{p})$ is a symmetric
      From Hotelling's lemma, we have $\frac{\partial \pi(\bar{p})}{\partial p} = y(\bar{p})$.
      Then $\frac{\partial^2 \pi(\bar{p})}{\partial p_1 \cdot \partial p_2} = \frac{\partial y_2}{\partial p_1} = \frac{\partial y_1}{\partial p_2} = \frac{\partial^2 \pi(\bar{p})}{\partial p_2 \cdot \partial p_1}$
      Therefore, $D^2\pi(\bar{p})$ is symmetric.

EX) $y_1(p_q, p_1, p_2) = p_q \cdot \frac{p_2}{p_1^2}$ and $y_2(p_q, p_1, p_2) = p_q \cdot \frac{p_1}{p_2^2}$ can not be factor demand functions of a profit maximization firm generally.
because $\frac{\partial y_1}{\partial p_2} = p_q \cdot \frac{1}{p_1^2} \neq p_q \cdot \frac{1}{p_2^2} = \frac{\partial y_2}{\partial p_1}$ unless $p_1 = p_2$.

(b) $Dy(\bar{p})$ is positive semidefinite

Because $\pi(p)$ is a convex function in p, the Hessian matrix $H = Dy(\bar{p}) = D^2\pi(\bar{p})$ is positive semidefinite.

Therefore, the first order principal minor $\frac{\partial^2 \pi(\bar{p})}{\partial p_1^2} = \frac{\partial y_1}{\partial p_1}$ has to be nonnegative.

(c) it implies the law of supply

(d) When $y(\cdot)$ is $\boxed{\text{not}}$ a differetiable function at $\bar{p}$, we can use revealed preference argument to show the law of supply.

$y_t \in y(p_t)$ and $y_s \in y(p_s)$

Then $p_t \cdot y_t \geq p_t \cdot y_s$ and $p_s \cdot y_s \geq p_s \cdot y_t$

$\triangle p \cdot \triangle y = (p_t - p_s) \cdot (y_t - y_s) = p_t \cdot y_t - p_t \cdot y_s + p_s \cdot y_s - p_s \cdot y_t \geq 0$

(e) $Dy(\bar{p}) \cdot \bar{p} = 0$

By Euler's theorem and the fact that $\pi(p)$ is HD of 1 in p, $D^2\pi(\bar{p}) \cdot \bar{p} = 0$

2.3 Aggregation

The absence of a budget constraint implies that individual supply is not subject to wealth effects. As prices change, there are only substitution effects along the production frontier so that the aggregation is simple.

2.4 Efficient Production

1. A production vector $y \in Y$ is efficient if there is no $y' \in Y$ such that $y' \geq y, y' \neq y$

   There is no other feasible production vector that generates as much output as y using as little as input.

2. If $y \in Y$ is profit maximizing for some $p >> 0$, then y is efficient.

   It is valid even if Y is not convex.

3. If Y is convex, then every efficient $y \in Y$ is profit maximizing for some $p \geq 0$

3 Competitive Equilibrium

3.1 Definition

3.1.1 Static Pure Exchange Economy

Remark 1. *An economy is a pure exchange economy if its only technological possibility is that of free disposal; that is, if for* $\forall j \in J$, $Y_j = -R^L_+$

3.1.2 Production Economy

1. Environment

   Commodity Space=R^L

   $X_i = R^L_+$ for $\forall i \in I$

   $\succsim_i$is a complete continuous preorder defined on $X_i = R^L_+$ for $\forall i \in I$

   $e_i \in R^L_+$

   $Y_j \subset R^L$ for $\forall j \in J$

   $\theta_{ij} \in [0,1]$ for $\forall i, j$ s.t. $\sum_{i\in I} \theta_{ij} = 1$ for $\forall j$

2. Feasible allocation

   $FAlloc = \left\{ \left(\{x_i\}_{i\in I}, \{y_j\}_{j\in J} \right) \in \Pi_{i\in I} X_i \cdot \Pi_{j\in J} Y_j \mid \sum_{i\in I} x_i \le \sum_{i\in I} e_i + \sum_{j\in J} y_j \right\}$

3. Competitive Equilibrium

   (a) In the production economy E_p, a Competitive Equilibrium consists of $\left\{ \{x_i^*\}_{i\in I}, \left\{ y_j^* \right\}_{j\in J} \right\}$ and $p^* \in R^L_+ \backslash \{0\}$ s.t.

      i. for $\forall i \in I, x_i^*$ is $\succsim_i -$maximal in $B_i(p^*, e_i) = \left\{ x_i \in X_i \mid p^* \cdot x_i \le p^* \cdot e_i + \sum_{j\in J} \theta_{ij} \cdot p^* \cdot y_j^* \right\}$

      $\Leftrightarrow x_i^* \in \arg\max_{x_i \in B_i(p^*, e_i)} u_i(x_i)$ s.t. $B_i(p^*, e_i) = \Big\{ x_i \in X_i \mid p^* \cdot x_i \le p^* \cdot e_i + \sum_{j\in J} \theta_{ij} \cdot p^* \cdot$

      ii. for $\forall j \in J$, $y_j^* \in Y_j$, $p^* \cdot y_j \le p^* \cdot y_j^*$ for $\forall y_j \in Y_j$

      iii. $\sum_{i\in I} x_i^* \le \sum_{i\in I} e_i + \sum_{j\in J} y_j^*$

   (b) In the production economy E_p, a price equilibrium with transfers consists of $\left\{ \{x_i^*\}_{i\in I}, \left\{ y_j^* \right\}_{j\in J} \right\}$ and $p^* \in R^L_+ \backslash \{0\}$ s.t.

      i. $\exists (w_1, w_2, ..., w_I)$ s.t. $\sum_{i\in I} w_i = \sum_{i\in I} p^* \cdot e_i + \sum_{j\in J} p^* \cdot y_j^*$

      ii. for $\forall i \in I$, x_i^* is $\succsim_i -$ maximal in $B_i(p^*, e_i) = \{x_i \in X_i \mid p^* \cdot x_i \le w_i\}$

      iii. for $\forall j \in J$, $y_j^* \in Y_j$, $p^* \cdot y_j \le p^* \cdot y_j^*$ for $\forall y_j \in Y_j$

iv. $\sum_{i\in I} x_i^* \leq \sum_{i\in I} e_i + \sum_{j\in J} y_j^*$

(c) In the production economy E_p, a price quasi-equilibrium with transfers consists of $\left\{\{x_i^*\}_{i\in I}, \left\{y_j^*\right\}_{j\in J}\right\}$ and $p^* \in R_+^L \backslash \{0\}$ s.t.

  i. $\exists(w_1, w_2, ..., w_I)$ s.t. $\sum_{i\in I} w_i = \sum_{i\in I} p^* \cdot e_i + \sum_{j\in J} p^* \cdot y_j^*$

  ii. for $\forall i \in I$, $x_i^* \in X_i = R_+^L$ s.t. $p^* \cdot x_i^* \leq w_i$,
  and $x_i \in X_i = R_+^L$ and $x_i \succ_i x_i^*$ s.t. $p^* \cdot x_i \geq w_i$ for $\forall x_i \in X_i, x_i \neq x_i^*$

  iii. for $\forall j \in J$, $y_j^* \in Y_j$, $p^* \cdot y_j \leq p^* \cdot y_j^*$ for $\forall y_j \in Y_j$

  iv. $\sum_{i\in I} x_i^* \leq \sum_{i\in I} e_i + \sum_{j\in J} y_j^*$

(d) Relationship among concepts

  i. In order to relate the idea of pareto optimality to supportability by means of price taking behavior, it is useful to introduce a notion of equilibrium that allows for a more general determination of consumer's wealth levels than that in a private ownership economy. By way of motivation, we can imagine a situation where a social planner is able to carry out (lump-sum) redistributions of wealth, and where society's aggregate wealth can therefore be redistributed among agents in any desired manner.

4 Pareto Optimality

4.1 Production Economy

1. Strict Pareto Optimal

   A feasible allocation $\{\hat{x}, \hat{y}\} \in X \times Y = R_+^{L(I+J)}$ is strictly pareto optimal if there is no other feasible allocation $\{\bar{x}, \bar{y}\} \in X \times Y = R_+^{L(I+J)}$ s.t.
   with $\sum_{i\in I} \hat{x}_i \leq \sum_{i\in I} e_i + \sum_{j\in J} \hat{y}_j$,
   for $\forall i \in I$, $\bar{x}_i \succsim_i \hat{x}_i$ and for at least one $i' \in I$, $\bar{x}_{i'} \succ_{i'} \hat{x}_{i'}$

2. Weakly Pareto Optimal

   A feasible allocation $\{\hat{x}, \hat{y}\} \in X \times Y = R_+^{L(I+J)}$ is weakly pareto optimal if there is no other feasible allocation $\{\bar{x}, \bar{y}\} \in X \times Y = R_+^{L(I+J)}$ s.t.
   with $\sum_{i\in I} \hat{x}_i \leq \sum_{i\in I} e_i + \sum_{j\in J} \hat{y}_j$,for $\forall i \in I$, $\bar{x}_i \succ_i \hat{x}_i$

4.2 Quasi-linear utility function

I consumers
1 firm

$X_i = R \cdot R_+^{L-1}$
$u_i(x_i) = x_1^i + v_i(x_{-1}^i)$
v_i is strictly concave and strictly monotone.
Y is described by a strictly convex and strictly monotone transformatioin function $F(y)$

1. Show that every pareto optimal allocation has the same allocation of all goods other than good 1.
2. Assuming at least one consumer with differetiable utility function v_i has an interior consumption of goods except good 1 in pareto optimal allocation of those goods. Show that the allocation of goods except good 1 and prices of all goods in an equilibrium are independent of distribution of endowments and ownership shares.

4.3 Existence of C.E.

4.3.1 Production Economy

- The applicability of the existence proof is not limited to exchange economies. If we allow for production sets that are closed, strictly convex, and bounded above(and if a positive aggregate consumption buncle is producible from the initial aggregate endowments), then the production inclusive convex-valued, upper-hemi continuous excess demand correspondence $Z(\cdot)$(or a continuous function) satisfies the properties of excess demand correspondence. Hence, $\exists$a C.E.

1. Easy Market equilibrium lemma
   (a) If a function $Z : R^L_{++}(\triangle) \rightarrow R^L$, $Z(p) == \left\{Z^1(\cdot), Z^2(\cdot), ..., Z^L(\cdot)\right\}$, satisfying
      i. continuity,
      ii. HD of zero in $p \in \triangle$,
      iii. Walras' law in $p \in \triangle$,
      iv. Bounded from below($\exists s > 0$ s.t. $Z^l > -s$ for all $l \in L$ and all p)
      v. Boundary condition(If $\boxed{e_i \in R^L_{++}}$, $\underline{\succsim_i \text{is strictly monotone}}$)
      $[p_n \rightarrow p \in \partial\triangle] \rightarrow \left[\left\|D_i\left(p, e_i + \sum_{j\in J}\theta_{ij}\cdot y_j\right)\right\| \text{ is unbounded} \Leftrightarrow \sup_{z^l\in Z_i(p_n)}|Z_i(p_n)| = \infty\right]$
   then $\exists p^* \in R^L_{++}(\triangle)$ s.t. $Z(p^*) = 0$
2. Extended Market equilibrium lemma
   <u>Lemma</u>
   - If a correspondence $Z : R^L_{++}(\triangle) \rightarrow R^L$, $Z(p) = \left\{Z^1(\cdot), Z^2(\cdot), ..., Z^L(\cdot)\right\}$ s.t..$Z^l(p) = \sum_{i\in I} x^l_i(p) - \sum_{i\in I} e^l_i - \sum_{j\in J} y^l_j$ satisfying

   (a) i. u.h.c. and convex-valued,
      ii. HD of zero in p,
      iii. Walras' law,
      iv. Bounded from below($\exists s > 0$ s.t. $D^l_i\left(p, e_i + \sum_{j\in J}\theta_{ij}\cdot y_j\right) > -s + e^l_i$ for all $l \in L$ and all p)
      v. Boundary condition(If $\boxed{e_i \in R^L_{++}}$, $\underline{\succsim_i \text{is strictly monotone}}$)
      $[p_n \rightarrow p \in \partial\triangle] \rightarrow \left[\left\|D_i\left(p, e_i + \sum_{j\in J}\theta_{ij}\cdot y_j\right)\right\| \text{ is unbounded} \Leftrightarrow \sup_{z^l\in Z_i(p_n)}|Z_i(p_n)| = \infty\right]$
   then $\exists p^* \in R^L_{++}(\triangle)$ s.t. $0 \in Z(p^*)$

3. Existence Theorem with Production

   $\boxed{\text{Theorem}}$

   In $E = \left[\left\{I, (\succsim_i, e_i)_{i\in I}\right\}, \left\{J, (Y_j)_{j\in J}\right\}\right]$ s.t.

   $X_i = R^L_+$ for $\forall i$ and $X = \{X_i\}_{i\in I}$

$$\succsim_i \left\{\begin{array}{ccc} \text{strict monotone(l.n.s)} & \rightarrow & \text{boundary condition(walras' law)} \\ \text{(weakly)convex} & \rightarrow & \text{convex-valued correspondence} \\ \text{continuous(+monotone)} & \rightarrow & \text{continuous utiity function} \rightarrow \text{existence of } D_i\left(p, e_i\right) \end{array}\right.$$

$$e_i \in \left\{\begin{array}{ccl} R^L_+ & \rightarrow & \text{for all the other conditions} \\ R^L_{++} & \rightarrow & \text{Boundary condition} \end{array}\right. \rightarrow R^L_{++} \Rightarrow \textstyle\sum_{i\in I} e_i >> 0$$

   $Y_j = R^L$ for $\forall j$ and $Y = \{Y_j\}_{j\in J}$ satisfying

$$\begin{array}{l} 0 \in Y_j \\ Y_j \text{ is closed and convex} \\ Y \cap R^L_+ = \emptyset \\ Y \cap -Y = \emptyset \end{array},$$

   the Market Equilibrium Lemma is applicable and

   $\exists$an equilibrium with production.

   $\boxed{\text{Proof}}$

   (a) The set of feasible allocations is compact
   Define $A = \left\{(x, y) \in \Pi^I_{i=1} X_i \times \Pi^J_{j=1} Y_j \mid \text{for } \forall i, j, x_i \in X_i \text{ and } y_j \in Y_j, \sum_{i\in I} x^*_i \leq \sum_{i\in I} e_i + \sum_{j\in}\right.$

   (b) Truncate the economy
   Let $\hat{Y_j}$ be the projection of A on producer j's coordinate; the compact set of feasible production plans for j. Choose a compact convex set $K \subset R^L$ s.t. $\hat{Y_j} \subset intK$ for $\forall j$
   Define $Y^K_j = Y_j \cap K$ which is closed, convex, and $0 \in Y^K_j$.

   (c) Existence of equilibrium in Y^K_j

      i. Define $Z_K(p) = \left\{Z^1_K(\cdot), Z^2_K(\cdot), ..., Z^L_K(\cdot)\right\}$ s.t..$Z^l_K(p) = \sum_{i\in I} x^l_{iK}(p) - \sum_{i\in I} e^l_i - \sum_{j\in J} y^l_{jK}$ for $\forall l$

      ii. Check assumptions for Market Equilibrium Lemma

         - $\sum_{i\in I} x^l_{iK}(p) \in D\left(p^*, e + \sum_{j\in J} y^*_j\right)$
           is nonempty-valued, convex-valued, u.h.c.,
           satisfies HD of 0 in p,
           satisfies Walras' law,
           is bounded from below,
           satisfies boundary condition

        - $\sum_{j\in J} y^{l}_{jK} \in Y^{K}$
          is nonempty-valued, convex-valued, compacted-valued

    iii. Therefore, $Z_K(p)$ satisfies all assumptions for Extended Market Equilibrium Lemma
    then $\exists p^* \in R^L_{++}(\triangle)$ s.t. $0 \in Z(p^*)$ and
    for $\forall i \in I, x^*_i \in D_i\left(p^*, e_i + \sum_{j\in J}\theta_{ij}\cdot y^*_j\right)$
    for $\forall j \in J, y^*_j \in Y^K_j$

(d) Equilibrium for truncated economy Y^K is also al equilibrium for the original economy Y
We need to show $y^* \in Y^K$ is also profit-maximizing with p^* in Y.
$\rightarrow$for $\forall j \in J, y^*_j \in Y_j$ s.t. $p^*\cdot y_j \leq p^*\cdot y^*_j$ for $\forall y_j \in Y_j$
Suppose not. Then $\exists y'_j \in Y_j$ s.t. $p^*\cdot y'_j \geq p^*\cdot y^*_j$
Consider $\lambda y^*_j + (1-\lambda)\,y'_j = y''_j$ for $\lambda \in [0,1]$
For $\lambda < 1$, $p^*\cdot y''_j > p^*\cdot y^*_j$ and λ close enough to 1, $y''_j \in K$ and $y''_j \in Y_j$ so that $y''_j \in Y_j \cap K = Y^K_j$
It implies $y^*_j \notin Y^K_j$ which is a contradiction.

4.4 Solving for Equilibrium with Production(PSet #2)

4.4.1 General Case

1. Environment

   Commodity Space=R^L

   I consumers

   $X_i = R^L_+$ for $\forall i \in I$

   $\succsim_i$is a complete continuous preorder defined on $X_i = R^L_+$ for $\forall i \in I$

   $\rightarrow$represented by $u_i(x_i)$

   $e_i \in R^L_+$

   one firm

   $Y \subset R^L$

   $\rightarrow$represented by $q \leq f(z)$ or $F(y) \leq 0$

   $\theta_{ij} \in [0,1]$ for $\forall i, j$ s.t. $\sum_{i \in I} \theta_{ij} = 1$ for $\forall j$

2. Problem

   $$\begin{array}{l} \max \ u_i(x_i) \\ s.t. \ p \cdot x_i \leq p \cdot e_i + \theta_i \cdot p \cdot y \end{array}$$

   $$\begin{array}{l} \max p \cdot y \\ s.t. \ F(y) \leq 0 \end{array} \text{ or } \begin{array}{l} \max p_q \cdot q - p_z \cdot z \\ s.t. \ q \leq f(z) \end{array}$$

   $$\sum_{i \in I} x^*_i \leq \sum_{i \in I} e_i + y$$

3. Checking points

   (a) If $u_i(x_i)$ is strongly monotone at least for one consumer, then $p^* >> 0$

   (b) If $u_i(x_i)$ is Cobb-Douglas or Logarithm function, then $x_i >> 0$ so that there is no corner solution.

   (c) If $F(y)$ is linear and one input and one output, divide p be 3 cases.$p_1 \lesseqqgtr p_2$

   (d) If $f(z)$ is constant return, then $\pi(p) = 0$

4. Solving$\left(\frac{\pi(p^*)}{p^*} \} \rightarrow x^* \rightarrow y^* \right)$

   (a) Putting $\pi(p) = p \cdot y$ into consumer's problem.

   (b) Solving x^*

   (c) Check feasibility in order to get y^*

4.4.2 Example(PSet #2)

1. $L = 3$

   $I = 2$

   $u_1(x_1, x_2, x_3) = 0.4 \ln x_2 + 0.6 \ln x_3$

   $u_2(x_1, x_2, x_3) = \ln x_2 + \ln x_3$

   $e_1 = e_2 = (6, 0, 0)$

   $J = 1$

   $Y = \{(y_1, y_2, y_3) \in R^3 \mid y_1 \leq 0, y_2 \geq 0, y_3 \geq 0, y_2 + 2y_3 \leq -4y_1\}$

   $\theta_1 = 1, \theta_2 = 0$

   - Find an equilibrium of this economy

     $\boxed{\text{Check}}$

     $x_2^1, x_3^1, x_2^2, x_3^2 > 0$ because of property of logarithm function→interior solution!!

     $\rightarrow y_2 > 0, y_3 > 0$

     Because consumers can not ger any utility from x_1, $y_1 = -12$.

     $p_2, p_3 > 0$ because utility function is strictly concave and strictly monotone.

   (a) $\pi(p^*)$ or p^*

   $\max p_1 \cdot y_1 + p_2 \cdot y_2 + p_3 \cdot y_3$

   s.t. $y_2 + 2y_3 \leq -4y_1$

   $y_2 > 0, y_3 > 0$

   Lagragian

   FOC

   $p_1 + 4\lambda = 0$

   $p_2 + \lambda = 0$

   $p_3 + 2\lambda = 0$

   $p_1 = 4, p_2 = 1, p_3 = 2$

   $y_1 = -12$

   $\pi(p) = -48 + y_2 + 2y_3 \leq -48 + 48 = 0$

   Therefore, $\pi(p^*) = 0$

   (b) $\boxed{\text{Consumer 1}}$

   $\max_{x^1} 0.4 \ln x_2^1 + 0.6 \ln x_3^1$

   s.t. $x_2^1 + 2x_3^1 \leq 24 + 1 \cdot 0$

   FOC

$MRS_{23}\left(x^1\right)=\frac{\frac{0.4}{x_2^1}}{\frac{0.6}{x_3^1}}=\frac{p_2}{p_3}=\frac{1}{2}$

$\{\frac{3x_2^1-4x_3^1=0}{x_2^1+2x_3^1=24}$ by stricy incresing utility function

$x_3^1=7,2$

$x_2^1=9.6$

$\boxed{\text{Consumer } 2}$

$\max_{x^2}\ln x_2^2+\ln x_3^2$

s.t. $x_2^2+2x_3^2\leq 24$

FOC

$MRS_{23}\left(x^2\right)=\frac{\frac{1}{x_2^2}}{\frac{1}{x_3^2}}=\frac{p_2}{p_3}=\frac{1}{2}$

$\{\frac{x_2^2-2x_3^2=0}{x_2^2+2x_3^2=24}$ by stricy incresing utility function

$x_2^2=12$

$x_3^2=6$

(c) Feasibility

$x_1^1+x_1^2=12+y_1$

$x_2^1+x_2^2=0+y_2$

$x_3^1+x_3^2=0+y_3$

(d) $y_1=-12, y_2=21.6, y_3=13.2$

$x_1^1=0, x_2^1=9.6, x_3^1=7,2$

$x_1^2=0, x_2^2=12, x_3^2=6$

- $p_1=4, p_2=1, p_3=2$
- What if the firm owned by consumer 2?
- What if there are two firms with
  $Y_1=\left\{(y_1,y_2,y_3)\in R^3\mid y_1\leq 0, y_2\geq 0, y_3=0, y_2\leq -4y_1\right\}$
  $Y_2=\left\{(y_1,y_2,y_3)\in R^3\mid y_1\leq 0, y_2=0, y_3\geq 0, 2y_3\leq -4y_1\right\}$?

(a) $\pi_1\left(p^*\right),\pi_2\left(p^*\right)$ or p^*

$\boxed{\text{Firm } 1}$

$\max p_1\cdot y_1+p_2\cdot y_2$

s.t. $y_2\leq -4y_1$

$y_2>0$

Lagragian

FOC
$p_1 + 4\lambda = 0$
$p_2 + \lambda = 0$
$p_1 = 4, p_2 = 1$
$\pi_1 (p) = 4y_1 + y_2 \leq 0$
Therefore, $\pi_1 (p^*) = 0$
$\boxed{\text{Firm 2}}$
$\max p_1 \cdot y_1 + p_3 \cdot y_3$
s.t. $2y_3 \leq -4y_1$
$y_3 > 0$
Lagragian
FOC
$p_1 + 4\lambda = 0$
$p_3 + 2\lambda = 0$
$p_1 = 4, p_3 = 2$
$\pi_2 (p) = 4y_1 + 2y_3 \leq 0$
Therefore, $\pi_2 (p^*) = 0$

(b) $\boxed{\text{Consumer 1}}$
$\max_{x^1} 0.4 \ln x_2^1 + 0.6 \ln x_3^1$
s.t. $x_2^1 + 2x_3^1 \leq 24 + 1 \cdot 0 + 1 \cdot 0$
FOC
$MRS_{23} \left(x^1\right) = \frac{\frac{0.4}{x_2^1}}{\frac{0.6}{x_3^1}} = \frac{p_2}{p_3} = \frac{1}{2}$
$\{\frac{3x_2^1 - 4x_3^1 = 0}{x_2^1 + 2x_3^1 = 24}$ by stricy incresing utility function
$x_3^1 = 7,2$
$x_2^1 = 9.6$
$\boxed{\text{Consumer 2}}$
$\max_{x^2} \ln x_2^2 + \ln x_3^2$
s.t. $x_2^2 + 2x_3^2 \leq 24$
FOC
$MRS_{23} \left(x^2\right) = \frac{\frac{1}{x_2^2}}{\frac{1}{x_3^2}} = \frac{p_2}{p_3} = \frac{1}{2}$
$\{\frac{x_2^2 - 2x_3^2 = 0}{x_2^2 + 2x_3^2 = 24}$ by stricy incresing utility function
$x_2^2 = 12$
$x_3^2 = 6$

(c) Feasibility
$x_1^1 + x_1^2 = 12 + y_1^1 + y_1^2$
$x_2^1 + x_2^2 = 0 + y_2^1$
$x_3^1 + x_3^2 = 0 + y_3^2$

(d) $y_2 = 21.6, y_3 = 13.2$
$y_2 = -4y_1^1 \rightarrow y_1^1 = -5.4$
$2y_3 = -4y_1^2 \rightarrow y_1^2 = -6.6$
$x_1^1 = 0, x_2^1 = 9.6, x_3^1 = 7,2$
$x_1^2 = 0, x_2^2 = 12, x_3^2 = 6$

- $p_1 = 4, p_2 = 1, p_3 = 2$

2. $L = 2$
$I = 2$
$X_1 = R_+^2$
$u_1(x_1, x_2) = x_1 + x_2$
$e_1 = (6, 2)$
$X_2 = R_+^2$
$u_2(x_1, x_2) = \min\{x_1, x_2\}$
$e_2 = (6, 4)$
$J = 1$
$Y = \left\{(y_1, y_2) \in R^2 \mid y_1 \leq 0, y_2 \leq 4\sqrt{-y_1}\right\}$
$\theta_1 = 0, \theta_2 = 1$

- Find an equilibrium of this economy
$\boxed{\text{Check}}$

(a) $\pi(p^*)$ or p^*
$\max p_1 \cdot y_1 + p_2 \cdot y_2$
s.t. $y_2 \leq 4\sqrt{-y_1}$

$$\begin{aligned}
\pi(p) &= p_1 \cdot y_1 + p_2 \cdot y_2 \\
&\leq p_1 \cdot y_1 + p_2 \cdot 4\sqrt{-y_1} \\
&= p_1 \cdot \left(-\left(\sqrt{(-y_1)}\right)^2 + \frac{p_2}{p_1} \cdot 4\sqrt{-y_1} - \left(\frac{p_2}{p_1} \cdot 2\right)^2 \right) + \frac{4P_2^2}{P_1} \\
&= -p_1 \cdot \left(\sqrt{(-y_1)} - \frac{2p_2}{p_1}\right)^2 + \frac{4P_2^2}{P_1}
\end{aligned}$$

When $\sqrt{(-y_1)} = \frac{2p_2}{p_1} \rightarrow y_1 = -\frac{4p_2^2}{p_1^2}$

$\pi(p)$ is maximized $\frac{4P_2^2}{P_1}$

Therefore, $\pi(p^*) = \frac{4P_2^2}{P_1}$

(b) $\boxed{\text{Consumer 1}}$

$\max_{x^1} x_1^1 + x_2^1$

s.t. $p_1 x_1^1 + p_2 x_2^1 \leq 6p_1 + 2p_2$

$x_1^1 + x_2^1$

$\leq x_1^1 + \frac{1}{p_2}\left(6p_1 + 2p_2 - p_1 x_1^1\right)$

$\leq (1 - \frac{p_1}{p_2})x_1^1 + \frac{1}{p_2}\left(6p_1 + 2p_2\right)$

FOC

case 1) $x_1^1, x_2^1 > 0$

$p_1 = p_2$

$x_1^1 + x_2^1 = 8$

case 2) $x_1^1 = 0, x_2^1 > 0$

$p_1 < p_2$

case 3) $x_1^1 > 0, x_2^1 = 0$

$p_1 > p_2$

$\boxed{\text{Consumer 2}}$

$\max_{x^2} \min\left\{x_1^2, x_2^2\right\}$

s.t. $p_1 x_1^2 + p_2 x_2^2 \leq 6p_1 + 4p_2 + \frac{4P_2^2}{P_1}$

By the property of utility function, $x_1^2 = x_2^2$

$x_1^2 = x_2^2 = \frac{6p_1 + 4p_2 + \frac{4P_2^2}{P_1}}{p_1 + p_2}$

case 1) $x_1^1, x_2^1 > 0$

$p_1 = p_2$

case 2) $x_1^1 = 0, x_2^1 > 0$

case 3) $x_1^1 > 0, x_2^1 = 0$

(c) Feasibility($p^* \rightarrow y^*$)

$x_1^1 + x_1^2 = 12 + y_1$

$x_2^1 + x_2^2 = 6 + y_2$

(d) $y_1 = -\frac{4p_2^2}{p_1^2} = -4, y_2 = 8$

$x_1^1 = 1, x_2^1 = 7$

$x_1^2 = x_2^2 = 7$

$p_1 = p_2 = 1$

- $x_1^1 = 0, x_2^1 = 15$
  $x_1^2 = x_2^2 = 3$
  $y_1 = -\frac{4p_2^2}{p_1^2} = -9, y_2 = 2$
  Is the pareto optimal?
  No.
  Production makes it possible for consumers to consume $(8, 14)$
  The allocation $(4.5, 10.5), (3.5, 4.5)$ is feasible and makes consumer 2 better off without hurting consumer 1's utility.

5 Welfare Theorem

5.1 Production Economy

5.1.1 First Welfare Theorem

1. First welfare theorem

   If preferences are l.n.s., and if (x^*, y^*, p^*) is a price equilibrium with transfers, then the allocation (x^*, y^*) is pareto optimal.

   $\boxed{\text{Proof}}$

   Suppose that (x^*, y^*, p^*) is a price equilibrium with transfers and the associated wealthes $(w_1, w_2, ..., w_I)$ s.t. $\sum_{i \in I} w_i = \sum_{i \in I} p^* \cdot e_i + \sum_{j \in J} p^* \cdot y_i^*$, but it is not pareto optimal; i.e. $\exists$a feasible allocation $(x, y) \in X \times Y$ s.t. for $\forall i \in I$, $x_i \succsim_i x_i^*$ and for at least one i', $x_{i'} \succ_{i'} x_{i'}^*$.

   Individual agents' preference maximization of the definition of price equilibrium with transfer implies that if $x_i \succ_i x_i^*$, then $p^* \cdot x_i > p^* \cdot x_i^* = w_i$ for $\forall i \in I$. With this fact and l.n.s. of preference implies that if $x_i \succsim_i x_i^*$, then $p^* \cdot x_i \geq p^* \cdot x_i^* = w_i$ for $\forall i \in I$.

   $x_i \succsim_i x_i^* \rightarrow p^* \cdot x_i \geq w_i$ for $\forall i \in I$
   Then $x_{i'} \succ_{i'} x_{i'}^* \rightarrow p^* \cdot x_{i'} > w_i$ for at least one i'
   $\rightarrow \quad \sum_{i \in I} p^* \cdot x_i > \sum_{i \in I} w_i = \sum_{i \in I} p^* \cdot e_i + \sum_{j \in J} p^* \cdot y_i^*$

   And from firms' profit maximization of the definition of equilibrium implies that $p^* \cdot y_j \leq p^* \cdot y_j^*$ for $\forall j \in J$.

   Therefore, $\sum_{i \in I} p^* \cdot x_i > \sum_{i \in I} w_i = \sum_{i \in I} p^* \cdot e_i + \sum_{j \in J} p^* \cdot y_i^* \geq \sum_{i \in I} p^* \cdot e_i + \sum_{j \in J} p^* \cdot y_j$
   $\Leftrightarrow \sum_{i \in I} p^* \cdot x_i > \sum_{i \in I} p^* \cdot e_i + \sum_{j \in J} p^* \cdot y_j$

   It is a contradiction because from the definition of feasible allocation,

   we have $\sum_{i \in I} x_i \leq \sum_{i \in I} e_i + \sum_{j \in J} y_j \rightarrow \sum_{i \in I} p^* \cdot x_i \leq \sum_{i \in I} p^* \cdot e_i + \sum_{j \in J} p^* \cdot y_j$.

2. Interpretation

   (a) Although the result appear to follow from very simple hypothetheses, note that we are already assuming universal price quoting of commodities(complete market) and price taking by agents. Under a certain circumstances like externality, market power, and asymmetric information, the first welfare theorem fail to be satisfied.

   (b) The first welfare theorem is entirely silent about the desirability of the equilibrium allocation from a distributional standpoint.

5.1.2 Second Welfare Theorem

There are two ways to prove this theorem.

A price quasiequilibrium with transfers

1. $\boxed{\text{Theorem}}$

   In $E = \left[\left\{I, (\succsim_i, e_i)_{i \in I}\right\}, \left\{J, (Y_j)_{j \in J}\right\}\right]$ s.t.

   $X_i = R^L_+$, $0 \in X_i$, and $\boxed{\text{convex}}$ for $\forall i$

   For $\forall i \in I$, $\succsim_i$is a preorder s.t.$\begin{cases} \text{l.n.s.} \\ \text{(weakly)convex:} \{x_i \in X_i \mid x_i \succsim_i x_i'\} \text{ is convex for } \forall x_i' \\ \text{continuous} \end{cases}$

   $(w_1, w_2, ..., w_I) \in R^L_{++}$

   $Y_j = R^L$ for $\forall j$ and $Y = \{Y_j\}_{j \in J}$ satisfying

   For $\forall j \in J$, Y_j is $\boxed{\text{convex}}$,

   for every pareto optimal allocation (x^*, y^*), $\exists p^* \in R^L$, $p^* \neq 0$ s.t. (x^*, y^*, p^*) is a price equilibrium with transfers.

   $\boxed{\text{Proof}}$→application of the separating hyperplane theorem for convex sets

   (a) Find an equilibrium candidate price system p^*

      i. Define, for $\forall i \in I, V_i = \{x_i \in X_i \mid x_i \succ_i x_i^*\}$
      and $V = \sum_{i \in I} V_i = \left\{\sum_{i \in I} x_i \in X = R^L \mid x_1 \in V_1, ..., x_I \in V_I\right\}$.
      For $\forall j \in J, Y = \sum_{j \in J} Y_j = \left\{\sum_{j \in J} y_i \in Y \mid y_1 \in Y_1, ..., y_J \in Y_J\right\}$ and $Y + \{e\}$ is the aggregate production set shifted to $\{e\}$

      ii. Convexity

         - For $\forall i \in I, V_i$ is convex because $\succsim_i$is weakly convex and transitive.
           And then V is convex because the sum of convex sets is convex.
           For $\forall j \in J, Y + \{e\}$ is convex because the sum of convex sets is convex

      iii. $V \cap Y + \{e\} = \emptyset$ ←Pareto optimality of (x^*, y^*)
      Suppose not. $\exists (x, y) \in V \cap Y + \{e\}$.
      Then for $\forall i \in I$, $x_i \succ_i x_i^*$ because $(x, y) \in V$ and (x, y) is feasible because $(x, y) \in Y + \{e\}$.
      Therefore, (x^*, y^*) is not pareto optimal which is a contradiction.

      iv. Separating Hyperplane Theorem
      By separating hyperplane theorem, $\exists p^* \in R^L, p^* \neq 0$ s.t.$\exists r$ s.t $p^* \cdot x \geq r$ for $\forall (x, y) \in V$ and $p^* \cdot x \leq r$ for $\forall (x, y) \in Y + \{e\}$

   (b) Show (x^*, y^*) with p^* consists of a price quasiequilibrium

      i. Show $\left[\text{For } \forall i \in I, x_i \succsim_i x_i^* \to \sum_{i \in I} p^* \cdot x_i \geq r\right]$
      Suppose for $\forall i \in I, x_i \succsim_i x_i^*$.

By l.n.s. of preferences for $\forall i$, $\exists \tilde{x}_i$ which is arbitrary close to x_i with $\tilde{x}_i \succ_i x_i$. Hence $\tilde{x}_i \in V$ and it implies $\sum_{i\in I} \tilde{x}_i \in V$ and $p^* \cdot \sum_{i\in I} \tilde{x}_i \geq r$
Take the limit as $\tilde{x}_i \to x_i$, and then $\sum_{i\in I} p^* \cdot x_i \geq r$

ii. Show $\left[\sum_{i\in I} p^* \cdot x_i^* = \sum_{i\in I} p^* \cdot e_i + \sum_{j\in J} p^* \cdot y_j^* = r\right]$
$\left[\text{For } \forall i \in I, x_i^* \succsim_i x_i^* \to \sum_{i\in I} p^* \cdot x_i^* \geq r\right]$
$\sum_{i\in I} x_i^* = \sum_{i\in I} e_i + \sum_{j\in J} y_j^* \in Y + \{e\} \to \sum_{i\in I} p^* \cdot x_i^* \leq r$
Therefore, $\sum_{i\in I} p^* \cdot x_i^* = \sum_{i\in I} p^* \cdot e_i + \sum_{j\in J} p^* \cdot y_j^* = r$

iii. Show $\left[\text{For } \forall j \in J, p^* \cdot y_j \leq p^* \cdot y_j^* \text{ for } \forall y_j \in Y_j\right]$
Consider for any $j \in J$, $\sum_{h\neq j} y_h^* + y_j + \{e\} \in Y + \{e\}$
$p^* \cdot \left[\sum_{h\neq j} y_h^* + y_j + \{e\}\right] \leq r = p^* \cdot \left[\sum_{i\in I} e_i + \sum_{j\in J} y_j^*\right]$
$p^* \cdot y_j \leq p^* \cdot y_j^*$

iv. Show [For $\forall i, x_i \succ_i x_i^* \to p^* \cdot x_i \geq p^* \cdot x_i^*$]
Suppose $x_i \succ_i x_i^*$.
By $\sum_{i\in I} p^* \cdot x_i \geq r$ and $\sum_{i\in I} p^* \cdot x_i^* = r$,
$p^* \cdot \left(x_i + \sum_{i'\neq i} x_{i'}^*\right) \geq r = p^* \cdot \left(x_i^* + \sum_{i'\neq i} x_{i'}^*\right) \to p^* \cdot x_i \geq p^* \cdot x_i^*$

v. Define $w_i = p^* \cdot x_i^*$ for $\forall i \in I$.
Then $\sum_{i\in I} w_i = \sum_{i\in I} p^* \cdot x_i^* = p^* \cdot \left(\sum_{i\in I} e_i + \sum_{j\in J} y_j^*\right)$

Therefore, for every pareto optimal allocation (x^*, y^*), $\exists p^* \in R^L$, $p^* \neq 0$ s.t. (x^*, y^*, p^*) is a price quasiequilibrium with transfers

(c) Any price quasiequilibrium (x^*, y^*, p^*) with $(w_1, w_2, ..., w_I) \in R^L_{++}$ is a price equilibrium (x^*, y^*, p^*) with $(w_1, w_2, ..., w_I) \in R^L_{++}$
NTS: $x_i \succ_i x_i^* \to p^* \cdot x_i > w_i$
Suppose not. $\exists x_i \in X_i$ s.t. $x_i \succ_i x_i^*$ and $p^* \cdot x_i = w_i$
By convexity of X_i, $\exists x_i' \in X_i$ s.t. $p^* \cdot x_i' < w_i$ and $\exists \bar{x}_i = \lambda x_i + (1-\lambda) x_i'$ s.t. $p^* \cdot \bar{x}_i < w_i$
By continuity of $\succsim_i, \exists \varepsilon > 0\,(\text{for } \lambda \text{ close enough to } 1), \exists \bar{x}_i = \lambda x_i + (1-\lambda) x_i' \in B(x_i)$ s.t. $p^* \cdot \bar{x}_i < w_i$ and $\bar{x}_i \succ_i x_i^*$.
It contradicts (x^*, y^*, p^*) is a price quasiequilibrium?!

2. Conclusion

(a) $\left.\frac{If\ \exists\ a\ cheaper\ consumption}{\text{If } (w_1,w_2,...,w_I)>>0 \text{ and } 0\in X_i}\right\}$, a price quasi-equilibrium is a price equilibrium with transfer. At the example, because there is no cheaper consumption, the price quasi-equilibrium was not a price equilibrium.

(b) The second welfare theorem identifies conditions under any pareto optimal allocation can be implemented through competitive markets and offers a strong conceptual affirmation of the use of competitive markets, even for dealing with distributional concerns.

(c) Some limitations on the use of welfare theorem

(d) A planning authority wishing to implement a particular pareto optimal allocation must be able to insure the supporting prices will be taken as given by consumers and firms.

(e) The authority must have good information to identify the pareto optimal allocation to be implemented and to compute the right supporting price vector. For this purpose, the authority must know at least the statistical joint distribution of preferences, endowments, and other relevant characteristics of consumers and firms. And moreover, in order to implement the correct transfer levels for each consumer, the authority must know who is who, which is unlikely in practice. As a result, most common transfer plan fails to be lump-sum transfers.

(f) Even if the authority observes all the required information, it must actually have the power to enforce the necessary transfers through some tax and transfer mechanism that individual can not evade.

(g) Because of informational and enforceability limitations, it is practically unlikely that extensive lump-sum taxation will be possible. If these transfer plan is not possible, then the second welfare theorem fails in the sense that for a typical economy, only a limited range of pareto optima are supportable by means of prices supplemented by the usual sort of taxation plan. For the typical economy, redistribution plans are distortionary; that is , they trade off distributional aims against pareto optimality.

(h) In summary, the second welfare theorem is a very useful theoretical reference point. But it is far from a direct prescription for policy practice. On the contrary, by pointing out what is necessary to achieve any desired pareto optimal allocation, it serves a cautionary purposes.

5.1.3 Failure of Second welfare theorem by Nonconvexity of Y

1. Nonconvex Production Technologies and Marginal Cost Pricing

   The second welfare theorem runs into difficulties in the presence of nonconvex production sets. In the first place, large nonconvexities caused by the presence of fixed costs or extensive increasing returns lead to a world of a small number of large firms(in the limit, natural monopoly), making price taking assumption unlikely.

   In figure 16.G.1, at the only relative prices that could support the production y^* locally, the firm sustains a loss and would rather avoid it by shutting down.

   And in figure 15.C.3, the allocation x^* maximizes the welfare of the consumer, but for the only value of relative prices that could support x^* as a utility maximizing bundle, the firm does not maximize profits even locally.

   Although nonconvexities may prevent us from supporting the pareto optimal production allocation as a profit maximizing choice, under the differentiability assumptions, we can use the first order necessary conditions derived there to formulate a weaker result that parallels the second welfare theorem.

2. Marginal cost price equilibrium with transfers

   In the production economy E_p, a marginal cost price equilibrium with transfers consists of $\left\{\{x_i^*\}_{i\in I}, \left\{y_j^*\right\}_{j\in J}\right\}$ and $p^* \in R_+^L \backslash \{0\}$ s.t.

   (a) $\exists\,(w_1, w_2, ..., w_I)$ s.t. $\sum_{i\in I} w_i = \sum_{i\in I} p^*\cdot e_i + \sum_{j\in J} p^*\cdot y_j^*$
   (b) for $\forall i\in I$, x_i^* is $\succsim_i -$ maximal in $B_i\left(p^*, e_i\right) = \{x_i \in X_i \mid p^*\cdot x_i \leq w_i\}$
   (c) for $\forall j \in J$, $p^* = \gamma_j F_j'\left(y_j^*\right)$ for some $\gamma_j > 0$ and $F_j\left(y_j^*\right) = 0$
   (d) $\sum_{i\in I} x_i^* \leq \sum_{i\in I} e_i + \sum_{j\in J} y_j^*$

   For two goods case, $p^* = \gamma_j F_j'\left(y_j^*\right)$ means that the price of input must equal the price of the output multiplied by the marginal productivity of the input.

   The price of output equals the marginal cost.

3. First welfare theorem fails.

   A marginal cost price equilibrium does not need to be pareto optimal.

   because marginal cost pricing neglects second order conditions and it may therefore happen that at the marginal cost price equilibrium, the second order conditions for the social utility maximization are not satisfied.

4. Generalized second welfare theorem

   Suppose $\succsim_i$is convex for $\forall i \in I(u_i(x_i)$ is C^2 and quasiconcave, $u_i'(x_i) >> 0$, and $u_i(0) = 0)$ and $F_j(y_j) = 0$ at transformation frontier, $F_j(y_j)$ is C^2, $F_j(0) \leq 0$, $F_j'(y_j) >> 0$

   Then if $(\hat{x}, \hat{y})$ is pareto optimal, then $\exists$a price vector p^* and wealth levels $(w_i)_{i \in I}$ with $\sum_{i \in I} w_i = \sum_{i \in I} p^* \cdot e_i + \sum_{j \in J} p^* \cdot \hat{y}_j$ s.t. $(\hat{x}, \hat{y})$ is a marginal cost price equilibrium allocation s.t.

   (a) for $\forall i \in I$, x_i^* is $\succsim_i -$ maximal in $B_i(p^*, e_i) = \{x_i \in X_i \mid p^* \cdot x_i \leq w_i\}$
   (b) for $\forall j \in J$, $p^* = \gamma_j F_j'\left(y_j^*\right)$ for some $\gamma_j > 0$
   (c) $\sum_{i \in I} x_i^* \leq \sum_{i \in I} e_i + \sum_{j \in J} y_j^*$

   →The firm incurs a loss at the prices that locally support the pareto optimal allocation.

   →If Y_j is convex or f_j is quasi-concave, condition b is equivalent to the profit maximization.

Marginal Cost Price Equilibrium Examples(**PSet #3= Midterm, My case,FP 1998 III-2**)

1. PSet #3=Midterm(MCP is pareto optimal)

   1) Is there an profit maximizing equilibrium?

   2) Find a marginal cost price equilibrium

   3) Verify whether the MCP equilibrium allocation is Pareto optimal

   $L = 2$

   $I = 2$

   $X_1 = R_+^2, u_1(x_1, x_2) = x_1^1 + 5x_2^1, e_1 = (0, 5)$

   $X_2 = R_+^2, u_2(x_1, x_2) = 4\ln\left(x_1^2 + 1\right) + x_2^2, e_2 = (10, 0)$

   $J = 1, Y = \left\{(y_1, y_2) \in R^2 \mid y_1 \leq 0, y_2 \leq \max(-y_2 - 2, 0)\right\}$

   $\theta_1 = 0, \theta_2 = 1$

   - Find an equilibrium of this economy
     $\boxed{\text{Check}}$
     $x_1^2 > 0$

     (a) $\pi(p^*)$ or p^*
         $\max p_1 \cdot y_1 + p_2 \cdot y_2$
         s.t. $y_2 \leq \max(-y_1 - 2, 0)$
         Normalize $p_1 = 1$.
         case 1) $1 = p_2$

$\pi(p) \leq 0$ and $y_1 = y_2 = 0$
case 2) $1 > p_2$
$\pi(p) \leq 0$ and $y_1 = y_2 = 0$
case 3) $1 < p_2$
$\pi(p) = p_1 \cdot y_1 + p_2 \cdot y_2 \leq p_1 \cdot y_1 + p_2 \cdot (-y_1 - 2) = (p_1 - p_2) \cdot y_1 - 2p_2 \rightarrow \infty$ as $y_1 \rightarrow -\infty$
Therefore, $p_2 \leq 1$, $\pi(p) = 0$, and $y_1 = y_2 = 0$

(b) $\boxed{\text{Consumer 1}}$
$\max_{x^1} x_1^1 + 5x_2^1$
s.t. $x_1^1 + p_2 x_2^1 \leq 5p_2$
$x_1^1 + 5x_2^1 \leq 5p_2 - p_2 x_2^1 + +5x_2^1 = -p_2\left(x_2^1 - 5\right) + 5x_2^1 \leq -\left(x_2^1 - 5\right) + 5x_2^1$
$x_1^1 = 0, x_2^1 = 5$
$\boxed{\text{Consumer 2}}$
$\max_{x^2} 4\ln\left(x_1^2 + 1\right) + x_2^2$
s.t. $x_1^2 + p_2 x_2^2 \leq 10 + \pi(p)$
$\rightarrow x_1^2 + p_2 x_2^2 \leq 10$
$4\ln\left(x_1^2 + 1\right) + x_2^2 \leq 4\ln\left(x_1^2 + 1\right) + \frac{10 - x_1^2}{p_2}$
$g\left(x^2\right) = 4\ln\left(x_1^2 + 1\right) + \frac{10 - x_1^2}{p_2}$
$g'\left(x^2\right) = \frac{4}{x_1^2 + 1} - \frac{1}{p_2} = 0$
$x_1^2 = 4p_2 - 1$
$x_2^2 = \frac{11 - 4p_2}{p_2}$

(c) Feasibility($p^* \rightarrow y^*$)
$x_1^1 + x_1^2 = 10 + y_1$
$x_2^1 + x_2^2 = 5 + y_2$

(d) $y_1 = 0, y_2 = 0$
$x_1^2 = 10 = 4p_2 - 1$
$p_2 = \frac{11}{4} > 1$ which is a contradiction!!!!
Therefore, there is no profit maximizing equilibrium.

- Find a marginal cost price equilibrium
  Note that $F(y) = \{\frac{y_2 + y_1 + 2 \text{ if } y_1 < -2}{y_2 \text{ if } y_1 \in [-2,0]}$

  (a) $y_1 < -2$
  $\boxed{\text{for } \forall j \in J,\ p^* = \gamma_j F_j'\left(y_j^*\right) \text{ for some } \gamma_j > 0 \text{ and } F_j\left(y_j^*\right) = 0}$
  $p^* = \gamma(1, 1).$
  Let $\gamma = 1$. Then $p_1^* = p_2^* = 1$
  $\pi(p) = y_1 + y_2 = y_1 + (-y_1 - 2) = -2$
  $\boxed{\text{for } \forall i \in I,\ x_i^* \text{ is } \succsim_i - \text{maximal in } B_i(p^*, e_i) = \{x_i \in X_i \mid p^* \cdot x_i \leq w_i\}}$

$\boxed{\text{Consumer 1}}$
$\max_{x^1} x_1^1 + 5x_2^1$
s.t. $x_1^1 + x_2^1 \leq 5$
$x_1^1 = 0, x_2^1 = 5$
$\boxed{\text{Consumer 2}}$
$\max_{x^2} 4\ln\left(x_1^2+1\right) + x_2^2$
s.t. $x_1^2 + x_2^2 \leq 10 - 2 = 8$
$4\ln\left(x_1^2+1\right) + x_2^2 \leq 4\ln\left(x_1^2+1\right) + 8 - x_1^2$
$g\left(x^2\right) = 4\ln\left(x_1^2+1\right) + 8 - x_1^2$
$g'\left(x^2\right) = \frac{4}{x_1^2+1} - 1 = 0$
$x_1^2 = 3, x_2^2 = 5$
$\boxed{\text{Feasibility}(p^* \to y^*)}$
$x_1^1 + x_1^2 = 10 + y_1 \to 0 + 3 = 10 + y_1 \to y_1 = -7$
$x_2^1 + x_2^2 = 5 + y_2 \to 5 + 5 = 5 + y_2 \to y_2 = 5$
$\Longrightarrow p_1^* = p_2^* = 1, x^1 = (0,5), x^2 = (3,5), y^* = (-7,5)$

(b) $y_1 \in [-2, 0]$
$\boxed{\text{for } \forall j \in J,\ p^* = \gamma_j F_j'\left(y_j^*\right) \text{ for some } \gamma_j > 0 \text{ and } F_j\left(y_j^*\right) = 0}$
$p^* = \gamma(0,1)$
But the utility function is strictly monotone so that $p^* \in R_{++}^2$

- Verify whether the MCP equilibrium allocation is Pareto optimal

  Yes, because consumer 1 consumes his endowment so that in the economy, it seems like there is only one consumer.

  And consumer 2 maximizes his utility with $(3,5)$.

2. My case(MCP is not Pareto optimal)

   1) Is there an profit maximizing equilibrium?

   2) Find a marginal cost price equilibrium

   3) Verify whether the MCP equilibrium allocation is Pareto optimal

   $L = 2$

   $I = 1$

   $X_1 = R_+^2, u_1(x_1, x_2) = x_1 x_2, e_1 = (6,2)$

   If $e = (6,1)$, then MCP is Pareto optimal!!!

   $J = 1, Y = \left\{(y_1, y_2) \in R^2 \mid y_1 \leq 0, y_2 \leq \max(-2y_2 - 6, 0)\right\}$

   - Find an equilibrium of this economy

$\boxed{\text{Check}}$

x_1 or $x_2 = 0$ is not an interesting case.

utility function is strictly monotone and strictly increasing so that $p \in R^2_{++}$‘.

(a) $\pi(p^*)$ or p^*

$\max p_1 \cdot y_1 + p_2 \cdot y_2$

s.t. $y_2 \leq \max(-2y_1 - 6, 0)$

Normalize $p_1 = 1$.

case 1) $\frac{1}{2} = p_2$

$\pi(p) \leq 0$ and $y_1 = y_2 = 0$

case 2) $\frac{1}{2} > p_2$

$\pi(p) \leq 0$ and $y_1 = y_2 = 0$

case 3) $\frac{1}{2} < p_2$

$\pi(p) = y_1 + p_2 \cdot y_2 \leq y_1 + p_2 \cdot (-2y_1 - 6) = (1 - 2p_2) \cdot y_1 - 2p_2 \to \infty$ as $y_1 \to -\infty$

Therefore, $p_2 \leq \frac{1}{2}$, $\pi(p) = 0$, and $y_1 = y_2 = 0$

(b) $\boxed{\text{Consumer 1}}$

$\max_{x^1} x_1 x_2$

s.t. $x_1^1 + p_2 x_2^1 \leq 6 + 2p_2$

$x_1 x_2 \leq (6 + 2p_2 - p_2 x_2) x_2 = -p_2 x_2^2 + (2p_2+6) x_2 = -p_2 \left(x_2^2 - 2\frac{(p_2+3)}{p_2} x_2 + \left(\frac{(p_2+3)}{p_2}\right)^2 \right) + \frac{(p_2+3)^2}{p_2}$

when $x_2 = \frac{(p_2+3)}{p_2}$, $x_1 = 3 + p_2$, utility is maximized.

(c) Feasibility($p^* \to y^*$)

$x_1^1 = 6 + y_1 = 6$

$3 + p_2 = 6 \to p_2 = 3$ which is a contradiction!!!

Therefore, there is no profit maximizing equilibrium.

- Find a marginal cost price equilibrium

  Note that $F(y) = \{\frac{y_2+2y_1+6 \text{ if } y_1<-3}{y_2 \text{ if } y_1\in[-3,0]}$

(a) $y_1 < -3$

$\boxed{\text{for } \forall j \in J,\ p^* = \gamma_j F_j' \left(y_j^*\right) \text{ for some } \gamma_j > 0 \text{ and } F_j\left(y_j^*\right) = 0}$

$p^* = \gamma(2, 1)$.

Let $\gamma = 1$. Then $p_1^* = 2, p_2^* = 1$

$\pi(p) = 2y_1 + y_2 = 2y_1 + (-2y_1 - 6) = -6$

$\boxed{\text{for } \forall i \in I,\ x_i^* \text{ is } \succsim_i - \text{maximal in } B_i(p^*, e_i) = \{x_i \in X_i \mid p^* \cdot x_i \leq w_i\}}$

$\boxed{\text{Consumer 1}}$

$\max_{x^1} x_1 x_2$

s.t. $2x_1^1 + x_2^1 \leq 12 + 2 - 6 = 8$

$x_1x_2 \leq x_1(8-2x_1) = -2x_1^2 + 8x_1 = -2\left(x_1^2 - 4x_1 + 4 - 4\right) = -2(x_1 - 2) + 8$
$x_1 = 2, x_2 = 4$
$\boxed{\text{Feasibility}(p^* \to y^*)}$
$x_1 = 6 + y_1 \to y_1 = -4$
$x_2 = 2 + y_2 \to y_2 = 2$
$\Longrightarrow p_1^* = 2, p_2^* = 1, x^1 = (2,4), y^* = (-4,2)$

(b) $y_1 \in [-2, 0]$
$\boxed{\text{for } \forall j \in J,\ p^* = \gamma_j F_j'\left(y_j^*\right) \text{ for some } \gamma_j > 0 \text{ and } F_j\left(y_j^*\right) = 0}$
$p^* = \gamma(0,1)$
But the utility function is strictly monotone so that $p^* \in R_{++}^2$

- Verify whether the MCP equilibrium allocation is $\boxed{\text{not}}$ Pareto optimal
  The initial endowment gives higher utility to consumer.
  $u(6,2) = 12 > 8 = u(2,4)$
  Therefore, $x^1 = (6,2), y^* = (0,0)$ pareto dominates MCP equilibrium.

3. FP 1998 III-2

1) Is there an profit maximizing equilibrium?

2) Find a marginal cost price equilibrium

3) Verify whether the MCP equilibrium allocation is Pareto optimal

$L = 2$

$I = 2$

$X_1 = R_+^2, u_1(x_1, x_2) = \ln x_1^1 + 3\ln x_2^1, e_1 = (8,4)$

$X_2 = R_+^2, u_2(x_1, x_2) = 3\ln x_1^2 + \ln x_2^2, e_2 = (6,4)$

$J = 1, Y = \left\{(y_1, y_2) \in R^2 \mid y_1 \leq 0, \sqrt{8y_2} \leq -y_1\right\}$

$\theta_1 = 1, \theta_2 = 0$

- Find an equilibrium of this economy
  $\boxed{\text{Check}}$
  $x_1^1, x_2^1, x_1^2, x_2^2 > 0$

(a) $\pi(p^*)$ or p^*
max $p_1 \cdot y_1 + p_2 \cdot y_2$
s.t. $\sqrt{8y_2} \leq -y_1$
And like figure 15.C.3, the allocation x^* maximizes the welfare of the consumers, but for the only value of relative prices that could support x^* as a utility maximizing bundle, the firm does not maximize profits even locally.
Therefore, there is no profit maximizing equilibrium.

- Find a marginal cost price equilibrium
  Note that $F(y) = y_1 + \sqrt{8y_2} \le 0$

  (a) $\boxed{\text{for } \forall j \in J,\ p^* = \gamma_j F_j'\left(y_j^*\right) \text{ for some } \gamma_j > 0 \text{ and } F_j\left(y_j^*\right) = 0}$
  $p^* = \gamma\left(1, \frac{\sqrt{2}}{\sqrt{y_2}}\right).$
  Let $\gamma = 1$. Then $p_1^* = 1, p_2^* = \frac{\sqrt{2}}{\sqrt{y_2}} \rightarrow y_2 = \frac{2}{p_2^2}, y_1 = -\frac{4}{p_2}$
  $p_2 = p$
  $\pi(p) = -\frac{4}{p} + \frac{2}{p} = -\frac{2}{p}$
  $\boxed{\text{for } \forall i \in I,\ x_i^* \text{ is } \succsim_i - \text{maximal in } B_i(p^*, e_i) = \{x_i \in X_i \mid p^* \cdot x_i \le w_i\}}$
  $\boxed{\text{Consumer 1}}$
  $\max_{x^1} \ln x_1^1 + 3\ln x_2^1$
  s.t. $x_1^1 + px_2^1 \le 8 + 4p - \frac{2}{p}$
  $\boxed{\text{Consumer 2}}$
  $\max_{x^2} 3\ln x_1^2 + \ln x_2^2$
  s.t. $x_1^2 + px_2^2 \le 6 + 4p$
  $MRS_{12}^1 = MRS_{12}^2 = \frac{1}{p}$
  $\frac{\frac{1}{x_1^1}}{\frac{3}{x_2^1}} = \frac{1}{p} = \frac{\frac{3}{x_1^2}}{\frac{1}{x_2^2}}$
  $px_2^1 = 3x_1^1 \rightarrow x_1^1 = 2 + p - \frac{1}{2p}$
  $x_1^2 = 3px_2^2 \rightarrow x_1^2 = \frac{9+6p}{2}$
  $\boxed{\text{Feasibility}(p^* \rightarrow y^*)}$
  $x_1^1 + x_1^2 = 14 + y_1 = 14 - \frac{4}{p}$
  $\rightarrow 2 + p - \frac{1}{2p} + \frac{9+6p}{2} = 14 - \frac{4}{p}$
  $\rightarrow 2p + p^2 - 0.5 + 4.5p + 3p^2 - 14p + 4 = 0$
  $\rightarrow 4p^2 - 7.5p + 3.5 = 0$
  $\rightarrow p = 1 \text{ or } \frac{7}{8}$
  $\rightarrow x^1 = (2.5, 7.5), x^2 = (7.5, 2.5), y^* = (-4, 2)$

- Verify whether the MCP equilibrium allocation is profit maximizing.
  No, $\pi(p) = -\frac{4}{p} + \frac{2}{p} = -\frac{2}{p} = -2$
  This firm would be better if shutting down and get 0 profit.

6 Pareto Optimality and Social Welfare Function

- Given a family of $u_i(\cdot)$ of continuous utility functions representing the preferences $\succsim_i$of the I consumers, we can capture the attainable vectors of utility levels for an economy specified by $E = \left[\left\{I, (\succsim_i, e_i)_{i\in I}\right\}, \left\{J, (Y_j)_{j\in J}\right\}\right]$ by means of the utility possibility set s.t.

$$U = \left\{(u_1, u_2, ..., u_I) \in R^L \mid \exists (x, y) \text{ s.t.} u_i(x_i) \geq u_i \text{ for } \forall i \in I\right\}$$

- In two consumers' economy, we can depict the utility possibiliity set which is closed under sufficient conditions.

- By the definition of pareto optimality, the utility levels of a pareto optimal allocation must belong to the boundary of the utility possibility set; the pareto frontier UP

  $UP = \left\{(u_1, u_2, ..., u_I) \in U \mid \nexists \left(u_1', ..., u_I'\right) \in U \text{ s.t.} u_i' \geq u_i \text{ for } \forall i \in I \text{ and } u_{i'}' > u_{i'} \text{ for at least one } i'\right\}$

- A feasible allocation (x, y) is a pareto optimum iff $(u_1(x_1), ..., u_I(x_I)) \in UP$.

- Note that if every X_i and Y_j is convex and the utility functions $u_i(\cdot)$ are concave, then the utility possibility set U is convex.

1. Representative agent's problem

   Under certain circumstances, the model can be justified as representing the outcome of a more general economy by interpreting the firm as a representative producer and the consumer as a representative consumer. The former is always possible, but the latter-that is, the existence of a representative consumer requires strong conditions. If, however, the economy is composed of many consumers with identical concave utility functions and identical initial endowments, and if society has a strictly concave social welfare function in which these consumers are treated symmetrically, then a representative consumer exists who has the same utility function as the consumers over levels of per capita consumption.

2. Social Planner's Problem

   (a) Suppose that society's distributional principles can be summarized in a social welfare function $W(u_1, u_2, ..., u_I) = \sum_{i\in I} \lambda_i u_i$ for some constant $\lambda = (\lambda_1, \lambda_2, ..., \lambda_I)$ s.t. $\lambda \geq 0$ because social welfare should be nondecreasing in the consumers' utility levels.

   (b) For economies with $\boxed{\text{convex}}$ utility possibility sets, there is a close relation between pareto optima and linear social welfare optima;

   every linear social welfare optimum with weights $\lambda >> 0$ is pareto optimal and

   every pareto optimal allocation is a social welfare optimum for some welfare weights $\lambda \geq 0$

(c) Let (x^*, y^*) be pareto optimal.
With $\lambda >> 0$, define a utility function $u_\lambda(\bar{x})$ on aggregate consumption vectors in X by

$$u_\lambda(\bar{x}) = \max_x \sum_{i \in I} \lambda_i \cdot u_i(x_i)$$

$$\text{s.t. } x_i \in X_i \text{ for all } i \text{ and } \sum_i x_i = \bar{x}$$

Then (x^*, y^*) is a solution to the problem

$$\max u_\lambda(\bar{x})$$

$$\text{s.t. } \bar{x} = \bar{e} + \bar{y},\ \bar{x} \in X \text{ and } \bar{y} \in Y$$

3. First order conditions for pareto optimality

With differentiability assumptions, we show how prices and the optimality properties of price taking behavior emerge from an examination of the first order condition of pareto optimality problems.

(a) Assumptions

i. $u_i(x_i)$ is twice differentiable and $u_i'(x_i) >> 0 \leftarrow$ preference strictly monotone, and $u_i(0) = 0$

ii. $Y_j = \left\{ y \in R^L \mid F_j(y_j) \leq 0 \right\}$
$F_j(y_j) = 0$ at transformation frontier, $F_j(y_j)$ is C^2, $F_j(0) \leq 0$, $F_j'(y_j) >> 0$

(b) Problem

The problem of identifying the pareto optimal allocations for this economy can be reduced to the selection of allocations that solve the following problem

$$\begin{array}{ll} \max u_1(x_1) & \\ \text{s.t. } u_i(x_i) \geq \bar{u}_i \text{ for } \forall i \neq 1 & \delta_i \\ \sum_{i \in i} x_i^l \leq \sum_{i \in I} e_i^l + \sum_{j \in J} y_j^l \text{ for } \forall l \in L & \mu_l \\ F_j(y_j) \leq 0 \text{ for } \forall j \in J & \gamma_j \end{array}$$

By solving it for varing required levels of utility for these other consumers $(\bar{u}_i)_{i=2}^I$, we can identify all the pareto optimal allocations for this economy.

i. Example(PSet #3-2)

1) Show that every pareto optimal allocation is a solution to this optimization problem for some $\bar{u}_i$

$\boxed{\text{Claim}}$ A pareto optimal allocation (x^*, y^*) is a solution to the problem for some choice of utility level $\bar{u}_i$

Proof

Suppose not; then $(\hat{x}, \hat{y})$ is P.O., but is not a solution to this problem.
Then $\exists (x', y')$ s.t.
feasible;$\sum_{i \in i} x'_i \leq \sum_{i \in I} e_i + \sum_{j \in J} y'_j$ and $F(y') \leq 0$ for $\forall j \in J$
$u_i(x'_i) \geq u_i(\hat{x}_i)$ for $\forall i \neq 1$
$u_1(x'_1) > u_1(\hat{x}_1)$ for $i = 1$
It implies that (x', y') pareto dominates $(\hat{x}, \hat{y})$ which is a contradiction!!!
Therefore, a pareto optimal allocation $(\hat{x}, \hat{y})$ is a solution to the problem for some choice of utility level $\bar{u}_i$
2) Assuming that utility function is strictly monotone, and continous, show that any allocation that is a solution to this maximization problem is Pareto optimal.

Proof

Suppose not; then (x^*, y^*) is a solution to this problem, but is not P.O.
Then $\exists (x', y')$ s.t.
feasible;$\sum_{i \in i} x'_i \leq \sum_{i \in I} e_i + \sum_{j \in J} y'_j$ and $F(y') \leq 0$ for $\forall j \in J$
$u_i(x'_i) \geq u_i(x^*_i)$ for $\forall i \neq k$
$u_k(x'_k) > u_k(x^*_k)$ for at least one $k \in I$
Because the utility function is strictly monotone and continuous, $u_k(x'_k) > u_k(x^*_k)$ for at least one $k \in I$ means that $x'_k \gneqq x^*_k$ and $\exists l \in L$ s.t. $x'_{kl} > x^*_{kl}$.
Let 1_l be a l^{th} unit vector.
And because the utility function is continuous, $\exists \varepsilon > 0$ s.t. $u_k(x'_k - \varepsilon \cdot 1_l) > u_k(x^*_k)$.
Based on it, let's construct a feasible allocation $(\bar{x}, \bar{y})$which shows (x^*, y^*) can not be a solution which will gives us a contradiction.
$\bar{x}_k = x'_k - \varepsilon \cdot 1_l$
$\bar{x}_i = x'_i$ for $\forall i \neq 1, i \neq k$
$\bar{x}_1 = x'_1 + \varepsilon \cdot 1_l$
$\bar{y}_j = y'_j$ for $\forall j \in J$
Then $\sum_{i \in I} \bar{x}_i = x'_k - \varepsilon \cdot 1_l + \sum_{i \neq 1, i \neq k} x'_i + x'_1 + \varepsilon \cdot 1_l$
$= \sum_{i \in I} x'_i \leq \sum_{i \in I} e_i + \sum_{j \in J} y'_j$ which is feasible.
And by strict monotonicity and continuity of utility functions,
$u_i(\bar{x}_i) = u_i(x'_i) \geq u_i(x^*_i)$ for $\forall i \neq 1, i \neq k$
$u_1(\bar{x}_1) = u_1(x'_1 + \varepsilon \cdot 1_l) > u_1(x^*_1)$
$u_k(\bar{x}_k) = u_k(x'_k - \varepsilon \cdot 1_l) > u_k(x^*_k)$
It implies that (x^*, y^*) is not a solution to this problem which is a contradiction!!!
3) If utility function is just l.n.s., then 2) does not satisfy.
A graph with
$(x_1^1, x_2^1) \succ (x_1'^1, x_2'^1)$ if $x_1^1 > x_1'^1$
$(x_1^2, x_2^2) \succ (x_1'^2, x_2'^2)$ if $x_2^2 > x_2'^2$
which is monotone so that l.n.s.

But some solutions to this problem is not pareto optimal.

(c) First order conditions

Under $\bar{u}_i \geq 0$ for $\forall i \neq 1$, all the constraints of problem will be binding at a solution.

$x_i^l; \delta_i \frac{\partial u_i}{\partial x_i^l} - \mu_l \begin{cases} \leq 0 \\ = 0 \text{ if } x_i^l > 0 \end{cases}$ for $\forall i, l$

$y_j^l; \mu_l - \gamma_j \cdot F_j'\left(y_j^l\right) = 0$ for $\forall j, l$

At the interior solution,

$MRS_i\left(l, l'\right) = MRS_{i'}\left(l, l'\right)$

$MRT_j\left(l, l'\right) = MRT_{j'}\left(l, l'\right)$

$MRS_i\left(l, l'\right) = MRT_{j'}\left(l, l'\right)$

(d) Interpretations

i. The multiplier μ_l at an optimal solution is exactly equal to the increase in consumer 1' utility derived from a relaxation of the corresponding constraint, that is, from a marginal increase in the available social endowment $\sum_{i \in I} e_i^l$; that is, the multiplier μ_l can be interpreted as the marginal value or shadow price of good l in terms of consumer 1's utility

ii. The multiplier δ_i equals the marginal change in consumer 1's utility if we decrease the utility requirement $\left(\bar{u}_i\right)_{i=2}^{I}$.
At the optimal interior allocation, weighted by the amount that relaxing consumer i's utility constraint is worth in terms of raising consumer 1's utility, the increase in the utility of any consumer i from receiving an additional unit of good l should be equal to the marginal value μ_l of good l.

iii. The multiplier γ_j is the marginal benefit from relaxing the jth production constraint(the marginal cost from tighting the jth production constraint)
At an optimum, the marginal cost is equated to the marginal benefit μ_l of good l for $\forall j \in J$

4. Welfare theorem and social planner's problem

(a) Assumptions

i. $\succsim_i$is $\boxed{\text{strictly monotone}}$, $\boxed{\text{convex}}$, continuous, complete preorder on $R^L \to u_i\left(x_i\right)$ is C^2 and quasiconcave, $u_i'\left(x_i\right) >> 0$, and $u_i\left(0\right) = 0$

ii. $F\left(y_j\right) = 0$ at transformation frontier, $F\left(y_j\right)$ is C^2, $F\left(0\right) \leq 0$, $F'\left(y_j\right) >> 0$, and a $\boxed{\text{convex function}}$(production sets Y_j is convex for $\forall j$)
It means we do not have to check the second order condition for having maximum.

(b) Problem

Let $\left(x^*, y^*, p^*\right)$ be a price equilibrium with transfers s.t $w_i = p^* \cdot e_i + p^* \cdot \sum_{j \in J} \theta_{ij} \cdot y_j^*$ for $\forall i \in I$ iff

i. $\begin{array}{l} \max u_i\left(x_i\right) \\ \text{s.t. } p^* \cdot x_i \leq w_i \text{ for } \forall i \in I \end{array} \rightarrow \alpha$

ii. $\begin{array}{l} \max p^* \cdot y_j \\ \text{s.t. } F\left(y_j\right) \leq 0 \text{ for } \forall j \in J \end{array} \rightarrow \beta$

iii. $\sum_{i \in I} x_i^* = \sum_{i \in I} e_i + \sum_{j \in J} y_j^*$

(c) First order conditions

$$x_i^l; u_i'\left(x_i^l\right) - \alpha_i \cdot p_l \begin{cases} \leq 0 \\ = 0 \text{ if } x_i^l > 0 \end{cases} \text{for } \forall i, l$$

$$y_j^l; p_l - \beta_j \cdot F'\left(y_j^l\right) = 0 \text{ for } \forall j, l$$

$$\Leftrightarrow \begin{array}{l} x_i^l; \delta_i \frac{\partial u_i}{\partial x_i^l} - \mu_l \begin{cases} \leq 0 \\ = 0 \text{ if } x_i^l > 0 \end{cases} \text{for } \forall i, l \\ y_j^l; \mu_l - \gamma_j \cdot F'\left(y_j^l\right) = 0 \text{ for } \forall j, l \end{array}$$

(d) Social welfare function

$$\begin{array}{ll} \max_x \sum_{i \in I} \lambda_i \cdot u_i\left(x_i\right) & \\ \text{s.t. } \sum_{i \in i} x_i^l \leq \sum_{i \in I} e_i^l + \sum_{j \in J} y_j^l \text{ for } \forall l \in L & \rightarrow \Psi \\ F\left(y_j\right) \leq 0 \text{ for } \forall j \in J & \rightarrow \Phi \end{array}$$

$$\Leftrightarrow \begin{array}{l} x_i^l; \lambda_i u_i'\left(x_i^l\right) - \Psi_l \begin{cases} \leq 0 \\ = 0 \text{ if } x_i^l > 0 \end{cases} \text{for } \forall i, l \\ y_j^l; \Psi_l - \Phi_j \cdot F'\left(y_j^l\right) = 0 \text{ for } \forall j, l \end{array}$$

if $\mu_l = p_l = \Psi_l$, $\frac{1}{\delta_i} = \alpha_i \cdot = \frac{1}{\lambda_i}$ and $\gamma_j = \beta_j = \Phi_j$

(e) Interpretations

i. Since both sets of conditions are necessary and sufficient for their respective problems, it means that the allocation (x^*, y^*) is pareto optimal iff it is a price equilibrium with transfers with respect to some price vector p^*. Note that the equilibrium price p_l exactly equal to μ_l, the marginal value of good l in the pareto optimal problem.

(f) Conclusion

Under the assumptions made about the economy E,

in particular $u_i(\cdot)$ is concave and $F_j(\cdot)$ is convex for $\forall i, j$,

every pareto optimal allocation (and, hence, every price equilibrium with transfers) maximizes a weighted sum of utilities subject to the resource and technological constraints. Moreover, the weight λ_i of the utility of the ith consumer equals the reciprocal of consumer i's marginal utility of wealth evaluated at the supporting prices and imputed wealth.

7 General Equilibrium with Public Good

We only focus on the case that the firm produce only public good h using inputs z. Therefore, we can describe production set Y by production function $f(z)$.

1. Environment

   In $E = \left[\left\{I, (\succsim_i, e_i)_{i\in I}\right\}, Y\right]$ s.t.

   $X_i = R_+^2$ for $\forall i$ and $X = \{X_i\}_{i\in I}$

   A Private good $x^1$and

   A Public good x^2: nondepletable and nonrivalrous

   $$\succsim_i \begin{cases} \text{l.n.s.} \\ \text{(weakly)convex} \\ \text{continuous} \end{cases}$$

   $e_i^1 \in R_{++}^1 \Rightarrow \sum_{i\in I} e_i^1 >> 0$ and no endowment on public good

   a single firm with production set $Y = R^2$ represented by a production function $f(z)$ which is increasing and concave.

2. Assumptions

   (a) $\succsim_i$is $\boxed{\text{strictly monotone}}$, $\boxed{\text{convex}}$, continuous, complete preorder on $R^L \rightarrow u_i(x_i)$ is C^2 and quasiconcave, $u_i'(x_i) >> 0$, and $u_i(0) = 0$

   (b) $f(z)$ is concave and C^2, $f(0) \leq 0 : f'(z) > 0, f''(z) < 0$(production sets Y is convex)

3. Feasible allocation

   $\left(\left(x^1, x^2\right), (z, q)\right)$ s.t. $q \leq f(z)$, $x^1 + z = e^1$, $x^2 = q$.

4. $\boxed{\text{Personalized prices}}$

   Now we try to reduce public goods to private goods.

   Define a personalized commodities x_i^2 for $\forall i \in I$ and then

   the single firm's production set is

   $Y = \left\{(-z, (h_1, h_2, ..., h_I)) \in R^L \mid h_1 = h_2 = ... = h_I \leq f(z), z \geq 0\right\}$

   With the representation, the general equilibrium model with public good is tranformed into a normal problem with private goods.

   $\rightarrow E^{public}$

5. Equilibrium with private provision of public goods

- Example of private provision of public good(normal price equilibrium is $\boxed{\text{not}}$ P.O.)
  We can get a result that a normal equilibrium concept with transfers can not be pareto optimal

  (a) A price equilibrium with transfers in E consists of $\left\{\{x_i^*, h_i^*\}_{i\in I}, \{z^*, h^*\}\right\}$and $\{p^*, p_h^*\} \in R_+^2 \backslash \{0\}$ s.t

    i. For $\forall i \in I$, $\{x_i^*, h_i^*\}_{i\in I}$ solves
       $\max_{(x_i, h_i)} u_i(x_i, h_i)$
       s.t. $p^* \cdot x_i + p_{h_i}^* \cdot h_i \le p^* \cdot e_i + \theta_i \cdot (\boxed{p_h^* \cdot h^*} + p^* \cdot z^*)$

    ii. $\{z^*, h^*\}$ solves
       $\max_{(z,h)} \boxed{p_h^* \cdot h} + p^* \cdot z$
       s.t. $h \le f(z)$

    iii. $\sum_{i\in I} x_i^* \le \sum_{i\in I} e_i + z^*$
       $\boxed{\sum_{i\in I} h_i^* = h^*}$

  (b) FOC

    i. First order conditions
       If the solution is interior,
       $\frac{\frac{\partial u_i}{\partial h_i}}{\frac{\partial u_i}{\partial x_i^l}} = \frac{p_h}{p_l} = \frac{\frac{\partial f}{\partial h}}{\frac{\partial f}{\partial z^l}}$ for $\forall i \in I$
       $\frac{\frac{\partial u_i}{\partial h_i}}{\frac{\partial u_i}{\partial x_i^l}} = \frac{\frac{\partial f}{\partial h}}{\frac{\partial f}{\partial z^l}} < \sum_{i\in I} \frac{\frac{\partial u_i}{\partial h_i}}{\frac{\partial u_i}{\partial x_i^l}}$
       Therefore, the normal equilibrium with transfer is not pareto optimal.

6. Lindahl Equilibrium
   A price equilibrium with transfers for this artificial economy is known as a Lindahl equilibrium.

   (a) A Lindahl equilibrium is a price equilibrium with transfers in E^{public} with personalized commodities.
   It consists of $\left\{\{x_i^*, h_i^*\}_{i\in I}, \{z^*, h^*\}\right\}$and $\left\{p^*, p_{h_i}^*\right\} \in R_+^{I+1} \backslash \{0\}$ s.t

     i. For $\forall i \in I$, $\{x_i^*, h_i^*\}_{i\in I}$ solves
        $\max_{(x_i, h_i)} u_i(x_i, h_i)$
        s.t. $p^* \cdot x_i + p_{h_i}^* \cdot h_i \le p^* \cdot e_i + \theta_i \cdot (p^* \cdot z^* + \sum_{i\in I} p_{h_i}^* \cdot h^*)$

     ii. $\{z^*, h^*\}$ solves
        $\max_{(z,h)} p^* \cdot z + \sum_{i\in I} p_{h_i}^* \cdot h$
        s.t. $h \le f(z)$

     iii. $\sum_{i\in I} x_i^* \le \sum_{i\in I} e_i + z^*$
        $h_i^* = h^*$ for $\forall i \in I$

7. Pareto optimality
   $\left(\left(\hat{x}_i, \hat{h}_i\right)_{i\in I}, \left(\hat{z}, \hat{h}\right)\right)$ is pareto optimal if $\nexists$another feasible $\left((x_i', h_i')_{i\in I}, (z', h')\right)$ s.t. for $\forall i \in I$, $(x_i', h_i') \succsim_i \left(\hat{x}_i, \hat{h}_i\right)$ and for at least one i', $(x_i', h_i') \succ_i \left(\hat{x}_i, \hat{h}_i\right)$
   First order condition for pareto optimality
   If $\left(\left(\hat{x}_i, \hat{h}_i\right)_{i\in I}, \left(\hat{z}, \hat{h}\right)\right)$ is interior, it satisfies
   $\sum_{i\in I} \frac{\frac{\partial u_i}{\partial h_i}}{\frac{\partial u_i}{\partial x_i^l}} = \frac{\frac{\partial f}{\partial h}}{\frac{\partial f}{\partial z^l}} \Leftrightarrow \sum_{i\in I} MRS_i\left(x_i^l, h_i\right) = MRT\left(z^l, h\right)$

8. Welfare Theorem

   (a) Every Lindahl equilibrium is pareto optimal
   Suppose $\succsim_i$is l.n.s. for $\forall i \in I$.
   Let $\left[\left\{\{x_i^*, h_i^*\}_{i\in I}, \{z^*, h^*\}\right\}, \left\{p^*, p_{h_i}^*\right\}\right]$ be a Lindahl equilibrium with transfers.
   Then $\left\{\{x_i^*, h_i^*\}_{i\in I}, \{z^*, h^*\}\right\}$ is pareto optimal.
   $\boxed{\text{Proof}}$
   Suppose not. Then $\exists$another feasible allocation $\left((x_i', h_i')_{i\in I}, (z', h')\right)$ s.t.
   for $\forall i \in I$, $(x_i', h_i') \succsim_i \left(\hat{x}_i, \hat{h}_i\right)$ and
   for at least one i', $(x_{i'}', h_{i'}') \succ_i \left(\hat{x}_{i'}, \hat{h}_{i'}\right)$.
   Then for at least one i',
   $\left(x_{i'}', h_{i'}'\right) \succ_i \left(\hat{x}_{i'}, \hat{h}_{i'}\right) \rightarrow p^*\cdot x_{i'}' + p_{h_i}^*\cdot h_{i'}' > p^*\cdot x_i^* + p_{h_i}^*\cdot h_i^* = p^*\cdot e_i + \theta_i\cdot\left(\sum_{i\in I} p_{h_i}^*\cdot h_i^* - p^*\cdot z^*\right)$
   By the fact above and l.n.s. of $\succsim_i$, for $\forall i \neq i'$,
   $(x_i', h_i') \succsim_i \left(\hat{x}_i, \hat{h}_i\right) \rightarrow p^*\cdot x_i' + p_{h_i}^*\cdot h_i' \geq p^*\cdot x_i^* + p_{h_i}^*\cdot h_i^* = p^*\cdot e_i + \theta_i\cdot\left(\sum_{i\in I} p_{h_i}^*\cdot h_i^* - p^*\cdot z^*\right)$
   Therefore, $\begin{array}{l} p^*\cdot\sum_{i\in I} x_i' + p_{h_i}^*\cdot\sum_{i\in I} h_i' > p^*\cdot\sum_{i\in I} e_i + \sum_{i\in I}\theta_i\cdot\left(\sum_{i\in I} p_{h_i}^*\cdot h_i^* - p^*\cdot z^*\right) \\ \rightarrow p^*\cdot\sum_{i\in I} x_i' + p_{h_i}^*\cdot\sum_{i\in I} h_i' > p^*\cdot\sum_{i\in I} e_i + \left(\sum_{i\in I} p_{h_i}^*\cdot h_i^* - p^*\cdot z^*\right)\end{array}$
   From the feasibility, $\begin{array}{l} p^*\cdot\left(\sum_{i\in I} e_i + z'\right) + p_{h_i}^*\cdot\sum_{i\in I} h_i' > p^*\cdot\sum_{i\in I} e_i + \left(\sum_{i\in I} p_{h_i}^*\cdot h_i^* + p^*\cdot z^*\right) \\ \rightarrow p_{h_i}^*\cdot\sum_{i\in I} h_i' + p^*\cdot z' > \sum_{i\in I} p_{h_i}^*\cdot h_i^* + p^*\cdot z^* \\ \rightarrow p_{h_i}^*\cdot\sum_{i\in I} h' + p^*\cdot z' > \sum_{i\in I} p_{h_i}^*\cdot h^* + p^*\cdot z^*\end{array}$
   It contradicts that $\{z^*, h^*\}$ solves firm's maximization problem.

   (b) Every pareto optimal allocation can be implemented using a Lindahl equilibrium with appropriate wealth transfers.
   Because each consumer is large with respect to the market in her personalized good, the price taking assumption is unlikely satisfied. Nevertheless, the second welfare theorem may still be of some interest. In particular, it tells us that if the planning authority has

a means to enforce the prices, then we have a mechanism involving voluntary purchases of the public good that achieves the desired pareto optimal allocation.

- Further difficulty
    i. To calculate the personalized prices, statistical information(e.g. information on the distribution of preferences across the economy) is not enough but we need private information. This information is difficult to get because individuals will often have incentive not to reveal their information truthfully.

    ii. For a personalized market voluntary mechanism to work, individuals must expect to receive the amount of public good they purchase. This requires that the public good be excludable. In many cases, it is difficult.

7.1 Example

1. SP 1999 IV-3

   $L = 2$; z is a private good, h is nonrivalrous public good

   $I = 2$

   Postive endowment $e_i = (z_i, 0)$ s.t. $z_i > 0$ for $\forall i = 1, 2$

   $J = 1$

   $Y = \left\{(-z, h) \in R^2 \mid z \geq 0, h \leq kz \text{ s.t. } k > 0\right\}$

   (a) Define Lindahl Equilibrium for this economy(profit function, budget constraints, price variables)

   A Lindahl equilibrium is a price equilibrium with transfers in E^{public} with personalized commodities.

   Let the shares of the firm of consumers be $\theta_i \in [0, 1]$ for $\forall i = 1, 2$ and $\sum_{i=1,2} \theta_i = 1$.

   It consists of $\left\{\{z_i^*, h_i^*\}_{i=1,2}, \{z^*, h^*\}\right\}$and $\left\{p_z^*, p_{h_i}^*\right\} \in R_+^2 \backslash \{0\}$ s.t

   i. For $\forall i = 1, 2$, $\{z_i^*, h_i^*\}_{i=1,2}$ solves
   $\max_{(z_i,h_i)} u_i (z_i, h_i)$
   s.t. $p^* \cdot z_i + p_{h_i}^* \cdot h_i \leq p^* \cdot w_i + \theta_i \cdot (\sum_{i=1,2} p_{h_i}^* \cdot h^* - p^* \cdot z^*)$

   ii. $\{z^*, h^*\}$ solves
   $\max_{(z,h)} - p^* \cdot z + \sum_{i=1,2} p_{h_i}^* \cdot h$
   s.t. $h \leq kz$ for $k > 0$ and $z \geq 0$

   iii. $\sum_{i=1,2} z_i^* \leq \sum_{i=1,2} w_i + z^*$
   $h_i^* = h^*$ for $\forall i = 1, 2$

(b) Pareto optimality

$\left(\left(\hat{z}_i, \hat{h}_i\right)_{i\in I}, \left(\hat{z}, \hat{h}\right)\right)$ is pareto optimal

if $\nexists$another feasible $\left((z_i', h_i')_{i\in I}, (z', h_i')\right)$ s.t.

for $\forall i \in I$, $(z_i', h_i') \succsim_i \left(\hat{z}_i, \hat{h}_i\right)$ and

for at least one i', $(z', h_i') \succ_i \left(\hat{z}_i, \hat{h}_i\right)$

First order condition for pareto optimality

If $\left(\left(\hat{z}_i, \hat{h}_i\right)_{i\in I}, \left(\hat{z}, \hat{h}\right)\right)$ is interior, it satisfies

$\sum_{i\in I} \frac{\frac{\partial u_i}{\partial h_i}}{\frac{\partial u_i}{\partial z_i}} = \frac{\frac{\partial f}{\partial h}}{\frac{\partial f}{\partial z}} \Leftrightarrow \sum_{i\in I} MRS_i\left(z_i, h_i\right) = MRT\left(z, h\right)$

(c) Every Lindahl equilibrium is pareto optimal

Suppose $\succsim_i$is l.n.s. for $\forall i \in I$.

Let $\left[\left\{\{z_i^*, h_i^*\}_{i\in I}, \{z^*, h^*\}\right\}, \left\{p^*, p_{h_i}^*\right\}\right]$ be a Lindahl equilibrium with transfers.

Then $\left\{\{z_i^*, h_i^*\}_{i\in I}, \{z^*, h^*\}\right\}$ is pareto optimal.

$\boxed{\text{Proof}}$

Suppose not. Then $\exists$another feasible allocation $\left((z', h_i')_{i\in I}, (z', h_i')\right)$ s.t.

for $\forall i \in I$, $(z', h_i') \succsim_i \left(\hat{z}_i, \hat{h}_i\right)$ and

for at least one i', $(z', h_i') \succ_i \left(\hat{z}_{i'}, \hat{h}_{i'}\right)$.

Then for at least one i',

$(z', h_{i'}') \succ_i \left(\hat{z}_{i'}, \hat{h}_{i'}\right) \rightarrow p^* \cdot z' + p_{h_i}^* \cdot h_{i'}' > p^* \cdot x_{i'}^* + p_{h_{i'}}^* \cdot h_{i'}^* = p^* \cdot e_{i'} + \theta_{i'} \cdot \left(\sum_{i\in I} p_{h_i}^* \cdot h_i^* - p^* \cdot z^*\right)$

By the fact above and l.n.s. of $\succsim_i$, for $\forall i \neq i'$,

$(z', h_i') \succsim_i \left(\hat{z}_i, \hat{h}_i\right) \rightarrow p^* \cdot z' + p_{h_i'}^* \cdot h_i' \geq p^* \cdot x_i^* + p_{h_i'}^* \cdot h_i^* = p^* \cdot e_i + \theta_i \cdot \left(\sum_{i\in I} p_{h_i'}^* \cdot h_i^* - p^* \cdot z^*\right)$

Therefore, $p^* \cdot \sum_{i\in I} z_i' + p_{h_i'}^* \cdot \sum_{i\in I} h_i' > p^* \cdot \sum_{i\in I} e_i + \sum_{i\in I} \theta_i \cdot \left(\sum_{i\in I} p_{h_i'}^* \cdot h_i^* - p^* \cdot z^*\right)$

$\rightarrow p^* \cdot \sum_{i\in I} z_i' + p_{h_i'}^* \cdot \sum_{i\in I} h_i' > p^* \cdot \sum_{i\in I} e_i + \left(\sum_{i\in I} p_{h_i'}^* \cdot h_i^* - p^* \cdot z^*\right)$

$p^* \cdot \left(\sum_{i\in I} e_i - z'\right) + p_{h_i'}^* \cdot \sum_{i\in I} h_i' > p^* \cdot \sum_{i\in I} e_i + \left(\sum_{i\in I} p_{h_i'}^* \cdot h_i^* - p^* \cdot z^*\right)$

From the feasibility, $\rightarrow p_{h_i'}^* \cdot \sum_{i\in I} h_i' - p^* \cdot z' > \sum_{i\in I} p_{h_i'}^* \cdot h_i^* - p^* \cdot z^*$

$\rightarrow p_{h_i'}^* \cdot \sum_{i\in I} h' - p^* \cdot z' > \sum_{i\in I} p_{h_i'}^* \cdot h^* - p^* \cdot z^*$

It contradicts that $\{z^*, h^*\}$ solves firm's maximization problem.

2. PSet #5-1, Midterm

$L = 2$; x is a private good, h is nonrivalrous public good

$I = 2$

$u_i\left(x_i, h_i\right) = x_i + a_i\sqrt{h_i}$ for $\forall i = 1,2$

$e_i = (10,0)$ for $\forall i = 1,2$

$\theta_i = \frac{1}{2}$ for $\forall i = 1,2$

$J = 1$

$Y = \left\{(-y,h) \in R^2 \mid y \geq 0, h \leq \sqrt{y}\right\}$

(a) Find CE with private provision of public good

A CE with private provision of public good in E consists of $\left\{\{x_i^*, h_i^*\}_{i\in I}, \{y^*, h^*\}\right\}$and $\{p^*, p_h^*\} \in R_+^2 \backslash \{0\}$ s.t

i. $\{y^*, h^*\}$ solves

$\max_{(z,h)} \boxed{p_h^* \cdot h} - p^* \cdot y$

s.t. $h \leq \sqrt{y}$ and $y \geq 0$

Since both utility function are strictly monotone, $p^* \in R_{++}^2$.

Normalize $p_1^* = 1$

$\max_{(z,h)} \boxed{p_h^* \cdot h} - y$

s.t. $h \leq \sqrt{y}$ and $y \geq 0$

$p_h \cdot h - y \leq p_h \cdot \sqrt{y} - y = -\left(\sqrt{y} - \frac{p_h}{2}\right)^2 + \left(\frac{p_h}{2}\right)^2$

$\pi\left(p^*\right)$ is maximized as $\frac{p_h^2}{4}$ when $y = \frac{p_h^2}{4}, h = \frac{p_h}{2}$

ii. For $\forall i \in I$, $\{x_i^*, h_i^*\}_{i\in I}$ solves

$\max_{(x_i,h_i)} x_i + a_i\sqrt{h_i}$

s.t. $p^* \cdot x_i + p_h^* \cdot h_i \leq p^* \cdot e_i + \theta_i \cdot \left(\boxed{p_h^* \cdot h^*} + p^* \cdot z^*\right)$

Because $a_1 < a_2$, $MU_1 < MU_2$ so that only agent 2 buys public good and consumer 1 will free ride.

$\boxed{\text{Consumer 2}}$

$\max_{(x_2,h_2)} x_2 + 4\sqrt{h_2}$

s.t. $x_2 + p_h \cdot h_2 \leq 10 + \frac{p_h^2}{8}$

$x_2 + 4\sqrt{h_2} \leq 10 + \frac{p_h^2}{8} - p_h \cdot h_2 + 4\sqrt{h_2}$

$= -p_h\left(h_2 - \frac{4}{p_h}\sqrt{h_2} + \frac{4}{p_h^2} - \frac{4}{p_h^2}\right) + 10 + \frac{p_h^2}{8}$

$= -p_h\left(\sqrt{h_2} - \frac{2}{p_h}\right)^2 + 10 + \frac{p_h^2}{8}$

Therefore, he maximizes his utility when $h_2 = \frac{4}{p_h^2}$

$x_2 = 10 + \frac{p_h^2}{8} - p_h \cdot h_2 = 10 + \frac{p_h^2}{8} - \frac{4}{p_h}$

$\boxed{\text{Consumer 1}}$

$\max_{(x_i,h_i)} x_1$

s.t. $x_1 \leq 10 + \frac{p_h^2}{8}$

she maximizes her utility when $x_1 = 10 + \frac{p_h^2}{8}$

iii. $\boxed{\sum_{i \in I} x_i^* \leq \sum_{i \in I} e_i + y^*}$

$10 + \frac{p_h^2}{8} + 10 + \frac{p_h^2}{8} - \frac{4}{p_h} \leq 20 - \frac{p_h^2}{4}$

$\frac{p_h^2}{2} \leq \frac{4}{p_h} \rightarrow p_h \leq 2$

$\boxed{\sum_{i \in I} h_i^* = h^*}$

$\frac{4}{p_h^2} \leq \frac{p_h}{2} \rightarrow p_h \geq 2$

Therefore, $p_h = 2$

$\pi(p^*) = 1$ when $y = 1, h = 1$

$h_2 = 1, x_2 = 8.5$

$x_1 = 10.5$

$u_1(10.5, 1) = 12.5$

$u_1(8.5, 1) = 12.5$

(b) Is it pareto optimal? NO!!!

To find Pareto optimal allocation,

$\max x_1 + 2\sqrt{h}$

s.t. $x_2 + 4\sqrt{h} \geq 12.5$ for $\forall i \neq 1$

$x_1 + x_2 \leq 20 - y$

$h \leq \sqrt{y}$

First order conditions will be(if the solution is interior)

$$\sum_{i \in I} \frac{\frac{\partial u_i}{\partial h_i}}{\frac{\partial u_i}{\partial x_i^l}} = \frac{\frac{1}{\sqrt{h}}}{1} + \frac{\frac{2}{\sqrt{h}}}{1} = \frac{1}{\frac{1}{2\sqrt{y}}} = \frac{\frac{\partial f}{\partial h}}{\frac{\partial f}{\partial y}}$$

$$\frac{1}{\sqrt{h}} + \frac{2}{\sqrt{h}} = 2\sqrt{y} = 2h$$

$$h = \left(\frac{3}{2}\right)^{\frac{2}{3}}$$

$$y = \left(\frac{3}{2}\right)^{\frac{4}{3}}$$

$$x_2 \geq 12.5 - 4\sqrt{\left(\frac{3}{2}\right)^{\frac{2}{3}}} = 12.5 - 4\left(\frac{3}{2}\right)^{\frac{1}{3}} \risingdotseq 7.921$$

$$x_1 = 20 - y - x_2 = 20 - \left(\frac{3}{2}\right)^{\frac{4}{3}} - 7.921 \risingdotseq 10.362$$

$$u_1\left(10.362, \left(\frac{3}{2}\right)^{\frac{2}{3}}\right) = 10.362 + 2\left(\frac{3}{2}\right)^{\frac{2}{3}} \risingdotseq 12.982 > 12.5$$

$$u_1\left(7.921, \left(\frac{3}{2}\right)^{\frac{2}{3}}\right) = 7.921 + 4\left(\frac{3}{2}\right)^{\frac{2}{3}} \risingdotseq 13.161 > 12.5$$

(c) A Lindahl equilibrium is a price equilibrium with transfers in E^{public} with personalized commodities.

Let the shares of the firm of consumers be $\theta_i \in [0, 1]$ for $\forall i = 1, 2$ and $\sum_{i=1,2} \theta_i = 1$.

It consists of $\left\{\{x_i^*, h_i^*\}_{i=1,2}, \{y^*, h^*\}\right\}$and $\{p_x^*, p_{h_i}^*\} \in R_+^2 \setminus \{0\}$ s.t

i. For $\forall i = 1, 2$, $\{x_i^*, h_i^*\}_{i=1,2}$ solves
$\max_{(x_i,h_i)} u_i\left(x_i, h_i\right)$
s.t. $p^* \cdot x_i + p_{h_i}^* \cdot h_i \leq p^* \cdot e_i + \theta_i \cdot \left(-p^* \cdot y^* + \sum_{i=1,2} p_{h_i}^* \cdot h^*\right)$

ii. $\{y^*, h^*\}$ solves
$\max_{(y,h)} -p^* \cdot y + \sum_{i=1,2} p_{h_i}^* \cdot h$
s.t. $h \leq \sqrt{y}$ and $y \geq 0$

iii. $\sum_{i=1,2} x_i^* \leq \sum_{i=1,2} e_i + y^*$

iv. $h_i^* = h^*$ for $\forall i = 1, 2$

3. Fall 1999 III-1

$L = 2$; x is a private good, h is nonrivalrous public good

$I = 2$

$u_i\left(x_i, h_i\right) = x_i + a_i\sqrt{h_i}$ for $\forall i = 1, 2$

$a_1 = 3, a_2 = 1$

$e_i = (3, 0), e_2 = (2, 0)$

$\theta_i = \frac{1}{2}$ for $\forall i = 1, 2$

$J = 1$

$Y = \left\{(-y, h) \in R^2 \mid y \geq 0, h \leq \sqrt{y}\right\}$

(a) $x_1 = x_2 = 2, y = 1, h = 1$
Is it pareto optimal?
To find Pareto optimal allocation,
$\max x_1 + 3\sqrt{h}$
s.t. $x_2 + \sqrt{h} \geq u_2$ for $\forall i \neq 1$
$x_1 + x_2 \leq 5 - y$
$h \leq \sqrt{y}$
First order conditions for Pareto optimality will be(if the solution is interior)
$\sum_{i \in I} \frac{\frac{\partial u_i}{\partial h_i}}{\frac{\partial u_i}{\partial x_i^l}} = \frac{\frac{3}{2\sqrt{h}}}{1} + \frac{\frac{1}{2\sqrt{h}}}{1} = \frac{1}{\frac{1}{2\sqrt{y}}} = \frac{\frac{\partial f}{\partial h}}{\frac{\partial f}{\partial y}}$
$\frac{3}{2\sqrt{h}} + \frac{1}{2\sqrt{h}} = 2\sqrt{y} = 2h$
$h = 1$
$y = 1$
$x_2 \geq u_2 - 1$
When $u_2 = 3$, $x_2 = 2$
$x_1 = 5 - 1 - x_2 = 2$

Because utility function is strictly monotone and continuous, any allocation that is a solution to this maximization problem is Pareto optimal.

(b) Find CE with private provision of public good

A CE with private provision of public good in E consists of $\left\{\{x_i^*, h_i^*\}_{i\in I}, \{y^*, h^*\}\right\}$and $\{p^*, p_h^*\} \in R_+^2 \backslash \{0\}$ s.t

$\{y^*, h^*\}$ solves

$\max_{(z,h)} \boxed{p_h^* \cdot h} - p^* \cdot y$

s.t. $h \le \sqrt{y}$ and $y \ge 0$

For $\forall i \in I$, $\{x_i^*, h_i^*\}_{i\in I}$ solves

$\max_{(x_i,h_i)} x_i + a_i\sqrt{h_i}$

s.t. $p^* \cdot x_i + p_h^* \cdot h_i \le p^* \cdot e_i + \theta_i \cdot (\boxed{p_h^* \cdot h^*} + p^* \cdot y^*)$

$\boxed{\sum_{i\in I} x_i^* \le \sum_{i\in I} e_i + y^*}$

$\boxed{\sum_{i\in I} h_i^* = h^*}$

i. $\{y^*, h^*\}$ solves

$\max_{(z,h)} \boxed{p_h^* \cdot h} - p^* \cdot y$

s.t. $h \le \sqrt{y}$ and $y \ge 0$

Since both utility function are strictly monotone, $p^* \in R_{++}^2$.

Normalize $p_1^* = 1$

$\max_{(z,h)} \boxed{p_h^* \cdot h} - y$

s.t. $h \le \sqrt{y}$ and $y \ge 0$

$p_h \cdot h - y \le p_h \cdot \sqrt{y} - y = -\left(\sqrt{y} - \frac{p_h}{2}\right)^2 + \left(\frac{p_h}{2}\right)^2$

$\pi(p^*)$ is maximized as $\frac{p_h^2}{4}$ when $y = \frac{p_h^2}{4}, h = \frac{p_h}{2}$

ii. For $\forall i \in I$, $\{x_i^*, h_i^*\}_{i\in I}$ solves

$\max_{(x_i,h_i)} x_i + a_i\sqrt{h_i}$

s.t. $p^* \cdot x_i + p_h^* \cdot h_i \le p^* \cdot e_i + \theta_i \cdot (\boxed{p_h^* \cdot h^*} + p^* \cdot z^*)$

Because $a_1 > a_2$, $MU_1 > MU_2$ so that only agent 1 buys public good and consumer 2 will free ride.

$\boxed{\text{Consumer 1}}$

$\max_{(x_i,h_i)} x_1 + 3\sqrt{h_1}$

s.t. $x_1 + p_h \cdot h_1 \le 3 + \frac{p_h^2}{8}$

$x_1 + 3\sqrt{h_1} \le 3 + \frac{p_h^2}{8} - p_h \cdot h_1 + 3\sqrt{h_1}$

$= -p_h\left(h_1 - \frac{3}{p_h}\sqrt{h_1} + \frac{9}{4p_h^2} - \frac{9}{4p_h^2}\right) + 3 + \frac{p_h^2}{8}$

$= -p_h\left(\sqrt{h_1} - \frac{3}{2p_h}\right)^2 + 3 + \frac{p_h^2}{8}$

Therefore, he maximizes his utility when $h_1 = \frac{9}{4p_h^2}$

$x_1 = 3 + \frac{p_h^2}{8} - p_h \cdot h_1 = 3 + \frac{p_h^2}{8} - \frac{9}{4p_h}$

$\boxed{\text{Consumer 2}}$

$\max_{(x_2,h_2)} x_2$

s.t. $x_2 \leq 2 + \frac{p_h^2}{8}$

she maximizes her utility when $x_2 = 2 + \frac{p_h^2}{8}$

iii. $\boxed{\sum_{i \in I} x_i^* \leq \sum_{i \in I} e_i + y^*}$

$2 + \frac{p_h^2}{8} + 3 + \frac{p_h^2}{8} - \frac{9}{4p_h} \leq 5 - \frac{p_h^2}{4}$

$\frac{p_h^2}{2} \leq \frac{9}{4p_h} \rightarrow p_h^3 \leq \frac{9}{2}$

$\boxed{\sum_{i \in I} h_i^* = h^*}$

$\frac{9}{4p_h^2} \leq \frac{p_h}{2} \rightarrow p_h^3 \geq \frac{9}{2}$

Therefore, $p_h = \left(\frac{9}{2}\right)^{\frac{1}{3}}$

$\pi(p^*)$ is maximized as $\frac{\left(\frac{9}{2}\right)^{\frac{2}{3}}}{4}$ when $y = \frac{\left(\frac{9}{2}\right)^{\frac{2}{3}}}{4}, h = \frac{\left(\frac{9}{2}\right)^{\frac{1}{3}}}{2}$

$h_2 = \frac{9}{4\left(\frac{9}{2}\right)^{\frac{2}{3}}}$,

$x_2 = 3 + \frac{p_h^2}{8} - \frac{9}{4p_h} = 3 + \frac{\left(\frac{9}{2}\right)^{\frac{2}{3}}}{8} - \frac{9}{4\left(\frac{9}{2}\right)^{\frac{1}{3}}}$

$x_1 = 5 - \frac{\left(\frac{9}{2}\right)^{\frac{2}{3}}}{4} - 3 - \frac{\left(\frac{9}{2}\right)^{\frac{2}{3}}}{8} + \frac{9}{4\left(\frac{9}{2}\right)^{\frac{1}{3}}}$

iv. Therefore, $x_1 = x_2 = 2, y = 1, h = 1$ can not be implemented as equilibrium allocation with private provision of the public good

(c) A Lindahl equilibrium is a price equilibrium with transfers in E^{public} with personalized commodities.

Let the shares of the firm of consumers be $\theta_i \in [0,1]$ for $\forall i = 1,2$ and $\sum_{i=1,2} \theta_i = 1$.

It consists of $\left\{ \{x_i^*, h_i^*\}_{i=1,2}, \{y^*, h^*\} \right\}$and $\left\{p_x^*, p_{h_i}^*\right\} \in R_+^3 \setminus \{0\}$ s.t

For $\forall i = 1,2$, $\{x_i^*, h_i^*\}_{i=1,2}$ solves

$\max_{(x_i,h_i)} u_i(x_i, h_i)$

s.t. $p^* \cdot x_i + p_{h_i}^* \cdot h_i \leq p^* \cdot e_i + \theta_i \cdot \left(\sum_{i=1,2} p_{h_i}^* \cdot h^* - p^* \cdot y^*\right)$

$\{y^*, h^*\}$ solves

$\max_{(y,h)} \sum_{i=1,2} p_{h_i}^* \cdot h - p^* \cdot y$

s.t. $h \leq \sqrt{y}$ and $y \geq 0$

$\sum_{i=1,2} x_i^* \leq \sum_{i=1,2} e_i + y^*$

$h_i^* = h^*$ for $\forall i = 1,2$

i. $\{y^*, h^*\}$ solves

$\max_{(z,h)} \sum_{i=1,2} p^*_{h_i} \cdot h - p^* \cdot y$

s.t. $h \leq \sqrt{y}$ and $y \geq 0$

Since both utility function are strictly monotone, $p^* \in R^2_{++}$.

Normalize $p^* = 1$

$\max_{(z,h)} (p^*_{h_1} + p^*_{h_2}) \cdot h - y$

s.t. $h \leq \sqrt{y}$ and $y \geq 0$

$$(p^*_{h_1} + p^*_{h_2}) \cdot h - y \leq (p^*_{h_1} + p^*_{h_2}) \cdot \sqrt{y} - y = -\left(\sqrt{y} - \frac{(p^*_{h_1}+p^*_{h_2})}{2}\right)^2 + \left(\frac{(p^*_{h_1}+p^*_{h_2})}{2}\right)^2$$

$\pi(p^*)$ is maximized as $\frac{(p^*_{h_1}+p^*_{h_2})^2}{4}$ when $y = \frac{(p^*_{h_1}+p^*_{h_2})^2}{4}$, $h = \frac{(p^*_{h_1}+p^*_{h_2})}{2}$

ii. For $\forall i \in I$, $\{x^*_i, h^*_i\}_{i \in I}$ solves

$\max_{(x_i,h_i)} x_i + a_i\sqrt{h_i}$

s.t. $p^* \cdot x_i + p^*_{h_i} \cdot h_i \leq p^* \cdot e_i + \theta_i \cdot \left(\sum_{i=1,2} p^*_{h_i} \cdot h^* - p^* \cdot y^*\right)$

$\boxed{\text{Consumer 1}}$

$\max_{(x_i,h_i)} x_1 + 3\sqrt{h_1}$

s.t. $x_1 + p_{h_1} \cdot h_1 \leq 3 + \frac{(p^*_{h_1}+p^*_{h_2})^2}{8}$

FOC

$x_1 : 1 - \lambda = 0$

$h_1 : \frac{3}{2}\frac{1}{\sqrt{h_1}} - \lambda p_{h_1} = 0$

$\frac{3}{2}\frac{1}{\sqrt{h_1}} = p_{h_1}$

$\boxed{\text{Consumer 2}}$

$\max_{(x_2,h_2)} x_2 + \sqrt{h_2}$

s.t. $x_2 + p_{h_2} \cdot h_2 \leq 2 + \frac{(p^*_{h_1}+p^*_{h_2})^2}{8}$

FOC

$x_2 : 1 - \delta = 0$

$h_2 : \frac{1}{2}\frac{1}{\sqrt{h_2}} - \delta p_{h_2} = 0$

$\frac{1}{2}\frac{1}{\sqrt{h_2}} = p_{h_2}$

iii. $\boxed{h^*_i = h^* \text{ for } \forall i = 1, 2}$

$\boxed{\sum_{i \in I} x^*_i \leq \sum_{i \in I} e_i + y^*}$

$\frac{\frac{3}{2}\frac{1}{\sqrt{h^*}} + \frac{1}{2}\frac{1}{\sqrt{h^*}}}{2} = h^*$

$\frac{2}{\sqrt{h^*}} = 2h^*$

$h^*\sqrt{h^*} = 1$

$h^* = 1 \rightarrow y^* = 1$

$\rightarrow p^*_{h_1} = 1.5$

$\rightarrow p^*_{h_2} = 0.5$

$\rightarrow x_1^* = 2$
$\rightarrow x_2^* = 2$
Or
$\sum_{i \in I} \frac{\frac{\partial u_i}{\partial h_i}}{\frac{\partial u_i}{\partial x_i^l}} = \frac{\frac{3}{2\sqrt{h}}}{1} + \frac{\frac{1}{2\sqrt{h}}}{1} = \frac{1}{\frac{1}{2\sqrt{y}}} = \frac{\frac{\partial f}{\partial h}}{\frac{\partial f}{\partial y}}$
$\frac{3}{2\sqrt{h}} + \frac{1}{2\sqrt{h}} = 2\sqrt{y} = 2h$
$h = 1$
$y = 1$

iv. Therefore, $x_1 = x_2 = 2, y = 1, h = 1$ can be implemented as Lindahl Equilibrium

8 General Equilibrium with Externality

1. Externality

   An externality is present whenever the well-being of a consumer or the production possibilities of a firm affected by the actions of another agent in the economy.

2. Environment

   In $E = \left[\left\{I, (\succsim_i, e_i)_{i\in I}\right\}, Y\right]$ s.t.

   $X_i = R_+^L$ for $\forall i$ and $X = \{X_i\}_{i\in I}$

   $$\succsim_i \begin{cases} \text{l.n.s.} \\ \text{(weakly)convex} \quad \text{represented by } u_i\left(x_i, h\right) \text{ s.t } \frac{\partial u_i}{\partial h} < 0 \\ \text{continuous} \end{cases}$$

   $e_i \in R_+^L \Rightarrow \sum_{i\in I} e_i >> 0$

   a single firm with production set $Y \subset R^L$ represented by a transformation function $F\left(y\right)$ (or a production function $f\left(z\right)$) s.t. $\frac{\partial F}{\partial h} < 0, \frac{\partial F}{\partial y} > 0$

3. Assumptions

   (a) $\succsim_i$is $\boxed{\text{strictly monotone}}$, $\boxed{\text{convex}}$, continuous, complete preorder on $R^L \rightarrow u_i\left(x_i\right)$ is C^2 and quasiconcave, $u_i'\left(x_i\right) >> 0$, and $u_i\left(0\right) = 0$

   (b) $F\left(y_j\right) = 0$ at transformation frontier, $F\left(y_j\right)$ is C^2, $F(0) \leq 0$, $F'\left(y\right) > 0$, and a $\boxed{\text{convex function}}$(production sets Y is convex)

4. Pareto optimality under externality

   Consider the following social planner's problem

   $\max \sum_{ieI} \lambda_i u_i\left(x_i, h\right)$

   s.t. $\sum_{i\in I} x_i \leq \sum_{i\in I} e_i + y$

   $F\left(y, h\right) \leq 0$

   $x_i \geq 0$ for $\forall i \in I$

   - First order condition for pareto optimality
     If the solutioin is interior, it satisfies
     $$\sum_{i\in I} \frac{\frac{\partial u_i}{\partial h}}{\frac{\partial u_i}{\partial x_i^l}} = \frac{\frac{\partial F}{\partial h}}{\frac{\partial F}{\partial z^l}} \Leftrightarrow \sum_{i\in I} MRS_i\left(x_i^l, h\right) = MRT\left(z^l, h\right)$$

5. Equilibrium with No regulation

- Example of No regulation(normal price equilibrium is $\boxed{\text{not}}$ P.O.)
  We can get a result that a normal equilibrium concept with transfers can not be pareto optimal

(a) An equilibrium with no regulation in E consists of $\left\{\{x_i^*\}_{i\in I}, \{y^*, h^*\}\right\}$and $\{p^*\} \in R_+^L \setminus \{0\}$ s.t

  i. For $\forall i \in I$, $\{x_i^*\}_{i\in I}$ solves
     $\max_{(x_i)} u_i\left(x_i, h^*\right)$
     s.t. $p^* \cdot x_i \leq p^* \cdot e_i + \theta_i \cdot p^* \cdot y^*$

  ii. $\{z^*, h^*\}$ solves
     $\max_{(z,h)} p^* \cdot y$
     s.t. $F\left(y, h\right) \leq 0$

  iii. $\sum_{i\in I} x_i^* \leq \sum_{i\in I} e_i + y^*$

(b) FOC

  i. First order conditions
     If the solution is interior,
     $\frac{\frac{\partial u_i}{\partial x_k}}{\frac{\partial u_i}{\partial x_i^l}} = \frac{p_k}{p_l} = \frac{F_k}{F_l}$ for $\forall i \in I$
     $\frac{\partial F(y,h^*)}{\partial h} = 0$ which is different from FOC.
     Therefore, equilibrium with no regulation is not pareto optimal(FWT fails.)

6. Remedies

(a) Pigouvian Tax
  Impose a tax to the firm at t_h per unit of externality.
  An equilibrium with Pigouvian tax in E consists of $\left\{\{x_i^*\}_{i\in I}, \{y^*, h^*\}\right\}$and $\{p^*\} \in R_+^L \setminus \{0\}$ s.t

  i. For $\forall i \in I$, $\{x_i^*\}_{i\in I}$ solves
     $\max_{(x_i)} u_i\left(x_i, h^*\right)$
     s.t. $p^* \cdot x_i \leq p^* \cdot e_i + \theta_i \cdot p^* \cdot y^*$

  ii. $\{z^*, h^*\}$ solves
     $\max_{(z,h)} p^* \cdot y - t_h \cdot h$
     s.t. $F\left(y, h\right) \leq 0$

  iii. $\sum_{i\in I} x_i^* \leq \sum_{i\in I} e_i + y^*$

  FOC

  i. First order conditions
     If the solution is interior,

$\frac{\frac{\partial u_i}{\partial x_k}}{\frac{\partial u_i}{\partial x_i^l}} = \frac{p_k}{p_l} = \frac{F_k}{F_l}$ for $\forall i \in I$

$\frac{t_h}{p_l} = -\frac{\frac{\partial F(y,h^*)}{\partial h}}{\frac{\partial F(y,h^*)}{\partial y_l}}$

In order for equilibrium with Pigouvian tax to be P.O. with externality, .

$\sum_{i\in I} \frac{\frac{\partial u_i}{\partial h}}{\frac{\partial u_i}{\partial x_i^l}} = \frac{\frac{\partial F}{\partial h}}{\frac{\partial F}{\partial z^l}} = -\frac{t_h}{p_l}$

Therefore, if we set pigouvian tax rate to satisfy the above equation, equilibrium with pigouvian tax is P.O.(FWT)

And any P.O. allocation can be decentralized as an equilibrium with a suitable pigouvian tax rate on the firm and lump-sum transfer to consumers.

(b) Market for externality

Assign Property right γ_i to consumer i s.t. $\sum_{i\in I} \gamma_i = 1$

Then an equilibrium with Property right Market in E consists of $\left\{\{x_i^*, h_i^*\}_{i\in I}, \{y^*, h^*\}\right\}$and $\{p^*, p_h^*\} \in R_+^L \setminus \{0\}$ s.t

i. For $\forall i \in I$, $\{x_i^*, h_i^*\}_{i\in I}$ solves
   $\max_{(x_i)} u_i(x_i, h_i)$
   s.t. $p^* \cdot x_i \le p^* \cdot e_i + \theta_i \cdot (p^* \cdot y^* - p_h^* \cdot h^*) + \gamma_i \cdot p_h^* \cdot h_i$

ii. $\{z^*, h^*\}$ solves
   $\max_{(z,h)} p^* \cdot y - p_h^* \cdot h$
   s.t. $F(y, h) \le 0$

iii. $\sum_{i\in I} x_i^* \le \sum_{i\in I} e_i + y^*$
   $\sum_{i\in I} h_i^* = h^*$

FOC

i. First order conditions

If the solution is interior,

$\sum_{i\in I} \frac{\frac{\partial u_i}{\partial h_i}}{\frac{\partial u_i}{\partial x_i^l}} = \frac{\frac{\partial F}{\partial h}}{\frac{\partial F}{\partial z^l}} = \frac{p_h}{p_l}$

Therefore, if we open market for externality, equilibrium with market for externality is P.O.(FWT)

8.1 Examples

1. PSet #4-2

   $L = 2$

   $I = 1$

   $u(x_1, x_2) = x_1 x_2$

$e = (4, 1)$
$J = 2$
$Y_1 = \left\{ \left(y_1^1, y_2^1\right) \in R^2 \mid y_1^1 \leq 0, y_2^1 \leq -y_1^1 + s_1, s_1 \in [0, 1] \right\}$
$Y_2 = \left\{ \left(y_1^2, y_2^2\right) \in R^2 \mid y_1^2 \leq 0, y_2^2 \leq -y_1^2 \left(2 - s_1\right) \right\}$
1) Find a CE with no regulation

2) Suppose the consumer decides to shut down firm 1 and operate only firm 2. Find an equilibrium

3) Verify whether equilibrium allocation a) and b) are pareto optimal. Find all pareto optimal allocations

4) If two firms merge, for any level of $y_1 \leq 0$, find the maximal level of output of y_2. Does it have a convex production set?

5) Find a Marginal cost pricing equilibrium in the equilibrium with merged firm.

(a) Find a CE with no regulation

   An equilibrium with no regulation in E consists of $\left\{ \{x^*\}, \left\{ y_j^*, s_j^* \right\}_{j \in J} \right\}$ and $\{p^*\} \in R_+^L \setminus \{0\}$ s.t
   $\{x^*\}$ solves
   $\max_{(x)} u(x)$
   s.t. $p^* \cdot x \leq p^* \cdot e + p^* \cdot y_1^* + p^* \cdot y_2^*$
   For $j = 1, \{y_1^*, s_1^*\}$ solves
   $\max_{(y,s)} p_1^* \cdot y_1^1 + p_2^* \cdot y_2^1$
   s.t. $y_2^1 \leq -y_1^1 + s_1, y_1^1 \leq 0, s_1 \in [0, 1]$
   For $j = 2, \{y_2^*\}$ solves
   $\max_{(y,s)} p_1^* \cdot y_1^2 + p_2^* \cdot y_2^2$
   s.t. $y_2^2 \leq -y_1^2 \left(2 - s_1\right), y_1^2 \leq 0$
   $\sum_{i \in I} x_i^* \leq \sum_{i \in I} e_i + y^*$

   i. $\pi(p)$
      Normalize $p_1 = 1$ and $p_2 = p$.
      For $j = 1, \{y_1^*, s_1^*\}$ solves
      $\max_{(y,s)} y_1^1 + p \cdot y_2^1$
      s.t. $y_2^1 \leq -y_1^1 + s_1, y_1^1 \leq 0, s_1 \in [0, 1]$
      $y_1^1 + p \cdot y_2^1 \leq y_1^1 + p \cdot \left(-y_1^1 + s_1\right) = y_1^1 (1 - p) + p \cdot s_1$
      In order to maximize the profit, this firm has to emit $s_1 = 1$.
      If $1 < p$, then as $y_1^1 \to -\infty$, $\pi_1(p) \to \infty$ which is not possible in equilibrium.
      Therefore, $p \leq 1$.

ii. $\boxed{\text{case 1) } p = 1}$

$\pi_1(p) = y_1^1 + y_2^1 \leq y_1^1 + (-y_1^1 + 1) = 1$ and $y_2^1 = -y_1^1 + 1$

For $j = 2$, $\{y_2^*\}$ solves

$\max_{(y,s)} y_1^2 + y_2^2$

s.t. $y_2^2 \leq -y_1^2, y_1^2 \leq 0$

$\pi_2(p) = y_1^2 + y_2^2 \leq y_1^2 - y_1^2 = 0$

Therefore, $\pi_2(p) = 0$ and $y_2^2 = -y_1^2$

$\max_{(x)} x_1 x_2$

s.t. $x_1 + x_2 \leq 5 + 1 + 0$

$x_1 x_2 \leq x_1(6 - x_1) = -(x_1 - 3)^2 + 9$

Therefore, $x_1 = 3, x_2 = 3$

A CE with no regulation is $\left\{(1,1); x = (3,3), y_2^1 = -y_1^1 + 1, y_1^1 \leq 0, y_2^2 = -y_1^2, y_1^2 \leq 0\right\}$

iii. $\boxed{\text{case 2) } p < 1}$

$\pi_1(p) = y_1^1 + p \cdot y_2^1 \leq y_1^1 + p \cdot (-y_1^1 + s_1) = y_1^1(1 - p) + p$

$y_1^1 = 0$ and $\pi_1(p) = p, y_2^1 = 1$

For $j = 2$, $\{y_2^*\}$ solves

$\max_{(y,s)} y_1^2 + p y_2^2$

s.t. $y_2^2 \leq -y_1^2, y_1^2 \leq 0$

$\pi_2(p) = y_1^2 + p y_2^2 \leq y_1^2 - p y_1^2 = (1 - p) y_1^2$

$y_1^2 = 0$ and $\pi_2(p) = 0, y_2^2 = 0$

$\max_{(x)} x_1 x_2$

s.t. $x_1 + p x_2 \leq 4 + p + p + 0$

$x_1 x_2 \leq x_2(4 + 2p - p x_2) = -p\left(x_2 - \left(\frac{2+p}{p}\right)\right)^2 + \frac{(2+p)^2}{p}$

Therefore, $x_2 = \frac{2+p}{p}, x_1 = 2 + p$

Feasibility

$x_1 = e_1 + y_1^1 + y_1^2 = 4 = 2 + p \rightarrow p = 2$ which is impossible because $p < 1$!!!

iv. Therefore, it is only equilibrium $\left\{(1,1); x = (3,3), y_2^1 = -y_1^1 + 1, y_1^1 \leq 0, y_2^2 = -y_1^2, y_1^2 \leq 0\right\}$

(b) Suppose the consumer decides to shut down firm 1 and operate only firm 2. Find an equilibrium

An equilibrium with no regulation in E consists of $\{x^*\}, \{y^*, s^*\}$and $\{p^*\} \in R_+^L \backslash \{0\}$ s.t

$\{x^*\}$ solves

$\max_{(x)} u(x)$

s.t. $p^* \cdot x \leq p^* \cdot e + p^* \cdot y_2^*$

For $j = 2$, $\{y_2^*\}$ solves

$\max_{(y,s)} p_1^* \cdot y_1^2 + p_2^* \cdot y_2^2$

s.t. $y_2^2 \leq -y_1^2(2 - s_1), y_1^2 \leq 0$

$\sum_{i \in I} x_i^* \leq \sum_{i \in I} e_i + y_1^* + y_2^*$

i. $\pi(p)$

Normalize $p_1 = 1$ and $p_2 = p$.
For $j = 2$, $\{y_2^*\}$ solves
$\max_{(y,s)} y_1^2 + p y_2^2$
s.t. $y_2^2 \leq -y_1^2 (2 - s_1), y_1^2 \leq 0$
First, $s_1 = 0$.
$\pi_2(p) = y_1^2 + p y_2^2 \leq y_1^2 - p \cdot 2 y_1^2 = (1 - 2p) y_1^2$
If $\frac{1}{2} < p$, then as $y_1^2 \to -\infty$, $\pi_2(p) \to \infty$ which is not possible in equilibrium.
Therefore, $p \leq \frac{1}{2}$.

ii. $\boxed{\text{case 1) } p = \frac{1}{2}}$

$\max_{(y,s)} y_1^2 + \frac{1}{2} y_2^2$
s.t. $y_2^2 \leq -2 y_1^2, y_1^2 \leq 0$
$\pi_2(p) = y_1^2 + \frac{1}{2} y_2^2 \leq y_1^2 - y_1^2 = 0$
Therefore, $\pi_2(p) = 0$ and $y_2^2 = -2 y_1^2$
$\max_{(x)} x_1 x_2$
s.t. $x_1 + \frac{1}{2} x_2 \leq 4.5 + 0$
$x_1 x_2 \leq x_2 \left(4.5 - \frac{1}{2} x_2\right) = -\frac{1}{2}\left(x_2 - \frac{9}{2}\right)^2 + \frac{81}{8}$
Therefore, $x_1 = \frac{9}{4}, x_2 = \frac{9}{2}$
$x_1 = e_1 + y_1^2 = 4 + y_1^2 \to y_1^2 = -\frac{7}{4}$
$x_2 = e_2 + y_2^2 = 1 + y_2^2 \to y_2^2 = \frac{7}{2}$
A CE with no regulation is $\left\{\left(1, \frac{1}{2}\right); x = \left(\frac{9}{4}, \frac{9}{2}\right), y_2 = \left(-\frac{7}{4}, \frac{7}{2}\right)\right\}$

iii. $\boxed{\text{case 2) } p < \frac{1}{2}}$

For $j = 2$, $\{y_2^*\}$ solves
$\max_{(y,s)} y_1^2 + p y_2^2$
s.t. $y_2^2 \leq -2 y_1^2, y_1^2 \leq 0$
$\pi_2(p) = y_1^2 + p y_2^2 \leq y_1^2 - p 2 y_1^2 = (1 - 2p) y_1^2$
$y_1^2 = 0$ and $\pi_2(p) = 0, y_2^2 = 0$
$\max_{(x)} x_1 x_2$
s.t. $x_1 + p x_2 \leq 4 + p + 0$
$x_1 x_2 \leq x_2 (4 + p - p x_2) = -p\left(x_2 - \left(\frac{4+p}{2p}\right)\right)^2 + \frac{(4+p)^2}{4p}$
Therefore, $x_2 = \frac{4+p}{2p}, x_1 = \frac{4+p}{2}$
Feasibility
$x_1 = e_1 + y_1^2 = 4 = \frac{4+p}{2p} \to p = \frac{4}{7}$ which is impossible because $p < \frac{1}{2}$!!!

iv. Therefore, it is only equilibrium $\left\{\left(1, \frac{1}{2}\right); x = \left(\frac{9}{4}, \frac{9}{2}\right), y_2 = \left(-\frac{7}{4}, \frac{7}{2}\right)\right\}$

(c) Verify whether equilibrium allocation a) and b) are pareto optimal. Find all pareto optimal allocations

First, the first equilibrium is not Pareto optimal because it is dominated by the second one.

In order to verify whether the second one is pareto optimal or not, we will consider the merged firm case later.

(d) If two firms merge, for any level of $y_1 \leq 0$, find the maximal level of output of y_2. Does it have a convex production set?

$Y_1 = \left\{\left(y_1^1, y_2^1\right) \in R^2 \mid y_1^1 \leq 0, y_2^1 \leq -y_1^1 + s_1, s_1 \in [0,1]\right\} \subset Y_1' = \left\{\left(y_1^1, y_2^1\right) \in R^2 \mid y_1^1 \leq 0, y_2^1 \leq -y$
$Y_2 = \left\{\left(y_1^2, y_2^2\right) \in R^2 \mid y_1^2 \leq 0, y_2^2 \leq -y_1^2 (2 - s_1)\right\} \subset Y_2' = \left\{\left(y_1^2, y_2^2\right) \in R^2 \mid y_1^2 \leq 0, y_2^2 \leq -2y_1^2\right\}$
$\rightarrow Y = \left\{\frac{\left\{(y_1,y_2) \in R^2 | y_1 \leq 0, y_2 \leq -y_1+1\right\} \text{ if } y_1 \in [-1,0]}{\{(y_1,y_2) \in R^2 | y_1 \leq 0, y_2 \leq -2y_1\} \text{ if } y_1 < -1}\right.$
It is not a convex set because $(0,1)$ and $(-2,4)$ included, but $\left(-1, \frac{5}{2}\right)$ is not included.
The equilibrium is the same as only firm 2 case $\left\{\left(1, \frac{1}{2}\right); x = \left(\frac{9}{4}, \frac{9}{2}\right), y_2 = \left(-\frac{7}{4}, \frac{7}{2}\right)\right\}$
Therefore, this equilibrium allocation is only pareto optimal.

(e) Find a Marginal cost pricing equilibrium in the equilibrium with merged firm.

- Note that $F(y) = \left\{\frac{y_2 + y_1 - 1 \text{ if } y_1 \in [-1,0]}{y_2 + 2y_1 \text{ if } y_1 < -1}\right.$
  (a) $y_1 < -1$
  $\boxed{\text{for } \forall j \in J,\ p^* = \gamma_j F_j'\left(y_j^*\right) \text{ for some } \gamma_j > 0 \text{ and } F_j\left(y_j^*\right) = 0}$
  $p^* = \gamma (2, 1)$.
  Let $\gamma = 1$. Then $p_1^* = 2, p_2^* = 1$
  $\pi(p) = 2y_1 + y_2 = 2y_1 - 2y_1 = 0$
  $\boxed{\text{for } \forall i \in I,\ x_i^* \text{ is } \succsim_i - \text{maximal in } B_i(p^*, e_i) = \{x_i \in X_i \mid p^* \cdot x_i \leq w_i\}}$
  $\boxed{\text{Consumer 1}}$
  $\max_{x^1} x_1 x_2$
  s.t. $2x_1^1 + x_2^1 \leq 9$
  $x_1 x_2 \leq x_1 (9 - 2x_1) = -2\left(x_1 - \frac{9}{4}\right)^2 + \frac{81}{8}$
  $x_1 = \frac{9}{4}, x_2 = \frac{9}{2}$
  $x_1 = 4 + y_1 \rightarrow y_1 = -\frac{7}{4}$
  $x_2 = 1 + y_2 \rightarrow y_2 = \frac{7}{2}$
  $\Longrightarrow p_1^* = 2, p_2^* = 1, x = \left(\frac{9}{4}, \frac{9}{2}\right), y = \left(-\frac{7}{4}, \frac{7}{2}\right)$
  (b) $y_1 \in [-1, 0]$
  $\boxed{\text{for } \forall j \in J,\ p^* = \gamma_j F_j'\left(y_j^*\right) \text{ for some } \gamma_j > 0 \text{ and } F_j\left(y_j^*\right) = 0}$
  $p^* = \gamma (1, 1)$.
  Let $\gamma = 1$. Then $p_1^* = 1, p_2^* = 1$
  $\pi(p) = y_1 + y_2 = y_1 - y_1 + 1 = 1$
  $\boxed{\text{for } \forall i \in I,\ x_i^* \text{ is } \succsim_i - \text{maximal in } B_i(p^*, e_i) = \{x_i \in X_i \mid p^* \cdot x_i \leq w_i\}}$

$\boxed{\text{Consumer 1}}$
$\max_x x_1 x_2$
s.t. $x_1^1 + x_2^1 \leq 5 + 1$
$x_1 x_2 \leq x_1 (6 - x_1) = -(x_1 - 3)^2 + 9$
$x_1 = 3, x_2 = 3$
$x_1 = 4 + y_1 \rightarrow y_1 = -1$
$x_2 = 1 + y_2 \rightarrow y_2 = 2$
$\Longrightarrow p_1^* = 1, p_2^* = 1, x = (3, 3), y = (-1, 2)$

(c) But, $p_1^* = 2, p_2^* = 1, x = \left(\frac{9}{4}, \frac{9}{2}\right), y = \left(-\frac{7}{4}, \frac{7}{2}\right)$ gives higher utility for the consumer so that it is MCP equilibrium.

- Verify whether the MCP equilibrium allocation is Pareto optimal
  Which is verified before.

9 Choice Under Uncertainty

In previous chapters, we studied choices that result in perfectly certain outcomes. In reality, many important economic decisions involve an element of risk. Uncertain alternatives have a structure that we can use to restrict the preferences that rational individual holds.

9.1 Preference over risky alternatives and Expected Utility Theorem

Alternatives with uncertain outcomes are describable by means of objective probabilities defined on an abstract set of possible outcomes. These representation of risky alternatives are called lotteries. We assume that the decision maker has a rational preference over these lotteries. We then proceed to derive the expected utility theorem. This theorem say that under certain conditions, we can represent preferences by utility function possesses expected utility form. The key assumption to this theorem is the independence axiom.

9.1.1 Preference over lotteries(risky alternatives)

1. Continuous Complete Preorder $\boxed{\text{Sp 1999 III.2.(c)}}$

   $\Longrightarrow$ (with monotonicity) Continuous Utility Function

   On a set $X = R_+^L$, a complete preorder $\succsim$ is continuous on X,

   if for $\forall\ y \in X$, $\{x \in X \mid x \succsim y\}$ and $\{x \in X \mid x \precsim y\}$ are closed subsets of X

2. On the space of simple lotteries L, the rational(complete and transitive) preference $\succsim$ is continuous

   if for any $l, l', l'' \in L$, $\{\alpha \in [0,1] \mid \alpha l + (1-\alpha)\, l' \succsim l''\}$ and $\{\alpha \in [0,1] \mid \alpha l + (1-\alpha)\, l' \precsim l''\}$ are closed.

3. On the space of simple lotteries L, the rational(complete and transitive) preference $\succsim$ satisfies the independence axiom,

   if for any $l, l', l'' \in L$, and $\alpha \in (0,1)$, $l' \succsim l \Leftrightarrow \alpha l + (1-\alpha)\, l'' \succsim \alpha l' + (1-\alpha)\, l''$

9.1.2 Expected utility Form

The utility function $U : L \to R$ has an expected utility form

if there is an assignment of numbers $(u_1, ..., u_N)$ to the N outcomes such that for every simple lottery $l = (p_1, ..., p_N) \in L$, then $U(l) = \sum_{n=1}^{N} u_n p_n$.

$\to$The utility function $U(\cdot)$ is linear and therefore has the expected utility form.

$\to U(\beta l + (1-\beta)\, l') = \beta U(l) + (1-\beta)\, U(l')$ for $\forall l, l' \in L$ and $\forall \beta \in [0,1]$

And a utility function with the expected utility form is called an expected utility function.

9.1.3 Expected Utility Theorem

The continuity axiom insures that preferences on lotteries can be represented by some utility function. Because the expected utility form is linear in the probabilities, representability by the expected utility form is equivalent to these indifference curves being straight, parallel lines.

1. If the preference $\succsim_i$ on L is represented by a utility function $U(\cdot)$ that has the expected utility form, then $\succsim_i$ satisfies the independence axiom.

   $\boxed{\text{Proof}}$

   Assume that $\succsim_i$ is represented by an expected utility function $U(L) = \sum_n u_n p_n$ for $\forall p = (p_1, ..., p_N) \in L$.

   Let $l = (p_1, ..., p_N)$, $l' = (p'_1, ..., p'_N)$, $l'' = (p''_1, ..., p''_N)$, and $\alpha \in (0,1)$.

   $l' \succsim l \Leftrightarrow \sum_n u_n p_n \left(= EU(l)\right) \geq \sum_n u_n p'_n \left(= EU(l')\right)$

   $\Leftrightarrow \alpha \sum_n u_n p_n + (1-\alpha) \sum_n u_n p''_n \geq \alpha \sum_n u_n p'_n + (1-\alpha) \sum_n u_n p''_n$ holds iff $\alpha l + (1-\alpha) l'' \succsim \alpha l' + (1-\alpha) l''$

   Hence $l' \succsim l \Leftrightarrow \alpha l + (1-\alpha) l'' \succsim \alpha l' + (1-\alpha) l''$

   Thus, the independence axiom holds.

2. If the rational preference $\succsim_i$ on the space of lotteries L, satisfies the continuity and independence axioms, then $\succsim_i$ is represented by the expected utility form. That is, we can assign a number u_n to each outcome $n = 1, 2, ..., N$ in such a manner that for any two lotteries $L = (p_1, ..., p_N)$ and $L' = (p'_1, ..., p'_N)$, we have $L \succsim_i L'$ iff $\sum_{n=1}^{N} u_n p_n \geq \sum_{n=1}^{N} u_n p'_n$

   $\boxed{\text{Proof}}$

   Define $\bar{l}$ and $\underline{l}$ as the best lottery and worst lottery in L respectively.

   (a) $l \succsim l'$ and $\alpha \in (0,1) \rightarrow l \succ \alpha l + (1-\alpha) l' \succ l'$ by independence axiom.

   (b) Let $\alpha, \beta \in [0,1]$. $\beta > \alpha \Leftrightarrow \beta \cdot \bar{l} + (1-\beta) \cdot \underline{l} \succ \alpha \bar{l} + (1-\alpha) \underline{l}$ by independence axiom
   Note that $\gamma = \frac{\beta - \alpha}{1 - \alpha} \in (0,1)$ and $\beta \cdot \bar{l} + (1-\beta) \cdot \underline{l} = \gamma \cdot \bar{l} + (1-\gamma) \cdot \left[\alpha \bar{l} + (1-\alpha) \underline{l}\right]$

   (c) For any $l \in L$, $\exists$a unique α_l s.t. $\alpha_l \bar{l} + (1-\alpha_l) \underline{l} \sim l$ by continuity axiom and by completeness of $\succsim$ and connectedness of $[0,1]$
   and the uniqueness comes from (b)

   (d) A function $U : L \rightarrow R$ s.t. assigns $U(l) = \alpha_l$ for $\forall l \in L$ represents the preference $\succsim$
   $l \succsim l' \rightarrow \alpha_l \geq \alpha_{l'}$ from (b) and (c)

   (e) The utility function $U(\cdot)$ that assigns $U(l) = \alpha_l$ $\forall l \in L$ is linear and therefore has the expected utility form.
   $U(\beta l + (1-\beta) l') = \beta U(l) + (1-\beta) U(l')$ for $\forall l, l' \in L$ and $\forall \beta \in [0,1]$

Note that $\beta l + (1-\beta)\, l' \sim [\beta U(l) + (1-\beta)\, U(l')] \cdot \bar{l} + [\beta U(l) + (1-\beta)\, U(l')] \cdot \underline{l}$
Then $\alpha_{\beta l + (1-\beta) l'} = \beta U(l) + (1-\beta)\, U(l')$ and from (d), $\alpha_{\beta l + (1-\beta) l'} = U(\beta l + (1-\beta)\, l')$

3. Examples

   (a) Consider the set of all lotteries with three outcomes y_1, y_2, y_3 s.t. $y_i \in R$ and $y_1 < y_2 < y_3$
   $l = (p_1, p_2, p_3) \in \triangle \in R^3$.
   Let $U : L (= \triangle) \to R$ s.t. $U(l) = p_3 \left(\sum_{n=1}^{3} y_1 p_n \right)$
   Does U satisfy the independence axiom?
   NTS: if for any $l, l', l'' \in L$, and $\alpha \in (0,1)$,
   $U(l) \geq U(l')$
   $\Leftrightarrow l' \succsim l$
   $\Leftrightarrow \alpha l + (1-\alpha)\, l'' \succsim \alpha l' + (1-\alpha)\, l''$
   $\Leftrightarrow U(\alpha l + (1-\alpha)\, l'') \geq U(\alpha l' + (1-\alpha)\, l'')$
   NO!!!
   Counerexample
   Let $l = (1,0,0)$, $l' = (0,1,0)$, and $l'' = (0,0,1)$
   Then $U(l) = 0 \geq 0 = U(l')$
   Let $\alpha = \frac{1}{2}$.
   Then $\alpha l + (1-\alpha)\, l'' = \left(\frac{1}{2}, 0, \frac{1}{2}\right)$ and $\alpha l' + (1-\alpha)\, l'' = \left(0, \frac{1}{2}, \frac{1}{2}\right)$
   And $U(\alpha l + (1-\alpha)\, l'') = \frac{1}{2} y_1 + \frac{1}{2} y_3 < \frac{1}{2} y_2 + \frac{1}{2} y_3 = U(\alpha l' + (1-\alpha)\, l'')$

   (b) Using the independence axiom, show that $\exists \bar{l}$ and $\underline{l}$ as the best lottery and worst lottery in L respectively among all lotteries on a finite set of outcomes.

9.2 Risk aversion and its measurement.

Risky alternatives can have either outcomes in terms of consumption plan or monetary outcomes. Therefore, x can be either a consumption plan $x = (x_1, x_2, ..., x_S)$ with $p = (p_1, p_2, ..., p_S)$ $\{= (\pi_1, \pi_2, \ldots, \pi_S)\}$ or $x = (\$500, \$0, .., \$100)$ with $p = (p_1, p_2, ..., p_S)$. But, from now on, we focus on monetary outcome.

9.2.1 Lotteries over Monetary Outcomes

1. Denote amounts of money by the continuous variable x. We can describe a monetary lottery by means of a cumulative distribution function F:R$\to [0,1]$. That is, for any x, $F(x)$ is the probability taht the realtized payoff is less than or equal to x. Note that if the distribution function of a lottery has a density function $f(\cdot)$ associated with it, then $F(x) = \int_{-\infty}^{x} f(t)\, dt$ for $\forall x$. The advantage of a formalism based on distribution fuctions over one based on density

functions is that we can deal with discrete case set of outcome. The final distribution of money, $F(\cdot)$, induced by a compund lottery $(l_1, l_2, ..., l_k; \alpha_1, ..., \alpha_k)$ is just the weighted average of the distributions induced by each of the lotteries $F(x) = \sum_{k=1}^{k} \alpha_k F_k(x)$ s.t. $F_k(\cdot)$ is the distribution of the payoff under lottery l_k.Therefore, we take the lottery space L to be the set of all distribution functions over nonnegative amounts of money.

$$U(F) = \int u(x)\, dF(x) = Eu(x)$$

2. $U(\cdot)$ is defined on lotteries, but $u(\cdot)$ is defined on sure amounts of money.
3. What kind of properties does $u(\cdot)$ have?
   - (a) $u(\cdot)$ is increasing, continuous, and bounded.
   - (b) $u(\cdot)$ is concave⇔risk averse
   - (c) $u(\cdot)$ is strictly concave⇔strictly risk averse

9.2.2 Risk aversion

1. Risk aversion(Jensen's inequaltiy)

   If preferences can be represented by an expected utility function $U(\cdot)$ with a bernoulli utility function $u(\cdot)$, then a decision maker is risk averse iff

   $$Eu(x)\left(= \int u(x)\, dF(x)\right) \leq u(E(x)) \left(= u\left(\int x dF(x)\right)\right) \text{ for } \forall \text{nondeterninistic variable } x (\forall F(\cdot)$$

   $\leftarrow u(\cdot)$ is concave; that is, if $u(\cdot)$ is C^2, then $u''(x) \leq 0$ for $\forall x$

   $\rightarrow^{?}$ $u(\cdot)$ is concave. To show it, it is Pratt theorem.

   A decision maker is risk averse if for any lottery(risk alternative) $F(\cdot)$, the degenerate lottery that yields the amount $\int x dF(x) = E(x)$ with certainty is at least as good as the lottery itself.

   →It means that a risk of gaining or losing a dollar with the same probability is not worth taking.

2. Risk neutrality

   An economic agent is risk-neutral iff $Eu(x) = \sum_n u_n(x)\, p_n \leq u\left(\sum_n x_n p_n\right) = u(E(x))$ for $\forall$ nondeterministic variable x

   If the decision maker is always indifferent between two lotteries, he is risk neutral and it means that his bernoulli utility function is linear.

3. FP 1998 II.3

   $w = \$1000$

   Two states of the world, head(H) and tail(T) s.t. $p_H = p_T = \frac{1}{2}$

   (a) Expected utility maximizing risk neutral

   Suppose an agent in an economy with uncertainty is an expected utilty maximizer with a Bernoulli utility function $u(\cdot)$ on a nondeterministic variable x which is a final outcome like either a consumption plan or a monetary outcome.
   An economic agent is risk-neutral iff $Eu(x) = \sum_n u_n(x) p_n \leq u\left(\sum_n x_n p_n\right) = u(E(x))$ for $\forall$ nondeterministic variable x

   i. $\leftarrow u(\cdot)$ is linear; that is, if $u(\cdot)$ is C^2, then $u''(x) = 0$ for $\forall x$
   $\overset{?}{\longrightarrow}$ $u(\cdot)$ is linear for $\forall x$

   ii. iff $c(x,u)\{c(F,u)\} = E(x) \left(= \int x dF(x)\right)$ for $\forall x (\forall F(\cdot))$

   iii. iff $\rho(x,z) = 0$ for $\forall x$ and $\forall z > 0$ with $E(z) = 0$

   (b) If the agent is offered a lottery that pays $\$500 = x_H$ in H and $\$0 = x_T$ in T, how much is he willing to pay for a lottery ticket?

   He is willing to pay the amount of certainty equivalence, $c(x,u)$, s.t. $c(x,u) = E(x) = x_H p_H + x_T p_T = \frac{1}{2}\$500 + \frac{1}{2}\$0 = \$250$

   (c) Suppose a lottery ticket costs \$300 and that the individual can now buy information, where information means the ability to know the state of the world before buyinh the lottery ticket. How much will the individual be willing to pay for the information?

   How much the information is worth for this agent?

   If the agent buys the information with $\$x$ and knows that

   i. H will happen, then he will buy the lottery ticket with \$300 and he will get \$500. Therefore, his final wealth will be $\$1000 - \$x - \$300 + \$500 = \$(1200 - x)$

   ii. L will happen, then he will not buy the lottery ticket and his final wealth will be $\$(1000 - x)$

   Then if his final wealth is as much as his initial wealth, he will values the information s.t. $\frac{1}{2}(\$1200 - \$x) + \frac{1}{2}(\$1000 - \$x) = \$1100 - \$x \geq \$1000$
   Therefore, he will pay for the information at most \$100.

   (d) If we drop the assumption of risk-neutrality, does your conclution to part (a) necessarily still hold?

   No. I will construct a counterexample.
   $x = (\$500, \$0)$ and $p = \left(\frac{1}{2}, \frac{1}{2}\right)$
   Let us assume that the agent is strictly risk averse so that his bernoulli utility function is strictly concave s.t. $u_n(x) = \sqrt{x_n}$.

$\sqrt{500} = 22.361$, $\sqrt{0} = 0$
Then $E(x) = \frac{1}{2}\$500 + \frac{1}{2}\$0 = \$250$ and $\frac{1}{2} * 22.361 + \frac{1}{2} * 0 = 11.181$
Then his certainty equivalence is $c(x,u)$ s.t. $\sqrt{c(x,u)} = 11.181$
and $c(x,u) = (11.181)^2 = 125.01 < 250$

9.2.3 Measure of intensity of risk aversion

1. Measures

   (a) Arrow-Pratt measure of A.R.A. $r_a(x)$
   A natural candidate for measuring risk aversion is the second derivative, but the second derivative is not invariant to affine transformation of $u(\cdot)$ so that

$$r_a(x) = -\frac{u''(x)}{u'(x)} \text{ for } \forall \text{nondeterministic variable } x$$

   s.t. $u'(x) \neq 0$
   Then $r_a(x)$ is invariant to increasing affine transformation of $u(\cdot)$.

   (b) Certainty equivalence $c(x,u)\{c(F,u)\}$ of $x\{F(\cdot)\}$
   The certainty equivalence of $x\{F(\cdot)\}$, denoted $c(x,u)\{c(F,u)\}$, is the amount of money for which the individual is indifferent between a risk alternative $x\{F(\cdot)\}$ and the certain amount $c(x,u)\{c(F,u)\}$;

$$Eu(x) = \int u(x)\,dF(x) = u[c(x,u)]\,(= u[c(F,u)]) \text{ for } \forall x \in R (\text{and } \forall F(\cdot))$$

   Therefore, the agent is risk averse iff $c(x,u)\{c(F,u)\} \leq \sum x_n p_n \left(\int x dF(x)\right) = E(x)$ for $\forall x$ (for $\forall F(\cdot)$ and $\forall x \in R$)
   $c(x,u)\{c(F,u)\} \leq \sum x_n p_n \left(\int x dF(x)\right) = E(x)$
   $\rightarrow Eu(x) \left(= \int u(x)\,dF(x)\right) = u[c(x,u)]\,(= u[c(F,u)]) \leq u(E(x)) \left(= u\left[\int x dF(x)\right]\right)$
   because $u(\cdot)$ is nondecreasing.

   (c) Risk compensation(Risk Premium)
   Let z be any random variable s.t. $E(z) = 0$ with $Var(z) = \sigma^2 = E(z^2) - [E(z)]^2 = E(z^2)$
   For any x and z, the Risk compensation(Risk Premium) for z at x is $\rho(x,z)$ s.t.

$$Eu(x+z) = u(x - \rho(x,z))$$

   so that $x - \rho(x,z) = c(x+z,u)$ of $x+z$

i. For small z with $E(z)=0$, $\rho(x,z)\simeq\frac{r_a(x)\cdot\sigma^2}{2}$

$\boxed{\text{Proof}}$

For each realization of random variable z, a quadratic approximation of $u(x+z)$

$$\begin{aligned} u(x+z) &\simeq u(x)+u'(x)z+u''(x)\frac{z^2}{2} \\ Eu(x+z) &\simeq Eu(x)+E\left(u'(x)\cdot z\right)+E\left(u''(x)\frac{z^2}{2}\right) \\ &\simeq u(x)+u''(x)\frac{\sigma^2}{2} \end{aligned}$$

$$u(x-\rho(x,z))\simeq u(x)-u'(x)\cdot\rho(x,z)$$

$$\begin{aligned} Eu(x+z) &= u(x-\rho(x,z)) \\ u(x)+u''(x)\frac{\sigma^2}{2} &\simeq u(x)-u'(x)\cdot\rho(x,z) \\ \rho(x,z) &\simeq -\frac{u''(x)}{u'(x)}\frac{\sigma^2}{2}=r_a(x)\frac{\sigma^2}{2} \end{aligned}$$

(d) Probability premium $\pi(x,\varepsilon,u)$

For any fixed amount of money x and $\forall\varepsilon>0$, Probability premium $\pi(x,\varepsilon,u)$ is the excess in winning probability over fair odds that makes the agent indifferent between the certain outcome x and the risk alternative between two outcomes $x+\varepsilon$ and $x-\varepsilon$;

$$u(x)=\left(\frac{1}{2}+\pi(x,\varepsilon,u)\right)u(x+\varepsilon)+\left(\frac{1}{2}-\pi(x,\varepsilon,u)\right)u(x-\varepsilon)$$

Therefore, the agent is risk averse iff $\pi(x,\varepsilon,u)\geq 0$ for $\forall x$ and $\forall\varepsilon>0$

2. Conclusion

Suppose an agent is an expected utility maximizer with bernoulli utility function $u(\cdot)$ on amounts of money x which is a continous variable. Then

(a) The agent is risk averse
iff $Eu(x)\left(=\int u(x)\,dF(x)\right)\leq u(E(x))\left(=u\left(\int x dF(x)\right)\right)$ for $\forall$ nondeterministic variables $x(\forall F(\cdot))$;
$\leftarrow u(\cdot)$ is concave; that is, if $u(\cdot)$ is C^2, then $u''(x)\leq 0$ for $\forall x$
$\rightarrow^{?}$ $u(\cdot)$ is concave. To show it, it is Pratt theorem.

(b) iff $c(x,u)\{c(F,u)\}\leq E(x)\left(=\int x dF(x)\right)$ for $\forall x(\forall F(\cdot))$

(c) iff $\rho(x,z)\geq 0$ for $\forall x$ and $\forall z>0$ with $E(z)=0$

(d) iff $\pi(x,z,u)\geq 0$ for $\forall x$ and $\forall z>0$

9.2.4 Pratt Theorem

1. Theorem

   These measures of risk aversion can be used to compare the risk aversion of two agents.

   Let u_1 and u_2 be C^2 with $u_1' > 0$ and $u_2' > 0$, and ρ_i, r_{ai} be the risk compensation and Arrow-Pratt measure of ARA respectively for agent i.

   Then the followings are equivalent.

   (a) an agent with $u_2(\cdot)$ is more risk averse than another agent with $u_1(\cdot)$

   (b) $r_a(x, u_2) \geq r_a(x, u_1)$ $[\Leftrightarrow r_{a2}(x) \geq r_{a1}(x)]$ for $\forall$random variable x

   (c) $u_2(\cdot)$ is more concave utility function than $u_1(\cdot)$
   iff $u_2(x) = \psi(u_1(x))$ s.t. $\exists\psi(\cdot)$ which is a concave and increasing transformation; that is, $\psi''(y) \leq 0$ for $\forall y \in R$

   (d) $\rho_2(x, z) \geq \rho_1(x, z)$ for $\forall x$ and $\forall$random variable z
   $\Leftrightarrow \pi(x, z, u_2) \geq \pi(x, z, u_1)$ for $\forall x$ and $\forall$random variable z

   (e) $c_2(x, u_2) \leq c_1(x, u_1)$ for $\forall x$

   (f) $Eu_2(x) \geq u_2(x') \rightarrow Eu_1(x) \geq u_1(x')$ for $\forall x$ and a riskless outcome x'

2. SP 1998 II-1 and Proof in Financial Economics

   $(a) \Leftrightarrow (b) \Leftrightarrow (c)$

   (a) $(a) \rightarrow (b)$
   Suppose $r_a(x, u_2) \geq r_a(x, u_1)$ $[\Leftrightarrow r_{a2}(x) \geq r_{a1}(x)]$ for $\forall x$
   NTS: $\exists\psi(\cdot)$ which is a concave and increasing transformation, $\psi''(y) \leq 0$ for $\forall y \in R$, s.t. $u_2(x) = \psi(u_1(x)) = \psi \cdot u_1(x)$
   Let $u_1(x) = t$
   Since $u_1' > 0$, $\exists$an inverse function u_1^{-1}.
   Define $\psi(x) = u_2\left(u_1^{-1}(x)\right)$.
   $\psi'(x) = \frac{u_2'\left(u_1^{-1}(x)\right)}{u_1'\left(u_1^{-1}(x)\right)} > 0$ because $u_2' > 0$ also so that $\psi(\cdot)$ is increasing.
   $\psi''(x) = \frac{\frac{u_2''(t)}{u_1'(t)} \cdot u_1'(t) - u_2'(t) \cdot \frac{u_1''(t)}{u_1'(t)}}{\left(u_1'(t)\right)^2}$
   $= \frac{u_2''(t) - u_2'(t) \cdot \frac{u_1''(t)}{u_1'(t)}}{\left(u_1'(t)\right)^2} = \left(r_{a2}(t) - r_{a1}(t)\right)\frac{u_2'(t)}{\left(u_1'(t)\right)^2} \leq 0$ so that $\psi(\cdot)$ is concave.

(b) $(b) \rightarrow (c)$

Suppose $u_2(\cdot)$ is more concave utility function than $u_1(\cdot)$

NTS: $\rho_2(x,z) \geq \rho_1(x,z)$ for $\forall x$ and $\forall$random variable z

By the definition of risk compensation, $Eu_i(x+z) = u_i[x - \rho_i(x,z)]$ for $\forall i$ and for any x and z

$u_2[x - \rho_2(x,z)] = Eu_2(x+z) = E[\psi(u_1(x+z))] \leq \psi[Eu_1(x+z)]$ by property of concave function $\psi(\cdot)$

$\psi[Eu_1(x+z)] = \psi[u_1[x - \rho_1(x,z)]] = u_2[x - \rho_1(x,z)]$

Because u_2 is increasing, $x - \rho_2(x,z) \leq x - \rho_1(x,z)$

It implies that $\rho_2(x,z) \geq \rho_1(x,z)$ for $\forall x$ and $\forall$random variable z

(c) $(c) \rightarrow (a)$

Suppose $\rho_2(x,z) \geq \rho_1(x,z)$ for $\forall x$ and $\forall$random variable z

NTS: $r_a(x,u_2) \geq r_a(x,u_1)\,[\Leftrightarrow r_{a2}(x) \geq r_{a1}(x)]$ for $\forall x$

Suppose not; $\exists x'$ s.t. $r_{a2}(x') < r_{a1}(x')$

Since r_{a2}, r_{a1} is continuous, $\exists$a interval I of x' s.t. $r_{a2}(x') < r_{a1}(x')$ for $\forall x' \in I$

Let $x \in I$.

$r_{a2}(x) < r_{a1}(x)$.

By a contraposition of the above argument$(\urcorner(b) \rightarrow \urcorner(a))$, we have $\psi(\cdot)$ is strictly convex.

Then $u_2[x - \rho_2(x,z)] = Eu_2(x+z) = E[\psi(u_1(x+z))] > \psi[Eu_1(x+z)] = \psi[u_1[x - \rho_1(x,z)]$
$u_2[x - \rho_1(x,z)]$

$\Leftrightarrow u_2[x - \rho_2(x,z)] > u_2[x - \rho_1(x,z)]$

$\Leftrightarrow x - \rho_2(x,z) > x - \rho_1(x,z)$

$\Leftrightarrow \rho_2(x,z) < \rho_1(x,z)$ for $x \in I$.

But, it contradicts the assumption.

Therefore, $(a) \Leftrightarrow (b) \Leftrightarrow (c)$

3. Proof in Mas-colell

$(a) \Leftrightarrow (b) \Leftrightarrow (c) \Leftrightarrow (d) \Leftrightarrow (e)$

Furthermore, we can extend what we have done.

(a) $(a) \rightarrow (d)$

Suppose $u_2(\cdot)$ is more concave utility function than $u_1(\cdot)$

NTS: $c_2(x,u_2) \leq c_1(x,u_1)$ for $\forall x$

By the definition of certainty equivalence, $Eu_i(x) = u_i[c_i(x,u_i)]$ for $\forall i$ and for any x

   $u_2[c_2(x,u_2)] = Eu_2(x) = E[\psi(u_1(x))] \le \psi[Eu_1(x)]$ by property of concave function $\psi(\cdot)$

   $\psi[Eu_1(x)] = \psi[u_1[c_1(x,u_1)]] = u_2[c_1(x,u_1)]$

   Because u_2 is increasing, $c_2(x,u_2) \le c_1(x,u_1)$ for $\forall x$

(b) $(d) \to (a)$

   We can use the same argument for the above case($(c) \to (a)$)

(c) $(d) \Leftrightarrow (e)$

   $\to$:

   Suppose $c_2(x,u_2) \le c_1(x,u_1)$ for $\forall x$.

   NTS: $Eu_2(x) \ge u_2(x') \to Eu_1(x) \ge u_1(x')$ for $\forall x$ and a riskless outcome x'

   $u_2[c_2(x,u_2)] = Eu_2(x) \ge u_2(x')$ and $u_2(\cdot)$ is increasing so that $c_2(x,u_2) \ge x'$

   Then $Eu_1(x) = u_1[c_1(x,u_1)] \ge u_1[c_2(x,u_2)] \ge u_1[x']$ because $u_1(\cdot)$ is also increasing.

   $\leftarrow$:

   Suppose $Eu_2(x) \ge u_2(x') \to Eu_1(x) \ge u_1(x')$ for $\forall x$ and a riskless outcome x'

   NTS: $c_2(x,u_2) \le c_1(x,u_1)$ for $\forall x$.

   $[Eu_2(x) \ge u_2(x')] \to [Eu_1(x) \ge u_1(x')]$

   $\Leftrightarrow \{[u_2[c_2(x,u_2)] \ge u_2(x')] \to [u_1[c_1(x,u_1)] \ge u_1(x')]\}$

   $\to \{[c_2(x,u_2) \ge x'] \to [c_1(x,u_1) \ge x']\}$

   $\to \{c_2(x,u_2) \le c_1(x,u_1)\}$

   We use here a mathematical fact that $\{[a \ge c] \to [b \ge c]\} \to \{a \le b\}$

4. Corollaries

   (a) $\rho(x,z)$ is decreasing(constant) in x for $\forall z$ iff $r_a(x)$ is decreasing(constant) in x.

   $\boxed{\text{Proof}}$

   Here x is the initial wealth.

   Let $x_1 > x_2$.

   $[\rho(x_1,z) < \rho(x_2,z)] \Leftrightarrow [r_a(x_1+z) < r_a(x_2+z)]$

   Define $u(x_i+z) = u_i(z)$ and $r_a(x_i+z) = r_i(z)$

   $u(x_i + c_i(z,u_i)) = u_i(c_i(z,u_i)) = Eu_i(z) = Eu(x_i+z) = u(c_i(x_i+z,u))$

   Therefore, $c_i(x_i+z,u) = c_i(z,u_i) + x_i$

$$\rho(x_1,z) < \rho(x_2,z) \text{ for } \forall z$$

$$\begin{aligned}
&\Leftrightarrow \quad x_1 - c_1(x_1+z,u) < x_2 - c_2(x_2+z,u) \text{ for } \forall\ z \\
&\Leftrightarrow \quad x_1 - [c_1(z,u_1)+x_1] < x_2 - [c_2(z,u_2)+x_2] \text{ for } \forall z \\
&\Leftrightarrow \quad c_1(z,u_1) > c_2(z,u_2) \text{ for } \forall z \\
&\Leftrightarrow \quad r_1(z) < r_2(z) \text{ for } \forall z \\
&\Leftrightarrow \quad -\frac{u_1''(z)}{u_1'(z)} < -\frac{u_2''(z)}{u_2'(z)} \text{ for } \forall z \\
&\Leftrightarrow \quad -\frac{u''(x_1+z)}{u'(x_1+z)} < -\frac{u''(x_2+z)}{u'(x_2+z)} \text{ for } \forall\ z \\
&\Leftrightarrow \quad r_a(x_1+z) < r_a(x_2+z) \text{ for } \forall\ z
\end{aligned}$$

9.2.5 Optimal Portfolio with Risk aversion

1. Insurance

   Strictly risk averse agent with an initial wealth w but confronts a loss of D dollars with probability π

   one unit of insurance costs q dollars and pays 1 dollar if the loss happens.

   $\boxed{\text{Solution}}$

   Assume that he buys α unit of insurance

   $$\text{Wealth}\begin{cases} w - \alpha q \text{ with } (1-\pi) \\ w - \alpha q + D \text{ with } \pi \end{cases}$$

   $\max_{\alpha\in[0,\frac{w}{q}]} (1-\pi)\,u(w-\alpha q) + \pi u(w-\alpha q - D + \alpha)$

   FOC.

   $$\alpha : \begin{cases} (1-\pi)\,u'(w-\alpha q)(-q) + \pi u'(w-\alpha q - D+\alpha)(1-q) \leq 0 \text{ if } \alpha < \frac{w}{q} \\ (1-\pi)\,u'(w-\alpha q)(-q) + \pi u'(w-\alpha q - D+\alpha)(1-q) \geq 0 \text{ if } \alpha > 0 \end{cases} \text{ with } =0 \text{ if } \alpha \in \left(0,\frac{w}{q}\right)$$

   (a) $q = \pi$

   $$\alpha : \begin{cases} (1-\pi)\,u'(w-\alpha q)(-q) + \pi u'(w-\alpha q - D+\alpha)(1-q) \leq 0 \text{ if } \alpha < \frac{w}{q} \\ (1-\pi)\,u'(w-\alpha q)(-q) + \pi u'(w-\alpha q - D+\alpha)(1-q) \geq 0 \text{ if } \alpha > 0 \end{cases} \text{ with } =0 \text{ if } \alpha \in \left(0,\frac{w}{q}\right)$$

   Since $u'(w-D) > u'(w)$, $\alpha^* > 0$.

   $u'(w-\alpha q - D + \alpha) = u'(w-\alpha q)$

   By strict concavity of $u(\cdot)$, $D = \alpha^*$

   Therefore, if insurance if actuarially fair(the price of insurance is equal to the expected cost of insurance), then the strict risk averse agent will completely insure.

Then it allows him to have $w - \pi D$ with certainty.

(b) $q > \pi$

Since $u'(w-D) > u'(w)\,,\ \alpha^* > 0.$

Then $(1-\pi)\,u'(w-\alpha q)\,(-q) + \pi u'(w-\alpha q - D + \alpha)\,(1-q) = 0$

$\rightarrow \frac{u'(w-\alpha q-D+\alpha)}{u'(w-\alpha q)} = \frac{(1-\pi)q}{\pi(1-q)} > 0$

$u'(w-\alpha q - D + \alpha) > u'(w-\alpha q)$

$\rightarrow w - \alpha^* q - D + \alpha^* < w - \alpha^* q$ by strict concavity of $u(\cdot)$

$\rightarrow \alpha^* < D$

Then he will not completely insure.

2. Optimal portfolio with one risky asset

(a) p_1 : current price of risky asset

p_2 : current price of risk free asset

α : portfolio weight of risky asset

β : portfolio weight of risk free asset

w : endowment$= \alpha + \beta$

a : amount of risky asset in portfolio

r' : return on risky asset

r : return on risk free asset

s.t. $Er' > r$

an agent utility function is $u(x) = -e^{-\theta x}$ for $x \in R$ and some $\theta > 0$

$$\begin{aligned} & \max_{\alpha\geq 0,\beta\geq 0} Eu\left(\alpha r' + \beta r\right) \\ = & \max_{\alpha\in[0,w]} Eu\left(\alpha r' + (w-\alpha)\,r\right) \\ = & \max_{\alpha\in[0,w]} Eu\left(wr + \alpha(r'-r)\right) \\ = & \max_{\alpha\in[0,w]} E\left[-e^{-\theta(wr+\alpha(r'-r)}\right] \end{aligned}$$

(b) Show that if $Er' > r$, then $\alpha^* > 0$

$$\alpha : \begin{cases} -\theta E\left[(r'-r)\cdot -e^{-\theta(wr+\alpha(r'-r)}\right] \leq 0 \text{ if } \alpha^* < w \\ -\theta E\left[(r'-r)\cdot -e^{-\theta(wr+\alpha(r'-r)}\right] \geq 0 \text{ if } \alpha^* > 0 \end{cases} \text{with } = 0 \text{ if } \alpha^* \in (o, w)$$

If $\alpha^* = 0 < w$, $-\theta E\left[(r'-r)\cdot -e^{-\theta wr}\right] = -\theta\left(-e^{-\theta wr}\right)E\left(r'-r\right) = \theta\left(e^{-\theta wr}\right)\left(Er'-r\right) > 0$ which is a contradiction

So that $0 < \alpha^* < w$ and then $-\theta E\left[(r'-r)\cdot -e^{-\theta(wr+\alpha(r'-r)}\right] = 0$

(c) With the natural restriction on asset holdings,
Show that α^* in the risky asset is a decreasing function of the measure of risk aversion.θ

$\boxed{\text{Solution}}$

First, Arrow-Pratt measure of absolute risk aversion $r_A(x) = -\frac{u''(x)}{u'(x)} = \theta > 0$ so that $u(\cdot)$ has constant absolute risk aversion
and the expected wealth of this agent is $\alpha r' + \beta r$
FOC.
$\alpha : -\theta E\left[(r'-r) \cdot -e^{-\theta(wr+\alpha(r'-r)}\right] = 0$ for $\alpha^* \in (o, w)$

Implicit function theorem

Let $f(\theta, \alpha^*) = -\theta E\left[(r'-r) \cdot -e^{-\theta(wr+\alpha(r'-r)}\right]$
$\frac{\partial f}{\partial \alpha} |_{\alpha=\alpha^*} = \theta^2 E\left[(r'-r)^2 \cdot -e^{-\theta(wr+\alpha(r'-r)}\right] < 0\,(\neq 0)$
Then $\exists$a C^1 function $g(\theta) = \alpha^*$ defined on an interval I around θ s.t.
$f(\theta, g(\theta)) = 0$ for $\forall \theta \in I$ and
with $\frac{\partial f}{\partial \theta} = \theta E\left[(wr + \alpha(r'-r)) \cdot (r'-r) \cdot -e^{-\theta(wr+\alpha(r'-r)}\right] - E\left[(r'-r) \cdot -e^{-\theta(wr+\alpha(r'-r)}\right]$
$= \theta\alpha E\left[(r'-r)^2 \cdot -e^{-\theta(wr+\alpha(r'-r)}\right] < 0,$
$\frac{\partial \alpha}{\partial w_1} = g'(w_1) = -\frac{\frac{\partial f}{\partial w_1}}{\frac{\partial f}{\partial \alpha}} < 0$
Therefore, the investor's optimal investment in the risky asset is a decreasing function of the measure of risk aversion.

Proof of $E\left[(r'-r)^2 \cdot -e^{-\theta(wr+\alpha(r'-r)}\right] < 0$

$E\left[(r'-r)^2 \cdot -e^{-\theta(wr+\alpha(r'-r)}\right] = -E\left[(r'-r)^2 \cdot e^{-\theta(wr+\alpha(r'-r)}\right]$
$= -E(r'-r)^2 \cdot E\left[e^{-\theta(wr+\alpha(r'-r)}\right]$
Since r' has a compact support; that is, $\exists B > 0$ s.t. $e^{-\theta(wr+\alpha(r'-r)} \geq B$, and $E(r'-r)^2 \geq 0$, therefore $E(r'-r)^2 \cdot E\left[e^{-\theta(wr+\alpha(r'-r)}\right] \geq B \cdot E(r'-r)^2 \geq 0$

(d) Show that α^* is independent of the investor's initial wealth if her utility function

$$\alpha : -\theta E\left[(r'-r)\cdot -e^{-\theta(wr+\alpha(r'-r))}\right] = 0 \text{ for } \alpha^* \in (o,w)$$

$$\begin{aligned}
0 &= -\theta E\left[\left(r'-r\right)\cdot -e^{-\theta(wr+\alpha(r'-r))}\right]\\
&= -\theta(Er'-r)\cdot\left[-e^{-\theta((w-\alpha)r+\alpha r')}\right]\\
&= -\theta(Er'-r)\cdot E\left[-e^{-\theta(w-\alpha)r}\cdot -e^{-\theta\alpha r'}\right]\\
&= -\theta(Er'-r)\cdot E\left[-e^{-\theta(w-\alpha)r}\cdot -e^{-\theta\alpha r'}\right]\\
&= -\theta\cdot\left(-e^{-\theta(w-\alpha)r}\right)\cdot Er'\cdot E\left(-e^{-\theta\alpha r'}\right)+\theta\cdot r\cdot\left(-e^{-\theta(w-\alpha)r}\right)E\left[-e^{-\theta\alpha r'}\right]
\end{aligned}$$

$$\begin{aligned}
\theta\cdot\left(-e^{-\theta(w-\alpha)r}\right)\cdot Er'\cdot E\left(-e^{-\theta\alpha r'}\right) &= \theta\cdot\left(-e^{-\theta(w-\alpha)r}\right)\cdot r\cdot E\left[-e^{-\theta\alpha r'}\right]\\
Er'\cdot E\left(-e^{-\theta\alpha r'}\right) &= r\cdot E\left[-e^{-\theta\alpha r'}\right]\\
(Er'-r)\cdot E\left(-e^{-\theta\alpha r'}\right) &= 0
\end{aligned}$$

Therefore, the value of α^* will be derived from here will not depend on w.

(e) If there is no arbitrage and portfolio is restricted to nonnegative, then the optimal choice of α is in a compact set and there is a solution.

If the agent is strictly risk averse, then the optimal portfolio is unique.

(f) In the optimal portfolio choice problem, if agent u_2 is more risk averse than another agent with u_1, then $\alpha_2 < \alpha_1$.
Let $\alpha_1, \alpha_2 \in (0, w_i)$

$$\max_{\alpha_i\in[0,w_i]} Eu_i\left(wr+\alpha_i(r'-r)\right)$$

Then $(Er'-r)\cdot E\left[u_i'(wr+\alpha_i^*(r'-r))\right] = 0$ for $\alpha_i^* \in (o, w_i)$
Let $f(\alpha_i^*) = (Er'-r)\cdot E\left[u_i'(wr+\alpha_i^*(r'-r))\right]$ is a decreasing function with respect to α_i^* because $u_i'(\cdot)$ is a decreasing function with respect to α_i^*.
$f_2(\alpha_2^*) = (Er'-r)\cdot E\left[u_2'(wr+\alpha_2^*(r'-r))\right]$
$f_2(\alpha_1^*) = (Er'-r)\cdot E\left[u_2'(wr+\alpha_1^*(r'-r))\right] = (Er'-r)\cdot E\,\{\psi'\left[u_1(wr+\alpha_1^*(r'-r))\right]\cdot u_1'(wr+\alpha$
(?)

(g) an agent consumers a single good at two dates $t = 1, 2$

endowment w_1 and uncertain endowment w_2

expected utility function with a strictly increasing time separablity $u : R_+ \to R$ with $\beta \in (0, 1)$

A single risk free asset with $(1 + r)$

The agent is strictly risk averse with $u(\cdot)$ which is C^2.

Show that his investment in this asset is an increasing function of w_1

$\boxed{\text{Solution}}$

The agent buys α unit of risk free asset in $t=1$ and get the return $(1+r)\alpha$ at $t=2$

$$\max_{\alpha\geq 0} u(w_1-\alpha)+\beta Eu[w_2+(1+r)\alpha]$$

FOC.

$\alpha : u'(w_1-\alpha^*)(-1)+\beta Eu'[w_2+(1+r)\alpha^*](1+r)=0$

<u>Implicit function theorem</u>

Let $f(w_1,\alpha^*)=-u'(w_1-\alpha^*)+\beta Eu'[w_2+(1+r)\alpha^*](1+r)$

$\frac{\partial f}{\partial \alpha}|_{\alpha=\alpha^*}=u''(w_1-\alpha^*)+\beta(1+r)^2 Eu''[w_2+(1+r)\alpha^*]<0\,(\neq 0)$

Then $\exists$a C^1 function $g(w_1)=\alpha^*$ defined on an interval I around w_1 s.t.

$f(w_1,g(w_1))=0$ for $\forall w_1\in I$ and

with $\frac{\partial f}{\partial w_1}=-u''(w_1-\alpha^*)>0$,

$\frac{\partial \alpha}{\partial w_1}=g'(w_1)=-\frac{\frac{\partial f}{\partial w_1}}{\frac{\partial f}{\partial \alpha}}>0$

Therefore, his investment in this asset is an increasing function of $w_1\in I$

$\rightarrow$The risk free asset is a normal good

(h) an agent consumes a single good at two dates $t=1,2$

endowment w at $t=1$

Without uncertainty, he consumes $w-\alpha'$ at $t=1$ and α' at $t=2$.

Now we introduce uncertainty in this economy.

expected utility function with a strictly increasing time separablity $u:R_+\rightarrow R$ with $\beta\in(0,1)$

the agent saves(invests) to risky asset α among w and at the second period, he gets $\alpha+y$

The agent is strictly risk averse with $u(\cdot)$ which is C^2.

Show that if $E[u'(\alpha'+y)]>u'(\alpha')$, then $\alpha^*>\alpha'$

$\boxed{\text{Solution}}$

The agent buys α unit of risky asset in $t=1$ and get the return $(1+r)\alpha$ with uncertainty at $t=2$

$$\max_{\alpha\geq 0} u(w-\alpha)+\beta Eu[\alpha+y]$$

FOC.

$\alpha' : u'(w-\alpha')=\beta u'[\alpha']$

$\alpha^* : u'(w-\alpha^*)=\beta Eu'[\alpha^*+y]$

Assume $E\left[u'\left[\alpha^* + y\right]\right] > u'\left(\alpha'\right)$
Suppose not; $\alpha^* \leq \alpha' \Leftrightarrow w - \alpha^* \geq w - \alpha'$
Then $u'\left(w - \alpha^*\right) \leq u'\left(w - \alpha'\right)$
$\Leftrightarrow Eu'\left[\alpha^* + y\right] \leq u'\left[\alpha'\right]$ which is a contradiction!!!
Therefore, if $E\left[u'\left(\alpha' + y\right)\right] > u'\left(\alpha'\right)$, then $\alpha^* > \alpha'$

9.3 Stochastic Order

Comparison of alternative distributions of monetary returns. We ask then one distribution of monetary returns can unambiguously better than another, and also when one distribution is more risky than another.

9.4 States of World

1. We extend it by allowing utility to depend on stats of nature underlying the uncertainty as well as on the monetary payoffs. In the process, we develop a framework for modeling uncertainty in terms of these underlying states.

9.5 Subjective probability

1. The assumption that uncertain prospects are offered to us with known objective probabilities is rarely in reality. The subjective probability framework offers a way of modeling choice under uncertainty in which the probabilities of different risky alternatives are not given to the decision maker in any objective way.

10 General Equilibrium under Uncertainty

So far we have treated our commodities which are distinct and physically tradable real objects. But, usually the usefulness of a commidity may depend on uncertain, external circumstances. To model this type of resource allocation problem, we can use the concept of a contingent commodity. A commodity such as medical care can be subdivided into many different artificial commodities, each of which has the interpretation "medical care is provided under circumstance s."

With I consumer, there are 2^I different states of nature, each corresponding to a different configuration of health across the population. We can therefore imagine 2^I different commodities called medical cares.

One of the strengths of general equilibrium theory is its ability to deal with arbitrary number of commodities. Therefore, even though it seems difficult to conceive of a very large number of markets for a very large number of contingent commodities, all the welfare theorems turn out to applicable to this uncertainty setting.

10.1 Contingent Commodities

We begin by formalizing uncertainty by means of states of the world and introducing the key idea of a contingent commodity: a commodity whose delivery is conditional on the realized state of the world.

1. Environment

   Commodity Space=R^L

   $X_i = R^L_+$ for $\forall i \in I$

   $Y_j \subset R^L$ for $\forall j \in J$

   $\theta_{ij} \in [0,1]$ for $\forall i, j$ s.t. $\sum_{i \in I} \theta_{ij} = 1$ for $\forall j$ →The firm's share is also state contingent, but here we assume it does not for simplicity.

   The new element is that technologies, endowment, and preferences are now uncertain. These depend on the state of the world. The state of the world is a complete description of a possible outcome of uncertainty. We assume that an exhaustive set S of states of the world is given.

2. Defintion

   For every physical commodity $l \in L$ and states $s \in S$, a unit of state contingent commodity ls is a title to receive a unit of the physical good l iff state s occurs. Accordingly, a state contingent commodity vector is specified by $x = \{(x_{11}, x_{21}, ..., x_{L1}), (x_{12}, x_{22}, ..., x_{L2}), ..., (x_{1S}, ..., x_{LS})\} \in R^{LS}$ and is understood as an entitlement to receive the commodity vector $(x_{1s}, x_{2s}, ..., x_{Ls})$ when state s occurs.

3. Charateristics of Economic agent depending on states of the world

(a) $e_i = \left\{ \left(e^i_{11}, e^i_{21}, ..., e^i_{L1}\right), \left(e^i_{12}, e^i_{22}, ..., e^i_{L2}\right), ..., \left(e^i_{1s}, ..., e^i_{Ls}\right) \right\} \in R^{LS}_+$

(b) $\succsim_i$defined on $X_i \subseteq R^{LS}$

the consumer evaluates contingent commodity vectors by first assigning to state s a probability π^i_s(could be subjective or objective), then evaluating the physical commodity vectors at state s according to Bernoullli state dependent utility function $u^i_s\left(x^i_{1s}, x^i_{2s}, ..., x^i_{Ls}\right)$, and finally computing the expected utility. That is the preferences of consumer i over two contingent commodity vectors $x_i, x'_i \in X_i \subseteq R^{LS}$ satisfy

$x_i \succsim_i x'_i$ iff $\sum_{s\in S} \pi^i_s u^i_s \left(x^i_{1s}, x^i_{2s}, ..., x^i_{Ls}\right) = EU_i\left(x_i\right) \geq EU_i\left(x'_i\right) = \sum_{s\in S} \pi^i_s u^i_s \left(x'^i_{1s}, x'^i_{2s}, ..., x'^i_{Ls}\right)$

It should be emphasized that the preferences $\succsim_i$are in the nature of ex ante preferences; the random variables describing possible consumptions are evaluated before the resolution of uncertainty.

(c) $Y_j \subseteq R^{LS}$

$y_j \in R^{LS}$ is a state contingent production plan if the input-output vector $\left(y^j_{1s}, ..., y^j_{Ls}\right)$ of physical commodities is feasible for firm j when state s occurs.

$y_j = \left(\left(y^j_{11}, y^j_{21}\right), \left(y^j_{12}, y^j_{22}\right)\right) \in R^{2\cdot 2}$

$(-1, 1, -1, 0)$ is feasible, but $(-1, 1, 0, 0)$ is not feasible. Input decision has to be made before uncertainty resolved.

(d) we say a vector $z \in R^{LS}$ is measurable with respect to the family of information partitions $(I_1, I_2, ..., I_T)$ if for every hts' and hts'', we have $z_{hts'} = z_{hts''}$ whenever s', s'' belong to the same element of the partition I_t like $I_t = \{(1, 2, 3), (s', 4, s''), (7)\}$

10.2 Arrow-Debreu Equilibrium

We use these tools to define the concept of an Arrow-Debreu equilibrium. It is a CE in which contingent commodities are traded. An Arrow-Debreu equilibrium results in Pareto optimla allocation of risk.

1. Environment

   Commodity Space=R^{LS}

   $X_i = R^{LS}_+$ for $\forall i \in I$

   $\succsim_i$defined on $X_i \subseteq R^{LS}$

   $\rightarrow x_i \succsim_i x'_i$ iff $\sum_{s\in S} \pi^i_s u^i_s \left(x^i_{1s}, x^i_{2s}, ..., x^i_{Ls}\right) = EU_i\left(x_i\right) \geq EU_i\left(x'_i\right) = \sum_{s\in S} \pi^i_s u^i_s \left(x'^i_{1s}, x'^i_{2s}, ..., x'^i_{Ls}\right)$

   $e_i = \left\{ \left(e^i_{11}, e^i_{21}, ..., e^i_{L1}\right), \left(e^i_{12}, e^i_{22}, ..., e^i_{L2}\right), ..., \left(e^i_{1s}, ..., e^i_{Ls}\right) \right\} \in R^{LS}_+$

   $Y_j \subset R^{LS}$ for $\forall j \in J$

   $\theta_{ij} \in [0, 1]$ for $\forall i, j$ s.t. $\sum_{i\in I} \theta_{ij} = 1$ for $\forall j$

2. Complete market

   We assume the existence of a market for every contingent commodity ls. These markets open before the resolution of uncertainty, at date 0.

   the price of commodity; p_{ls}.

   what is purchased in the market for the contingent commodity ls is commitment to deliver amounts of the physical good l if and when state of the world s occurs.

   Observe that although deliveries are contingent, the payments are not.

   Also note that information should be symmetric across economic agents so that they know that the state of the world s occurs.

3. Definition

   (a) In the production economy E_p with uncertainty, a Competitive Equilibrium, which we call an Arrow-Debreu equilibrium, consists of $\left\{ \{x_i^*\}_{i\in I}, \left\{y_j^*\right\}_{j\in J} \right\} \in R_+^{LSI} \cdot R^{LSJ}$ and $p^* \in R_+^{LS} \backslash \{0\}$ s.t.

      i. for $\forall i \in I, x_i^*$ is $\succsim_i -$ maximal in $B_i\left(p^*, e_i\right) = \left\{ x_i \in X_i \mid p^* \cdot x_i \leq p^* \cdot e_i + \sum_{j\in J} \theta_{ij} \cdot p^* \cdot y_j^* \right\}$
      $\Leftrightarrow x_i^* \in \arg\max_{x_i \in B_i(p^*, e_i)} EU_i\left(x_i\right)$ s.t.
      $B_i\left(p^*, e_i\right) = \left\{ x_i \in X_i \mid p^* \cdot x_i \leq p^* \cdot e_i + \sum_{j\in J} \theta_{ij} \cdot p^* \cdot y_j^* \right\}$

      ii. for $\forall j \in J$, $y_j^* \in Y_j$, $p^* \cdot y_j \leq p^* \cdot y_j^*$ for $\forall y_j \in Y_j$

      iii. $\sum_{i\in I} x_i^* \leq \sum_{i\in I} e_i + \sum_{j\in J} y_j^*$

   (b) In the pure exchange economy E with uncertainty,
   an Arrow-Debreu equilibrium consists of $\{x_i^*\}_{i\in I} \in R_+^{LSI}$ and $p^* \in R_+^{LS} \backslash \{0\}$ s.t.

      i. for $\forall i \in I$, x_i^* is $\succsim_i -$ maximal in $B_i\left(p^*, e_i\right) = \left\{x_i \in X_i \mid p^* \cdot x_i \leq p^* \cdot e_i\right\}$
      $\Leftrightarrow x_i^* \in \arg\max_{x_i \in B_i(p^*, e_i)} EU_i\left(x_i\right)$ s.t. $B_i\left(p^*, e_i\right) = \left\{x_i \in X_i \mid p^* \cdot x_i \leq p^* \cdot e_i\right\}$

      ii. $\sum_{i\in I} x_i^* \leq \sum_{i\in I} e_i$

4. Final 1999 #2

   Without uncertainty and With uncertainty

   L, I, S.

   Commodity Space$=R^{LS}$

   $X_i = R_+^{LS}$ for $\forall i \in I$

   Consumers have the same probabilities $\pi_s \in (0,1)$ for $s \in S$.

   $\succsim_i$defined on $X_i \subseteq R^{LS}$

$\rightarrow x_i \succsim_i x_i'$ iff $\sum_{s\in S}\pi_s u^i\left(x_{si}\right) = EU_i\left(x_i\right) \geq EU_i\left(x_i'\right) = \sum_{s\in S}\pi_s u^i\left(x_{si}'\right)$
and $u_i\left(\cdot\right)$ is strictly concave, strictly monotone, and continuous.
$e_i = \left\{\left(e_{1si}, e_{2si}, ..., e_{Lsi}\right)\right\}_{s\in S} \in R_+^{LS}$ and $e_{lsi} = e_{ls'i}$ for $\forall s \in S$; that is, no aggregate risk.

(a) Without uncertainty
$S = 1$
Let $p^* \in R_+^L$ and $\left\{x_i^*\right\}_{i\in I} \in R_+^{LI}$ be an Arrow equilibrium in this case.
Then it must solve that

$$\max_{x_i} u_i\left(x_i\right)$$

$$s.t.\ px_i \leq pe_i$$

and

$$\sum_{i\in I} x_{li}^* \leq \sum_{i\in I} e_{li} \text{ for } \forall l \in L$$

F.O.N.S.C.s are
$\frac{\frac{\partial u_i\left(x_{li}^*\right)}{\partial x_{li}}}{\frac{\partial u_i\left(x_{mi}^*\right)}{\partial x_{mi}}} = \frac{p_l^*}{p_m^*}$ for $\forall l, m \in L$

(b) With uncertainty
The consumer i's maximization problem is

$$\max_{x_{si}} \sum_{s\in S} \pi_s u_i\left(x_{si}\right)$$

$$s.t.\ \sum_{s\in S} p_s x_{si} \leq \sum_{s\in S} p_s e_{si}$$

and

$$\sum_{i\in I} x_{lsi} = \sum_{i\in I} e_{lsi} \text{ for } \forall l \in L \text{ and } \forall s \in S$$

F.O.N.S.C.s are
$\frac{\pi_s \frac{\partial u_i\left(x_{lsi}\right)}{\partial x_{lsi}}}{\pi_s \frac{\partial u_i\left(x_{msi}\right)}{\partial x_{msi}}} = \frac{p_{ls}}{p_{ms}}$ for $\forall l, m \in L$ at $\forall s \in S$
and $\frac{\pi_s \frac{\partial u_i\left(x_{lsi}\right)}{\partial x_{lsi}}}{\pi_{s'} \frac{\partial u_i\left(x_{ls'i}\right)}{\partial x_{ls'i}}} = \frac{p_{ls}}{p_{ls'}}$ for $\forall l \in L$, $\forall s, s' \in S$, $\forall i \in I$

$\boxed{\text{Claim}}$
$\tilde{p} \in R_+^{LS}$ and $\left\{\tilde{x}_i\right\}_{i\in I} \in R_+^{LSI}$ s.t. $\tilde{p}_s = \pi_s p^*$ for $\forall s$ and $\tilde{x}_{si} = x_i^*$ for $\forall s$ and $\forall i$ constitutes an Arrow-Debreu equilibrium under uncertainty.

i. $\dfrac{\pi_s \frac{\partial u_i(\tilde{x}_{lsi})}{\partial x_{lsi}}}{\pi_{s'} \frac{\partial u_i(\tilde{x}_{ls'i})}{\partial x_{ls'i}}} = \frac{\tilde{p}_{ls}}{\tilde{p}_{ls'}}$ for $\forall l \in L$, $\forall s, s' \in S$, $\forall i \in I$

$\dfrac{\pi_s \frac{\partial u_i(\tilde{x}_{lsi})}{\partial x_{lsi}}}{\pi_{s'} \frac{\partial u_i(\tilde{x}_{ls'i})}{\partial x_{ls'i}}} = \frac{p_{ls}}{p_{ls'}} = \dfrac{\pi_s \frac{\partial u_j(\tilde{x}_{lsj})}{\partial x_{lsj}}}{\pi_{s'} \frac{\partial u_j(\tilde{x}_{ls'j})}{\partial x_{ls'j}}}$ has to be satisfied.

Suppose not; $\frac{\partial u_i(\tilde{x}_{lsi})}{\partial x_{lsi}} \neq \frac{\partial u_i(\tilde{x}_{ls'i})}{\partial x_{ls'i}}$ for some $l \in L$, some $i \in I$.

W.l.o.g., we have $\frac{\partial u_i(\tilde{x}_{lsi})}{\partial x_{lsi}} > \frac{\partial u_i(\tilde{x}_{ls'i})}{\partial x_{ls'i}}$ for i

$\rightarrow \frac{\partial u_j(\tilde{x}_{lsj})}{\partial x_{lsj}} > \frac{\partial u_j(\tilde{x}_{ls'j})}{\partial x_{ls'j}}$ for $\forall j \neq i$

$\rightarrow \tilde{x}_{lsi} < \tilde{x}_{ls'i}$ for $\forall i \in I$

Then $\sum_{i\in I} \tilde{x}_{lsi} = \sum_{i\in I} e_{lsi} < \sum_{i\in I} \tilde{x}_{ls'i} = \sum_{i\in I} e_{ls'i}$

which contradicts to the assumption that there is no aggregate risk.

Therefore, $\tilde{x}_{lsi} = \tilde{x}_{ls'i} = \tilde{x}_{li} = x^*_{li}$ for $\forall i \in I, \forall l \in L$

ii. $\dfrac{\pi_s \frac{\partial u_i(x_{lsi})}{\partial x_{lsi}}}{\pi_{s'} \frac{\partial u_i(x_{ls'i})}{\partial x_{ls'i}}} = \dfrac{\pi_s \frac{\partial u_i(x^*_{li})}{\partial x_{li}}}{\pi_{s'} \frac{\partial u_i(x^*_{li})}{\partial x_{li}}} = \frac{\pi_s}{\pi_{s'}} = \frac{p_{ls}}{p_{ls'}}$ for $\forall l \in L$, $\forall s, s' \in S$

$\dfrac{\pi_s \frac{\partial u_i(x_{lsi})}{\partial x_{lsi}}}{\pi_s \frac{\partial u_i(x_{msi})}{\partial x_{msi}}} = \dfrac{\pi_s \frac{\partial u_i(x^*_{li})}{\partial x_{li}}}{\pi_s \frac{\partial u_i(x^*_{mi})}{\partial x_{mi}}} = \frac{\pi_s}{\pi_s} \cdot \frac{p^*_l}{p^*_m} = \frac{p_{ls}}{p_{ms}}$ for $\forall l, m \in L$ at $\forall s \in S$

$\tilde{p} \in R^{LS}_+$ s.t. $\tilde{p}_s = \pi_s p^*$ for $\forall s$

iii. This allocation is feasible automatically by construction.

(c) What if at least one consumer's subjective probabilities are different from π?

$\boxed{\text{Counterexample}}$

$L = 1, I = 2, S = 2$ with $u_i(x_{si}) = \ln x_{si}$ for $\forall s, i$.

$\pi_1 = \left(\frac{1}{2}, \frac{1}{2}\right) = \pi$ and $\pi_2 = \left(\frac{2}{3}, \frac{1}{3}\right)$

Then for equilibrium condition,

the following has to be satisfied.

$\frac{\pi_{11} u_1'(x_{11})}{\pi_{21} u_1'(x_{21})} = \frac{p_1}{p_2} \rightarrow \frac{\frac{1}{x_{11}}}{\frac{1}{x_{21}}} = \frac{p_1}{p_2}$

and $\frac{\pi_{12} u_2'(x_{12})}{\pi_{22} u_2'(x_{22})} = \frac{p_1}{p_2} \rightarrow 2\frac{\frac{1}{x_{12}}}{\frac{1}{x_{22}}} = \frac{p_1}{p_2}$

$\rightarrow \frac{x_{21}}{x_{11}} = 2\frac{x_{22}}{x_{12}}$

Therefore, we can not have $x_{11} = x_{21} = x_1$ and $x_{12} = x_{22} = x_2$ at the same time so that the above result is not satisfied.

5. Pareto Optimality

(a) $\boxed{\text{Every Pareto optimal allocation is state-independent(PSet \#8-1)}}$

Each consumer's preferences over state contingent commodities have an expected utility representation with $\boxed{\text{common probabilities}}\left(\pi_1^1 = \pi_1^2 = \pi_1, \pi_2^1 = \pi_2^2 = \pi_2\right)$.

$\boxed{\text{Consumers are strictly risk averse}}$, and

there is $\boxed{\text{no aggregate risk}}\left(e_1^1 + e_1^2 = e_1 = e_2 = e_2^1 + e_2^2 = e \text{ for } \forall s = 1, 2\right)$; i.e. the aggregate endowment is state-independent.

$\boxed{\text{Proof}}$

Suppose not; i.e., $\exists$a Pareto optimal allocation $x = \left\{\left(x^1, x^2\right)_{s=1}, \left(x^1, x^2\right)_{s=2}\right\} \in R_+^{2\cdot 2\cdot 2} = R_+^{LSI}$ s.t. $x_1^1 \neq x_2^1$ and $x_1^2 \neq x_2^2$.
Construct $x' = \left\{\left(x'^1, x'^2\right)_{s=1}, \left(x'^1, x'^2\right)_{s=1}\right\} \in R_+^{2\cdot 2\cdot 2}$ s.t. $x_1'^1 = x_2'^1 = \pi_1 \cdot x_1^1 + \pi_2 \cdot x_2^1$ and $x_1'^2 = x_2'^2 = \pi_1 \cdot x_1^2 + \pi_2 \cdot x_2^2$.
We will show that x' pareto dominates x.and it is feasible.

i. Because consumers are strictly risk averse, the expected utility functions are strictly concave for both agents.
$EU_1\left(x^{1\prime}\right) = \pi_1 u_1\left(x_1'^1\right) + \pi_2 u_1\left(x_2'^1\right) = \left(\pi_1 + \pi_2\right) u_1\left(x_1'^1\right) = u_1\left(x_1'^1\right) = u_1\left(\pi_1 \cdot x_1^1 + \pi_2 \cdot x_2^1\right)$
$> \pi_1 u_1\left(x_1^1\right) + \pi_2 u_1\left(x_2^1\right) = EU_1\left(x^1\right)$
$EU_2\left(x^{2\prime}\right) = \pi_1 u_2\left(x_1'^2\right) + \pi_2 u_2\left(x_2'^2\right) = \left(\pi_1 + \pi_2\right) u_2\left(x_1'^2\right) = u_2\left(x_1'^2\right) = u_2\left(\pi_1 \cdot x_1^2 + \pi_2 \cdot x_2^2\right)$
$> \pi_1 u_2\left(x_1^2\right) + \pi_2 u_2\left(x_2^2\right) = EU_2\left(x^2\right)$
Therefore, x' pareto dominates x

ii. $x_1'^1 + x_1'^2 = \pi_1 \cdot x_1^1 + \pi_2 \cdot x_2^1 + \pi_1 \cdot x_1^2 + \pi_2 \cdot x_2^2 = \pi_1 \cdot e_1 + \pi_2 \cdot e_2 = \left(\pi_1 + \pi_2\right) \cdot e = e$
$x_2'^1 + x_2'^2 = \pi_1 \cdot x_1^1 + \pi_2 \cdot x_2^1 + \pi_1 \cdot x_1^2 + \pi_2 \cdot x_2^2 = \pi_1 \cdot e_1 + \pi_2 \cdot e_2 = \left(\pi_1 + \pi_2\right) \cdot e = e$
Therefore, x' is feasible.
which is a contradiction!!!!

iii. Important Examples
1) If agents do $\boxed{\text{not}}$ have the common probabilities, then a pareto optimal allocation can be state-dependent.
2) If there is an aggregate risk($e^1 + e^2 = (2, 1)$), then at any point of pareto set, the common marginal rate of substitution is smaller than the ratio of probabilities
$\left(\frac{\pi_1}{\pi_2} > \frac{\frac{\partial u_1}{\partial x_1^1}}{\frac{\partial u_1}{\partial x_2^1}} = \frac{\frac{\partial u_2}{\partial x_1^2}}{\frac{\partial u_2}{\partial x_2^2}} = \frac{p_1}{p_2}\right)$
$\rightarrow \frac{p_2}{\pi_2}$ is negatively correlated with total endowment of good 2
→contingent instruments(a unit of contingent consumption) are comparatively more valuable if their returns(the amount of consumption in the different states) are negatively correlated with the market return(the aggregate initial endowment)

(b) $L = 1$ or $2, I = 2, S = 2$
$e^i \in R_{++}^2$ or $R_{++}^{2\cdot 2}$for $\forall i = 1, 2$
Each consumer's preferences over state contingent commodities have an expected utility representation with $\boxed{\text{common probabilities}}\left(\pi_1^1 = \pi_1^2 = \pi_1, \pi_2^1 = \pi_2^2 = \pi_2\right)$.

$\boxed{\text{Consumers are strictly risk averse}}$, and

there is $\boxed{\text{no aggregate risk}}$ $\left(e_1^1 + e_1^2 = e_1 = e_2 = e_2^1 + e_2^2 = e \text{ for } \forall s = 1,2\right)$

show that an allocation $\left(x^1, x^2\right) \in R_+^{2\cdot 2}$ with $x_s^i = \pi_1 e_1^i + \pi_2 e_2^i$ for $\forall i = 1,2$, and $\forall s = 1,2$ together with price $(p_1, p_2) = (\pi_1, \pi_2)$ constitute an Arrow-Debreu equilibrium for this economy.

$\boxed{\text{Solution}}$ Without production,

In the pure exchange economy E with uncertainty, a Competitive Equilibrium, which we call an Arrow-Debreu equilibrium, consists of $\{x_i^*\}_{i\in I} \in R_+^{LSI}$ and $p^* \in R_+^{LS} \backslash \{0\}$ s.t.

for $\forall i \in I$, x_i^* is $\succsim_i -$ maximal in $B_i(p^*, e_i) = \left\{x_i \in X_i \mid p^* \cdot x^i \leq p^* \cdot e^i\right\}$

$\Leftrightarrow x_i^* \in \arg\max_{x_i \in B_i(p^*, e_i)} EU_i(x_i)$ s.t. $B_i(p^*, e_i) = \left\{x_i \in X_i \mid p^* \cdot x^i \leq p^* \cdot e^i\right\}$

$\sum_{i\in I} x_i^* \leq \sum_{i\in I} e_i$

For $\forall i = 1,2$,

In this case, $EU_i(x_i) = \pi_1 u_i\left(x_1^i\right) + \pi_2 u_i\left(x_2^i\right)$ with $p_1^* \cdot x_1^i + p_2^* \cdot x_2^i \leq p_1^* \cdot e_1^i + p_2^* \cdot e_2^i$

FOC

$x_1^i : \pi_1 u_i'\left(x_1^i\right) - \lambda p_1^* = 0$

$x_2^i : \pi_2 u_i'\left(x_2^i\right) - \lambda p_2^* = 0$

$\rightarrow \frac{\pi_1 u_i'\left(x_1^i\right)}{\pi_2 u_i'\left(x_2^i\right)} = \frac{p_1^*}{p_2^*}$

$\rightarrow u_i'\left(x_1^i\right) = u_i'\left(x_2^i\right)$

$\rightarrow x_1^i = x_2^i = x^i$

$\pi_1 x^i + \pi_2 x^i = \pi_1 x_1^i + \pi_2 x_2^i = p_1^* \cdot x_1^i + p_2^* \cdot x_2^i = p_1^* \cdot e_1^i + p_2^* \cdot e_2^i = \pi_1 e_1^i + \pi_2 e_2^i$

$\rightarrow (\pi_1 + \pi_2) x^i = \pi_1 e_1^i + \pi_2 e_2^i$

$\rightarrow x^i = \pi_1 e_1^i + \pi_2 e_2^i$

We need to feasibility.

$x_1^1 + x_1^2 = x^1 + x^2 = \pi_1 e_1^1 + \pi_2 e_2^1 + \pi_1 e_1^2 + \pi_2 e_2^2 = \pi_1 e_1 + \pi_2 e_2 = (\pi_1 + \pi_2)\, e = e!!!$

(c) Final 1998

$L = 1$ or $2, I = 2, S = 2$

Each consumer's preferences over state contingent commodities have an expected utility representation with $\boxed{\text{common probabilities}}$ $\left(\pi_1^1 = \pi_1^2 = \pi_1, \pi_2^1 = \pi_2^2 = \pi_2\right)$.

$\boxed{\text{Consumer 1 is risk-neutral}}$ and $\boxed{\text{Consumer 2 is strictly risk averse}}$.

Consumption allocation is restricted to be nonnegative.

Show that the consumption plan of the strictly risk averse consumer in every pareto optimal allocation is state-independent.

$\boxed{\text{Proof}}$

Each consumer's preferences over state contingent commodities have an expected utility representation with $\boxed{\text{common probabilities}}$ $\left(\pi_1^1 = \pi_1^2 = \pi_1, \pi_2^1 = \pi_2^2 = \pi_2\right)$

Then each consumer's expected utililty can be represented by $U_i\left(x_1^i, x_2^i\right) = \pi_1 u_i\left(x_1^i\right) + \pi_2 u_i\left(x_2^i\right)$ for $\forall i = 1, 2$
Because consumer 2's expected utility function is strictly concave and strictly increasing, from his expected utility maximization, we will get the interior solution as consumption plan.
Let $x = \left\{\left(x^1, x^2\right)_{s=1}, \left(x^1, x^2\right)_{s=2}\right\} \in R_+^{1\cdot 2\cdot 2}$ be a pareto optimal allocation.
Then the interior pareto optimality condition gives us that
$\frac{\pi_1 u_2'\left(x_1^2\right)}{\pi_2 u_2'\left(x_2^2\right)} = \frac{p_1}{p_2}$
Because consumer 1 is risk neutral, he has a linear expected utility function and then
$\frac{\pi_1 u_1'\left(x_1^1\right)}{\pi_2 u_1'\left(x_2^1\right)} = \frac{\pi_1}{\pi_2} = \frac{p_1}{p_2}$
It implies that $u_2'\left(x_1^2\right) = u_2'\left(x_2^2\right)$
Then $x_1^2 = x_2^2$.
Therefore, consumer 2's consumption plan is state-independent in any Pareto optimal allocation.

6. Welfare Theorems

The welfare theorems apply without modification to the Arrow-Debreu equilibrium.

Especially, the convexity assumptions appears in terms of risk aversion and in the expected utility setting, the preference $\succsim_i$is convex if the expected utility function is concave.

The pareto optimality implication of Arrow-Debreu equilibrium means that the possibility of trading in contingent commodities leads to an efficient allocation of risk.

(a) First Welfare Theorem

$\boxed{\text{Claim}}$With monotonicity of preference $\succsim_i$ for $\forall i \in I$, an Arrow-Debreu equilibrium allocation $\left\{\{x_i^*\}_{i\in I}, \left\{y_j^*\right\}_{j\in J}\right\} \in R_+^{LSI} \times R^{LSJ}$ with $p^* \in R_+^{LS} \backslash \{0\}$ is pareto optimal.

$\boxed{\text{Proof}}$

Suppose not; then $\exists$another allocation $\left\{\{x_i'\}_{i\in I}, \left\{y_j'\right\}_{j\in J}\right\} \in R_+^{LSI} \times R^{LSJ}$ s.t.
$\sum_{i\in I} x_i' = \sum_{i\in I} e_i + \sum_{j\in J} y_j'$(equality is coming from monotonicity of $\succsim_i$)
$x_i' \succsim_i x_i^*$ for $\forall i \in I$ and
$x_{i'}' \succ_i x_{i'}^*$ for at least one $i' \in I$.
By monotonicity of $\succsim_i$and revealed preference axiom,
$x_{i'}' \succ_i x_{i'}^* \rightarrow p^* \cdot x_{i'}' > p^* \cdot x_{i'}^*$ for at least one $i' \in I$. and
$x_i' \succsim_i x_i^* \rightarrow p^* \cdot x_i' \geq p^* \cdot x_i^*$ for $\forall i \in I$

By summing up for all agents, $\sum_{i\in I} p^* \cdot x_i' > \sum_{i\in I} p^* \cdot x_i^* = \sum_{i\in I} p^* \cdot e_i + \sum_{j\in J} p^* \cdot y_j^*$

By definition of Arrow-Debreu equilibrium,

$\sum_{i\in I} p^* \cdot x_i' > \sum_{i\in I} p^* \cdot e_i + \sum_{j\in J} p^* \cdot y_j^* \geq \sum_{i\in I} p^* \cdot e_i + \sum_{j\in J} p^* \cdot y_j'$

$\rightarrow \sum_{i\in I} p^* \cdot x_i' > \sum_{i\in I} p^* \cdot e_i + \sum_{j\in J} p^* \cdot y_j'$

$\rightarrow \sum_{i\in I} x_i' > \sum_{i\in I} e_i + \sum_{j\in J} y_j'$ which is a contradiction!!!!

Therefore, an Arrow-Debreu equilibrium allocation $\left\{ \{x_i^*\}_{i\in I}, \left\{ y_j^* \right\}_{j\in J} \right\} \in R_+^{LSI} \times R^{LSJ}$ with $p^* \in R_+^{LS} \backslash \{0\}$ is pareto optimal.

An important reinterpretation of the concept of Arrow-Debreu equilibrium. We show taht under the assumptions of self-fulfilling or rational expectations, Arrow-debreu equilibria can be implemented by combining trade in a certain restricted set of contingent commodities with spot trade that occurs after the resolution of uncertainty. This results in a significant reduction in the number of ex-ante markets that must operate.

Instead of trading contingent commodities prior to the resolution of uncertainty, agents now trade assets; and instead of an Arrow-Debreu equilibrium we have the notion of Radner equilibrium. We discuss here the important notion of arbitrage among assets.

We briefly illustrate some of the welfare difficulties raised by the possibility of incomplete markets, that is, by the possibility of there being too few asset markets to guarantee a fully Pareto optimal allocation of risk.

(b) Arrow-Debreu equilibrium is Ex-ante P.O.

Ex-ante P.O.→Ex-post P.O. for $\forall s \in S$

$\boxed{\text{Proof}}$

$L = 2, I = 1, S = 2$

Suppose x^* is ex-ante P.O, but it is not ex-post P.O.

then $\exists s \in S$ and. $\exists \left\{ \{x_i'\}_{i\in I} \right\} \in R_+^{LSI}$ s.t.x_s' pareto dominates x_s^*.

Then $\left\{ \left(x_{-s}^*, x_s' \right) \right\} \in R_+^{LSI}$ is ex-ante pareto dominate x^* which is a contradiction.

In summary, at $t = 0$, the consumers can trade directly to an overall Pareto optimal allocation; hence there is no reason for further trade to take place. In other words, Ex-ante P.O. is Ex-post P.O. and then there is ex-post trade.

10.3 Radner Equilibrium

At an Arrow-Debreu equilibrium, all trade take place simultaneously and before the uncertainty is resolved. Trade is one shot affair. However, in reality, trade take place to a large extent sequentially over time, and frequently as a consequence of information disclosures.

And if not all LS contingent commodity markets are available at $t = 0$, then the initial trade to a pareto optimal allocation may not be feasible and it is quite possible that ex post(after the realization of state s) the resulting consumption allocation is not pareto optimal. There would be an incentive to reopen the markets and retrade.

The aim of this section is to introduce a first model of sequential trade and show that Arrow-Debreu equilibria can be reinterpreted by means of trading precesses that actually unfold through time. Arrow observed that even if not all the contingent commodities are available at $t = 0$, it may still be the case under some conditions that the retrading possibilities at $t = 1$ gurantee that pareto optimality is reached. We shall verify that the is the case whenever at least one physical commodity can be traded contingently at $t = 0$ if spot markets occur at $t = 1$ and the spot equilibrium prices are correctly anticipated at $t = 0$.

If spot trade can occur within each state, then the only task remaining at $t = 0$ is to transfer the consumer's overall purchasing power efficiently across states. This can be accomplished using contingent trade in a single commodity. By such a procedure we are able to reduce the number of required forward markets for LS to S.

Faced with prices $q \in R^S$ at $t = 0$ and expected spot prices $(p_1, p_2, ..., p_S) \in R^{LS}$ at $t = 1$, and with no endowment at $t = 0$ and $e_i = \left\{\left(e_{11}^i, e_{21}^i, ..., e_{L1}^i\right), \left(e_{12}^i, e_{22}^i, ..., e_{L2}^i\right), ..., \left(e_{1s}^i, ..., e_{Ls}^i\right)\right\} \in R^{LS}$ at $t = 1$, every consumer i formulates a trading plan $\left(z_1^i, ..., z_S^i\right) \in R^S$ for contingent commodities at $t = 0$, as well as a set of spot market consumption plans $\left(x_1^i, ..., x_S^i\right) \in R^{LS}$ for the different states taht may occur at $t = 1$ satisfying

$$\max_{(z,x)} EU_i\left(x_1^i, ..., x_S^i\right)$$

$$\begin{aligned} s.t. \sum_s q_s z_s^i &\leq 0 \\ p_s x_s^i &\leq p_s e_s^i + p_{1s} z_s^i \end{aligned}$$

If $z_s^i < -e_s^i$, then we say that at $t = 0$, consumer i is selling good 1 short. This is because he is selling at $t = 0$, contingent on state s occuring, more than he endows at $t = 1$ if s occurs. Hence if s occurs, he will actually have to buy in the spot market the extra amount $\left(-(z_s^i + e_s^i)\right)$of the first good required for the fulfillment of his commitments.

To define an appropriate notion of sequential trade, we assume that consumer's expectations of the prices that will clear the spot markets for the different states s do actually clear them once date $t = 1$ has arrived and a state s occurs. It is crucial

1. Environment

   Pure exchange economy

   Commodity Space=R^L

   $t = 0, 1$

   $\exists S$ states in date $t = 1$

There is no information and no consumption at $t = 0$

Uncertainty resolved at date $t = 1$

$X_i = R_+^{LS}$ for $\forall i \in I$

$\succsim_i$defined on $X_i = R^{LS}$

$\rightarrow x_i \succsim_i x_i'$ iff $\sum_{s\in S} \pi_s^i u_s^i \left(x_{1s}^i, x_{2s}^i, ..., x_{Ls}^i\right) = EU_i\left(x_i\right) \geq EU_i\left(x_i'\right) = \sum_{s\in S} \pi_s^i u_s^i \left(x_{1s}'^i, x_{2s}'^i, ..., x_{Ls}'^i\right)$

There is no initial endowment of the contingent commodity market.

$e_i = \left\{\left(e_{11}^i, e_{21}^i, ..., e_{L1}^i\right), \left(e_{12}^i, e_{22}^i, ..., e_{L2}^i\right), ..., \left(e_{1s}^i, ..., e_{Ls}^i\right)\right\} \in R^{LS}$

2. Radner equilibrium with Arrow Securities

   With no endowment at $t = 0$ and $e_i = \left\{\left(e_{11}^i, e_{21}^i, ..., e_{L1}^i\right), \left(e_{12}^i, e_{22}^i, ..., e_{L2}^i\right), ..., \left(e_{1s}^i, ..., e_{Ls}^i\right)\right\} \in R^{LS}$ at $t = 1$,

   a Radner equilibrium $\{(q, p); (z, x)\}$ consists of a collection $(q, p)$of a price vector $q \in R^S$ for contingent first goods at $t = 0$ and expected spot prices $p = (p_1, p_2, ..., p_S) \in R^{LS}$ for every state s at $t = 1$, and a collection (z, x) of a consumption plan $(z_1, ..., z_S) \in R^{SI}$ for contingent commodities at $t = 0$ and a set of spot market consumption plans $(x_1, ..., x_S) \in R^{LSI}$ for the different states s at $t = 1$ s.t.

   (a) For all $\forall i \in I$, $\left(z^{i*}, x^{i*}\right)$ is $\succsim_i$ −maximal in
   $B_i^R = \left\{x^i \in X_i \mid \text{for } \forall i \in I\ \exists z^{i*} \in R^S \text{ s.t. } \sum_s q_s z_s^i \leq 0 \text{ and for } \forall s \in S,\ p_s x_s^i \leq p_s e_s^i + \sum_k p_{1s} z_k^i r\right.$

   (b) For $\forall s \in S$,
   $\sum_{i\in I} z_s^{i*} \leq 0$
   $\sum_{i\in I} x_s^{i*} \leq \sum_{i\in I} e_s^i$

3. Radner equilibrium with Asset-Markets

   With no endowment at $t = 0$ and $e_i = \left\{\left(e_{11}^i, e_{21}^i, ..., e_{L1}^i\right), \left(e_{12}^i, e_{22}^i, ..., e_{L2}^i\right), ..., \left(e_{1s}^i, ..., e_{Ls}^i\right)\right\} \in R^{LS}$ at $t = 1$,

   a Radner equilibrium $\{(q, r, p); (z, x)\}$ consists of

   a collection $(q, r, p)$of a price vector $q \in R^K$ and a return vector $r \in R^K$ for assets at $t = 0$, and expected spot prices $p = (p_1, p_2, ..., p_S) \in R^{LS}$ for every state s at $t = 1$ and

   a collection (z, x) of a trading portfolio $(z_1, ..., z_K) \in R^{KI}$ for assets at $t = 0$ and a set of spot market consumption plan $(x_1, ..., x_S) \in R^{LSI}$ for the different states s at $t = 1$ s.t.

   (a) For all $\forall i \in I$, $\left(z^{i*}, x^{i*}\right)$ is $\succsim_i$ −maximal in
   $B_i^R = \left\{x^i \in X_i \mid \text{for } \forall i \in I\ \exists z^{i*} \in R^K \text{ s.t. } \sum_k q_k z_k^i \leq 0 \text{ and for } \forall s \in S,\ p_s x_s^i \leq p_s e_s^i + \sum_k p_{1s} r_s\right.$

   (b) For $\forall k \in K, \sum_{i\in I} z_k^{i*} \leq 0$
   For $\forall s \in S, \sum_{i\in I} x_s^{i*} \leq \sum_{i\in I} e_s^i$

- All the budget constraints at every state are HD of 0 in prices so that we normalize the first goods price at every states $p_{1s} = 1$ and $\left(q_1 = 1 \text{ or } \sum_{s\in S} q_s = 1\right)$.
- Return matrix R
  $$R = [r_1, ..., r_K] = \begin{bmatrix} r_{11} & \cdots & r_{1K} \\ \vdots & \ddots & \vdots \\ r_{S1} & \cdots & r_{SK} \end{bmatrix} = (S \cdot K) \text{ matrix}$$

4. Radner equilibrium with Arrow securities⇔Radner equilibrium with Assets

   Suppose the asset market is complete

   and with $S \times K$ return matrix R, $\{(q, r, p); (z, x)\}$ is a Radner equilibrium with assets

   Let $R' = I$ be the $S \times S$ return identity matrix of an Arrow securities..

   $\{(q', r, p); (z', x)\}$ is a Radner equilibrium with an Arrow security.

   If Range$R$ =RangeR', then $x \in \{(q, r, p); (z, x)\} \Leftrightarrow x \in \{(q', r, p); (z', x)\}$

   $\boxed{\text{Proof}}$

   Range$R = \left\{v \in R^S \mid v = Rz \text{ for some } z \in R^K\right\} \subset R^S$

   If Range$R$ =Range$I$, then $\exists z' \in R^S$ s.t. $Rz^i = Iz'^i$

   And because, in the Radner equilibrium, the asset portfolio q, q' are arbitrage free, we have $q = \mu \cdot R$ with $\mu \in R^S_{++}$ and $q' = \mu \cdot I$ with $\mu \in R^S_{++}$

   $q \cdot z^i = \mu \cdot R \cdot z^i = \mu \cdot I \cdot z'^i = q' \cdot z'^i$.

   This gives what we want.

   $\boxed{\text{Example}}$Look at the above

5. SP 1998 III-2

   Consider an exchange economy with $t = 0, 1$ s.t.at $t = 1$, states S, $L = 1$, $I = 2$.

   $\exists$two Arrow Securities traded at $t = 0$

   (a) Radner equilibrium with Asset-Markets
   With no endowment at $t = 0$ and $e_i = \left\{\left(e^i_{11}, e^i_{21}, ..., e^i_{L1}\right), \left(e^i_{12}, e^i_{22}, ..., e^i_{L2}\right), ..., \left(e^i_{1s}, ..., e^i_{Ls}\right)\right\} \in R^{LS}$ at $t = 1$,
   a Radner equilibrium $\{(q, r, p); (z, x)\}$ consists of
   a collection $(q, r, p)$of a price vector $q \in R^K$ and a return vector $r \in R^K$ for assets at $t = 0$, and expected spot prices $p = (p_1, p_2, ..., p_S) \in R^S$ for every state s at $t = 1$ and
   a collection (z, x) of a trading portfolio $(z_1, ..., z_K) \in R^{KI}$ for assets at $t = 0$ and a set of spot market consumption plan $(x_1, ..., x_S) \in R^{SI}$ for the different states s at $t = 1$ s.t.

i. For all $\forall i \in I$, $\left(z^{i*}, x^{i*}\right)$ is $\succsim_i$ −maximal in
$B_i^R = \{x^i \in X_i \mid$ for $\forall i \in I$ $\exists z^{i*} \in R^K$ s.t. $\sum_k q_k z_k^i \leq 0$ and for $\forall s \in S$, $p_s x_s^i \leq p_s e_s^i + \sum_k$

ii. For $\forall k \in K, \sum_{i \in I} z_k^{i*} \leq 0$
For $\forall s \in S, \sum_{i \in I} x_s^{i*} \leq \sum_{i \in I} e_s^i$

(b) $K = S$.

$\exists z^1, z^2$ s.t. r is indentity matrix.

Consumer 1 is risk-neutral and consumer 2 is strictly risk-averse.

Show that an equilibrium consumption plan of the strictly risk-averse Consumer 2 is state-independent in any Radner equilibrium(and in any pareto optimal allocation) in which consumption plans of both consumers are interior.

$\boxed{\text{Proof}}$

Each consumer's preferences over state contingent commodities have an expected utility representation with $\boxed{\text{common probabilities}}$ $\left(\pi_1^1 = \pi_1^2 = \pi_1, \pi_2^1 = \pi_2^2 = \pi_2\right)$

The consumers expected utility maximization problems are

$$\max_{x^i} \sum_{s=1}^{S} \pi_s u_i\left(x_s^i\right)$$

$$\begin{aligned} s.t. \sum_s q_s z_s^i &\leq 0 \\ \sum_s p_s x_s^i &\leq \sum_s p_s e_s^i + \sum_s p_s z_s^i \end{aligned}$$

Because consumer 2's expected utility function is strictly concave and strictly increasing, from his expected utility maximization, we will get the interior solution as consumption plan.

FOC.

$x_s^2 : \pi_s u_2'\left(x_s^2\right) - \lambda_1 p_s = 0$ for $\forall s \in S$.

$x_s^1 : \pi_s u_1'\left(x_s^1\right) - \lambda_2 p_s = 0$ for $\forall s \in S$.

$\mu_2 q_s = \lambda_2 p_s$ for $\forall s \in S$

$\mu_1 q_s = \lambda_1 p_s$ for $\forall s \in S$

$\frac{\pi_s u_2'\left(x_s^2\right)}{\pi_{s'} u_2'\left(x_{s'}^2\right)} = \frac{\pi_s u_1'\left(x_s^1\right)}{\pi_{s'} u_1'\left(x_{s'}^1\right)} = \frac{p_s}{p_{s'}} = \frac{q_s}{q_{s'}}$

Because consumer 1 is risk neutral, he has a linear expected utility function and then

$\frac{\pi_s u_1'\left(x_s^1\right)}{\pi_{s'} u_1'\left(x_{s'}^1\right)} = \frac{\pi_s}{\pi_{s'}} = \frac{p_s}{p_{s'}} = \frac{q_s}{q_{s'}}$

It implies that $u_2'\left(x_s^2\right) = u_2'\left(x_{s'}^2\right)$ for $\forall s, s' \in S$

Then $x^2_s = x^2_{s'}$.
Therefore, consumer 2's consumption plan is state-independent in any Radner equilibrium.

6. FP 1999 III-2

   Consider an exchange economy with $t = 0, 1$ s.t.at $t = 1$, states S, L, I.
   $\exists K$ assets traded at $t = 0$ with $r_{sk} \in R^L_+$

   (a) State a definition of Radner equilibrium in such asset market economy.

   (b) Given commodity price vectors $p \in R^{LS}_+$, an asset price vector $q \in R^K$ is called arbitrage free if there is no portfolio $z \in R^K$ s.t. $qz = \sum_k q_k z^i_k \leq 0$ and $\sum_k p_{1s} r_{sk} z^i_k = p_{1s} R z \geq 0$ with $r_s z^i \neq 0$ for $\forall s \in S$, with at least one strict inequality for some s.
   Show that an asset price vector q is arbitrage free s.t. $q_k = \sum_s \mu_s p_s r_{sk} = (S \times 1) \,.*\, (S \times 1) \,.*\, (S \times 1)$ for each $k \in K$ with $\mu >> 0$

   $\boxed{\text{Intuition}}$

   All the budget constraints at every state are HD of 0 in prices so that we normalize the first goods price at every states $p_{1s} = 1$.

   $q_k = \sum_s \mu_s p_s r_{sk}$ means that we can assign values $(\mu_1, ..., \mu_S) >> 0$ to units of wealth in the different states so that the prices of a unit of asset k is simple equal to the sum of the value of the returns across states of the asset k.

   $\mu_s p_s$ can be interpreted as the implicit price of state-contingent commodity that pays one unit of good 1 if state s occurs.

   In words, the fact that an asset price vector q is arbitrage free means that there is no portfolio that is budgetarily feasible and that yields a nonnegative return in every state and a strictly positive return in some state. Note that whether an asset price vector is arbitrage free or not depends only on the returns of the assets and not on preferences.

   $\boxed{\text{Claim}}$

   $\Leftrightarrow$If there is a vector of multipliers $\mu = (\mu_1, ..., \mu_S) >> 0$ satisfying $q_k = \sum_s \mu_s p_s r_{sk}$ for each $k \in K$, then the asset price vector $q \in R^K$ is arbitrage free

   $\boxed{\text{Proof}}$

   (?)

   (c) $\boxed{\text{Claim}}$

   If the asset price vector $q \in R^K$ is arbitrage free, then there is a vector of multipliers $\mu = (\mu_1, ..., \mu_S) >> 0$ satisfying $q_k = \sum_s \mu_s p_s r_{sk}$ for each $k \in K$

   $\boxed{\text{Proof}}$

   Note to begin with that since we deal with assets having nonnegative, nonzero returns, and aribitrage free price vector $q_k > 0$ for $\forall k \in K$.

Also, without loss of generality, we assume that no row of the return of return matrix R has all of its entries equal to 0.

Given an arbitrage free asset price vector $q \in R^K$,

consider a convex set $V = \{v \in R^S \mid v = Rz \text{ for some } z \in R^K \text{ with } qz = 0\}$

The arbitrage freeness of q implies that $V \cap \{R^S_+ \backslash \{0\}\} = \emptyset$.

Since V and $\{R^S_+ \backslash \{0\}\}$ are convex sets and the origin belongs to V, we can apply the separating hyperplane theorem to obtain a nonzero vector $\mu' = (\mu'_1, ..., \mu'_S)$ s.t. $\mu' \cdot v \leq 0$ for any $v \in V$ and $\mu' \cdot v \geq 0$ for any $v \in R^S_+ \backslash \{0\}$.

Note that it must be that $\mu' \geq 0$.

Moreover, because $v \in V$ implies $-v \in V$, it follows that $\mu' \cdot v = 0$ for any $v \in V$.

We now argue that the row vector q^T must be proportional to the row vector $\mu' R \in R^K$.

The entries of μ' and of R are all nonnegative and no row of R is null.

Therefore, $\mu' R \geq 0^T$ and $\mu' R \neq 0^T$. If q^T is not proportional to $\mu' R$, then we can find $z' \in R^K$ s.t. $qz' = 0$ and $\mu' R z' > 0$

But letting $v = Rz'$, we would then have $v \in V$ and $\mu' v \neq 0$, which we ahve just seen cannot happen. Hence q^T must be proportional to $\mu' R$; that is, $q^T = \alpha \mu' R$ for some real number $\alpha > 0$. Letting $\mu = \alpha \mu'$, we have the conclusion of the lemma.

(d) Imposing suitable conditions on consumer's preferences, show that asset prices are arbitrage free in a Radner equilibrium;

that is for every vector $q \in R^K$ of asset prices arising in a Radner equilibrium, we can find $\mu = (\mu_1, ..., \mu_S) \geq 0$ satisfying $q_k = \sum_s \mu_s p_s r_{sk}$ for each $k \in K$

$\boxed{\text{Proof}}$

If we assume that preferences are strictly monotone, then an equilibrium asset price vector $q \in R^K$ must be arbitrage free; if it were not, it would be possible to increase utility merely by adding a portfolio yielding an arbitrage opportunity to any current portfolio. Because there are no restrictions on short sales, this addition is always feasible.

If in a Radner equilibrium, the asset prices are not arbitrage-free, then $\exists$a portfolio $\bar{z} \in R^K$ s.t. $q \cdot \bar{z} = \sum_k q_k \bar{z}^I_k < 0$ and $\sum_k p_{1s} r_{sk} \bar{z}^i_k = p_{1s} R \bar{z} = 0$ with $r_s \bar{z}^i \neq 0$ for $\forall s \in S$, with at least one strict inequality for some s with $p_{1s} = 1$.

For an arbitrary consumption plan x and portfolio z, let z^* be a portfolio with positive and nonzero return $Rz^* > 0$.

Because there are no restrictions on short sales, there exists every a real number $\alpha > 0$ s.t. $\alpha q \bar{z} = q z^*$.

But, then portfolio $z + z^* - \alpha \bar{z}$ which satisfies $q \cdot (z + z^* - \alpha \bar{z}) \leq 0$ and $\sum_k p_{1s} r_{sk} (z^i_k + z^{i*}_k - \alpha \bar{z}^i_k) \geq 0$ so that consumption plan $x + (z^* - \alpha \bar{z}) R = x + z^* R$ are budget feasible and strictly preferred.

Thus (z, x) is not in a Radner equilibrium which is a contradiction!!!

10.4 Equivalence between Arrow-Debreu equilibrium and Radner equilibrium

1. Arrow Securities

   An Arrow Security is a unit of an asset which is a title to receive a return $r_s \cdot z_s^i$ of a contingent commodity iff state s occurs for consumer i and $r_s = 1$ for $\forall s \in S$. It serves the purpose of transferring wealth across the states of the world that will be revealed in the future. If spot trade can occur within each state, then the only task remaining at $t = 0$ is to transfer the consumer's overall purchasing power efficiently across states. It can be accomplished using contingent trades in a single commodity w.l.o.g. good 1; i.e. every consumer i formulates a trading plan $\left(z_1^i, ..., z_S^i\right) \in R^S$ of good 1 and $r_s = 1$ for $\forall s \in S$..

2. With Arrow-Securities

   (a) If an Arrow-Debreu equilibrium consists of $\left\{x^{i*}\right\}_{i \in I} \in R_+^{LSI}$ and $p \in R_{++}^{LS}$, then $\exists q \in R_{++}^S$ and $z^* \in R^{SI}$ s.t. $\left\{(q,p)\,;\left(z^{i*}, x^{i*}\right)_{i \in I}\right\}$ constitutes a Radner equilibrium.

   (b) If a Radner equilibrium consists of $\left\{(q,p)\,;\left(z^{i*}, x^{i*}\right)_{i \in I}\right\}$,.then $\exists \mu = (\mu_1, ..., \mu_S) \in R_{++}^S$ s.t $\left\{\mu p, \left\{x^{i*}\right\}_{i \in I}\right\} \in R_{++}^{LS} \times R_+^{LSI}$ constitutes an Arrow-Debreu equilibrium.

   $\boxed{\text{Proof}}$

   First, note that budget constraints in each equilibrium is
   $B_i^{AD} = \left\{x^i \in X_i \mid \sum_{s \in S} p_s \cdot x_s^i \leq \sum_{s \in S} p_s \cdot e_s^i\right\}$
   $B_i^R = \left\{x^i \in X_i \mid \text{for } \forall i \in I \ \exists z^{i*} \in R^S \text{ s.t. } \sum_s q_s z_s^i \leq 0 \text{ and for } \forall s \in S,\ p_s x_s^i \leq p_s e_s^i + p_{1s} z_s^i\right\}$
   and feasibility conditions are
   for $\forall s \in S$, $\sum_{i \in I} x_s^{i*} \leq \sum_{i \in I} e_s^i$
   for $\forall s \in S$, $\sum_{i \in I} z_s^{i*} \leq 0$ and $\sum_{i \in I} x_s^{i*} \leq \sum_{i \in I} e_s^i$
   In order to show the equivalence of both equilibria, we need to show that
   i) budget constraints are the same
   ii) feasibilities are satisfied.

   (a) Suppose $\left\{x^{i*}\right\}_{i \in I} \in R_+^{LSI}$ and $p \in R_{++}^{LS}$ is an Arrow-Debreu equilibrium

      i. Define q and z s.t $q_s = p_{1s}$ and $z_s^{i*} = \frac{p_s}{p_{1s}}\left(x_s^{i*} - e_s^i\right)$ for $\forall s \in S$
      Then $p_{1s} z_s^{i*} = p_{1s} \frac{p_s}{p_{1s}}\left(x_s^{i*} - e_s^i\right) = p_s\left(x_s^{i*} - e_s^i\right) \leq 0$ because $p \in R_{++}^{LS}$, $e \in R_+^{LS}$, and $\sum_{s \in S} p_s \cdot x_s^i \leq \sum_{s \in S} p_s \cdot e_s^i$
      Then $\sum_s q_s z_s^{i*} = \sum_s p_{1s} \frac{p_s}{p_{1s}}\left(x_s^{i*} - e_s^i\right) = \sum_s p_s\left(x_s^{i*} - e_s^i\right) \leq 0$
      Hence, $x^* \in B_i^{AD} \rightarrow x^* \in B_i^R$

And $\sum_{s\in S} p_s(x_s^{i*} - e_s^i) = \sum_{s\in S} p_s \frac{p_{1s}}{p_s} z_s^{i*} = \sum_{s\in S} q_s z_s^{i*} \leq 0$
Hence, $x^* \in B_i^R \rightarrow x^* \in B_i^{AD}$
Therefore, $B_i^{AD} = B_i^R$

ii. $\sum_{i\in I} z_s^{i*} = \sum_{i\in I} \frac{p_s}{p_{1s}} \left(x_s^{i*} - e_s^i\right) = \frac{p_s}{p_{1s}} \sum_{i\in I} \left(x_s^{i*} - e_s^i\right) \leq 0$

Then $\exists q \in R_{++}^S$ and $z^* \in R^{SI}$ s.t. $\left\{(q,p);\left(z^{i*},x^{i*}\right)_{i\in I}\right\}$ is a Radner equilibrium.

(b) Suppose $\left\{(q,p);\left(z^{i*},x^{i*}\right)_{i\in I}\right\}$ is a Radner equilibrium.

i. Let $\mu \in R_{++}^S$ s.t. $\mu_s p_{1s} = q_s$ and for $\forall s \in S$, $\mu_s p_s = p_s^{AD}$
Define $z_s^{i*} = \frac{\mu_s p_s}{\mu_s p_{1s}} \left(x_s^{i*} - e_s^i\right)$ for $\forall s \in S$
For $\forall s \in S$, $\mu_s p_s (x_s^i - e_s^i) \leq \mu_s p_{1s} z_s^i = q_s z_s^i$
$\rightarrow \sum_s p_s^{AD}(x_s^i - e_s^i) \leq \sum_s q_s z_s^i \leq 0$
Hence, $x^* \in B_i^R \rightarrow x^* \in B_i^{AD}$
And $p_s\left(x_s^{i*} - e_s^i\right) = \frac{p_s^{AD}}{\mu_s}\left(x_s^{i*} - e_s^i\right) \leq 0$ because $p \in R_{++}^{LS}$, $e \in R_{+}^{LS}$, and $\sum_{s\in S} p_s^{AD} \cdot x_s^i \leq \sum_{s\in S} p_s^{AD} \cdot e_s^i$
Then $\sum_s q_s z_s^{i*} = \sum_s \mu_s p_{1s} \frac{\mu_s p_s}{\mu_s p_{1s}} \left(x_s^{i*} - e_s^i\right) = \sum_s p_s^{AD}\left(x_s^{i*} - e_s^i\right) \leq 0$
Hence, $x^* \in B_i^{AD} \rightarrow x^* \in B_i^R$

ii. feasibility condition is automatically satisfied.

(c) $S = 2, L = 2, I = 2$
with $\pi_1^1 = \pi_2^1 = \pi_1^2 = \pi_2^2 = \frac{1}{2}$ and
For both $s = 1, 2$,
$u_1\left(x_{11}^1, x_{21}^1\right) = \ln x_{11}^1 + \ln x_{21}^1$ and $u_1\left(x_{12}^1, x_{22}^1\right) = \ln x_{12}^1 + \ln x_{22}^1$
$u_2\left(x_{11}^2, x_{21}^2\right) = \ln x_{11}^2 + \ln x_{21}^2$ and $u_2\left(x_{11}^2, x_{21}^2\right) = \ln x_{12}^2 + \ln x_{22}^2$
For $s = 1$, $e_1^1 = (2,2), e_1^2 = (0,0)$
For $s = 2$, $e_2^1 = (0,0), e_2^2 = (2,2)$

i. Find an Arrow-Debreu equilibrium with complete contingent commodity markets
Because there is no aggregation risk and consumers have the same probability assessments,
$\frac{\pi_1}{\pi_2} = \frac{p_{11}}{p_{12}} = \frac{p_{21}}{p_{22}} = 1$ which is the price of good 1 at $s = 1, 2$.
Therefore, $p_{11} = p_{12} = p_{21} = p_{22}$
Then $p_{11} == p_{12} = p_{21} = p_{22} = 1$ will support the following allocation as an Arrow-Debreu equilibrium.
$$\frac{\frac{\partial u_1}{\partial x_{11}^1}}{\frac{\partial u_1}{\partial x_{12}^1}} = \frac{\frac{1}{x_{11}^1}}{\frac{1}{x_{12}^1}} = \frac{\frac{\partial u_1}{\partial x_{21}^1}}{\frac{\partial u_1}{\partial x_{22}^1}} = \frac{\frac{1}{x_{21}^1}}{\frac{1}{x_{22}^1}} = \frac{\frac{\partial u_2}{\partial x_{11}^2}}{\frac{\partial u_2}{\partial x_{12}^2}} = \frac{\frac{1}{x_{11}^2}}{\frac{1}{x_{12}^2}} = \frac{\frac{\partial u_2}{\partial x_{21}^2}}{\frac{\partial u_2}{\partial x_{22}^2}} = \frac{\frac{1}{x_{21}^2}}{\frac{1}{x_{22}^2}} = 1$$
First, $x_{11}^1 = x_{12}^1 = x_{21}^1 = x_{22}^1 = x_{11}^2 = x_{12}^2 = x_{21}^2 = x_{22}^2$; that is, the states are symmetric.
Therefore, let' consider state 1's consumption only.

$x_{12}^1 = x_{11}^1$ and $x_{21}^2 = x_{11}^2$
From the feasibility,
$$\rightarrow \begin{cases} x_{12}^1 + x_{21}^2 = 2 \\ \quad 2x_{21}^2 = 2 \end{cases}$$
$x^1 = \{(1,1),(1,1)\}, x^2 = \{(1,1),(1,1)\}$
Therefore, an Arrow-Debreu equilibrium is
$\left\{(1,1)_{s=1},(1,1)_{s=2};\left(x^1,x^2\right)_{s=1} = \{(1,1),(1,1)\},\left(x^1,x^2\right)_{s=2} = \{(1,1),(1,1)\}\right\}$

ii. Find a Radner equilibrium with complete contingent commodity markets
Define q and z s.t $q_s = p_{1s} = 1$ and $z_s^{i*} = \frac{p_s}{p_{1s}}\left(x_s^{i*} - e_s^i\right) = \left(x_s^{i*} - e_s^i\right)$ for $\forall s \in S$
Then a Radner equilibrium is
$\{[(1,1),(1,1)_{s=1},(1,1)_{s=2}];\left[\left(z_1^1,z_1^2\right),\left(z_2^1,z_2^2\right)\right] = [(-1,1),(1,-1)],\left(x^1,x^2\right)_{s=1} = \{(1,1),(1,$

3. Assets

A unit of an asset is a title to receive a return r_s of good 1 at $t = 1$ when state s occurs. An asset is therefore charaterized by its return vector $r = (r_1, ..., r_S) \in R^S$

- Examples

  (a) Safe(riskless) Asset
  $r = (1, 1, ..., 1)$
  This asset promises the future noncontingent delivery of one unit of good 1.
  In the case of $L = 1$, it is the safe asset.
  But if $L \geq 2$, then this asset is not risk-free anymore since its return in terms of purchasing power depends on the spot prices of all other goods.

  (b) Arrow Securities

  (c) Options(Derivative Asset)
  This asset's return is somehow derived from the returns of another asset(primary asset)
  Suppose there is a primary asset with return vector $r \in R^S$. Then a (*European*) call option(derivative asset) on the primary asset at the strike price $c \in R$ is itself an asset. A unit of this asset gives the option to buy a unit of primary asset at price c after state s is revealed.
  In state s, the option will be worthy iff $r_s > c$ at state s.
  Hence, $r(c) = (\max\{0, r_1 - c\}, \max\{0, r_2 - c\}, ..., \max\{0, r_S - c\})$
  For a primary asset with returns $r = (4, 3, 2, 1)$ s.t. $r_1 > r_2 > r_3 > r_4$
  $r(3.5) = (0.5, 0, 0, 0)$
  $r(2.5) = (1.5, 0.5, 0, 0)$
  $r(1.5) = (2.5, 1.5, 0.5, 0)$

  (d) Pricing an Option
  $S = 2$

uncontigent asset $r_1 = (1,1) : q_1 = 1$
contingent asset $r_2 = (3+\alpha, 1-\alpha) : q_2$
Now consider an option on the second asset that has strike price $c \in (1,3)$
Then the option $r(c) = (3+\alpha-c, 0) =?$

$$R = \begin{bmatrix} 1 & 3+\alpha \\ 1 & 1-\alpha \end{bmatrix}$$

$$q = \mu R \to (1, q_2) = (\mu_1, \mu_2) \cdot \begin{bmatrix} 1 & 3+\alpha \\ 1 & 1-\alpha \end{bmatrix}$$

$$R^{-1} = \frac{-1}{2+2\alpha} \begin{bmatrix} 1-\alpha & -3-\alpha \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{-1+\alpha}{2+2\alpha} & \frac{3+\alpha}{2+2\alpha} \\ \frac{1}{2+2\alpha} & \frac{-1}{2+2\alpha} \end{bmatrix}$$

$$\mu = (1, q_2) \cdot \begin{bmatrix} \frac{-1+\alpha}{2+2\alpha} & \frac{3+\alpha}{2+2\alpha} \\ \frac{1}{2+2\alpha} & \frac{-1}{2+2\alpha} \end{bmatrix} = \left(\frac{q_2-1+\alpha}{2+2\alpha}, \frac{3+\alpha-q_2}{2+2\alpha} \right)$$

$$q^{option}(c) = \mu \cdot r(c) = \left(\frac{q_2-1+\alpha}{2+2\alpha}, \frac{3+\alpha-q_2}{2+2\alpha} \right) (3+\alpha-c, 0)^T = \frac{(q_2-1+\alpha)\cdot(3+\alpha-c)}{2+2\alpha}$$

Note that if the prices of two assets r_1, r_2 are arbitrage free, then we must have $3+\alpha \geq q_2 \geq 1-\alpha$
therefore, we know that $q^{option}(c) = \frac{(q_2-1+\alpha)\cdot(3+\alpha-c)}{2+2\alpha} \geq 0$, decreasing in c, and increasing in q_2.
We also can know that if the asset price q_2 stays constant but α increases, then the option becomes more valuable.

- Return matrix R

  $$R = [r_1, ..., r_K] = \begin{bmatrix} r_{11} & \cdots & r_{1K} \\ \vdots & \ddots & \vdots \\ r_{S1} & \cdots & r_{SK} \end{bmatrix} = (S \cdot K) \text{ matrix}$$

- We assume that unlimited short sale is possible

  (?) : Namely we will establish that knowledge of the return matrix R suffices to place significant restrictions on the asset price vector q that could arise at equilibrium.

- We now generalize this to show that this equivalence holds for any family of S or more assets, provided that at least S of them have returns that are linearly independent.

- Complete asset market

  An asset market with an $S \times K$ return matrix R is complete if rank$R = S$; that is, if there is some set of S assets in an asset market with linearly independent returns.

  For example, the case of S contingent commodities(Arrow securities) is $S \times S$ identity matrix, which shows a complete market.

  A complete asset market can be generated by using options; an asset market consisting of a primary asset plus three options with strike prices 3.5, 2.5, 1.5 is complete.

$$\begin{array}{l} r=(4,3,2,1) \\ r\left(3.5\right)=(0.5,0,0,0) \\ r\left(2.5\right)=(1.5,0.5,0,0) \\ r\left(1.5\right)=(2.5,1.5,0.5,0) \end{array} \longrightarrow \begin{bmatrix} 4 & 3 & 2 & 1 \\ 0.5 & 0 & 0 & 0 \\ 1.5 & 0.5 & 0 & 0 \\ 2.5 & 1.5 & 0.5 & 0 \end{bmatrix}$$

With a complete asset structure, economic agents are in effect unrestricted in their wealth transfers across states under budget constraints. Therefore, at the equilibrium, their portfolio choices induce the same consumption plan as an Arrow Debreu equilibrium so that it is pareto optimal.

4. With Assets whose market structure is complete,

   (a) If an Arrow-Debreu equilibrium consists of $\left\{x^{i*}\right\}_{i\in I}\in R_{+}^{LSI}$ and $p\in R_{++}^{LS}$, then $\exists q\in R_{++}^{K}$ and $z^{*}\in R^{KI}$ s.t. $\left\{(q,p)\,;\left(z^{i*},x^{i*}\right)_{i\in I}\right\}$ constitutes a Radner equilibrium.

   (b) If a Radner equilibrium consists of $\left\{(q,p)\,;\left(z^{i*},x^{i*}\right)_{i\in I}\right\}$,.then $\exists\mu=(\mu_{1},...,\mu_{S})\in R_{++}^{S}$ s.t $\left\{\mu p,\left\{x^{i*}\right\}_{i\in I}\right\}\in R_{++}^{LS}\times R_{+}^{LSI}$ constitutes an Arrow-Debreu equilibrium.

$\boxed{\text{Proof}}$

First, note that budget constraints in each equilibrium is

$B_{i}^{AD}=\left\{x^{i}\in X_{i}\mid\sum_{s\in S}p_{s}\cdot x_{s}^{i}\leq\sum_{s\in S}p_{s}\cdot e_{s}^{i}\right\}$

$B_{i}^{R}=\left\{x^{i}\in X_{i}\mid \text{for }\forall i\in I\ \exists z^{i*}\in R^{K}\text{ s.t. }\sum_{k}q_{k}z_{k}^{i}\leq0\text{ and for }\forall s\in S,\ p_{s}x_{s}^{i}\leq p_{s}e_{s}^{i}+\sum_{k}p_{1s}r_{sk}z_{k}^{i}\right\}$

and feasibility conditions are

for $\forall s\in S$, $\sum_{i\in I}x_{s}^{i*}\leq\sum_{i\in I}e_{s}^{i}$

for $\forall k\in K$, $\sum_{i\in I}z_{k}^{i*}\leq0$ and for $\forall s\in S,\sum_{i\in I}x_{s}^{i*}\leq\sum_{i\in I}e_{s}^{i}$

In order to show the equivalence of both equilibria, we need to show that

i) budget constraints are the same

ii) feasibilities are satisfied.

(a) Suppose $\left\{x^{i*}\right\}_{i\in I}\in R_{+}^{LSI}$ and $p\in R_{++}^{LS}$ is an Arrow-Debreu equilibrium

   i. Define q s.t $q_{k}=\sum_{s}p_{1s}r_{sk}$ for $\forall k\in K$
      Denote by Π the $S\times S$ diagonal matrix whose s diagonal entry is p_{1s}.
      Then $q^{T}=1_{s}\cdot\Pi\cdot R$, where 1 is a a column vector with 1.
      For $\forall i\in I$, $m^{i}=\left\{p_{1}\left(x_{1}^{i}-e_{1}^{i}\right),...,p_{S}\left(x_{S}^{i}-e_{S}^{i}\right)\right\}^{T}$ s.t. $1_{s}\cdot m^{i}=0$ for $\forall i$(budget constraint) and $\sum_{i}m^{i}=0$(feasibility).
      By completeness of the asset market, $\text{rank}\Pi\cdot R=S$ and

therefore, we can find vectors $z^{i*} \in R^K$ s.t. $z^{i*} = (\Pi R)^{-1} m^i$ for $\forall i \in I$ which allows consumer i to reach the Arrow-Debreu consumptions in the different states at the spot prices $(p_1, ..., p_S)$.
Note again $qz^{i*} = 1_s \cdot \Pi \cdot R \cdot z^{i*} = 1_s \cdot m^i = 0$ for $\forall i$ and
it implies $p_s(x_s^{i*} - e_s^i) \leq \sum_k p_{1s} r_{sk} z_k^{i*} = 0$ for $\forall s \in S$
because $p \in R_{++}^{LS}$ and $e \in R_+^{LS}$.
Hence, $x^* \in B_i^{AD} \rightarrow x^* \in B_i^R$
And $p_s(x_s^{i*} - e_s^i) \leq \sum_k p_{1s} r_{sk} z_k^{i*} = 0$ for $\forall s \in S$ and $qz^{i*} = 0$ implies
$\sum_{s\in S} p_s(x_s^{i*} - e_s^i) = \sum_{s\in S} \sum_k p_{1s} r_{sk} z_k^{i*} = \sum_k \left[\sum_{s\in S} p_{1s} r_{sk}\right] z_k^{i*} = \sum_k q_k z_k^{i*} = 0$
Hence, $x^* \in B_i^R \rightarrow x^* \in B_i^{AD}$
Therefore, $B_i^{AD} = B_i^R$

ii. $\sum_k q_k z_k^{i*} = \sum_k \left[\sum_{s\in S} p_{1s} r_{sk}\right] z_k^{i*} = \sum_{s\in S} \sum_k p_{1s} r_{sk} z_k^{i*} = \sum_{s\in S} p_s(x_s^{i*} - e_s^i) = 0$ and $q \in R_{++}^K$ implies $\sum_{i\in I} z_k^{i*} = 0$

Then $\exists q \in R_{++}^S$ and $z^* \in R^{SI}$ s.t. $\left\{(q, p); \left(z^{i*}, x^{i*}\right)_{i\in I}\right\}$ is a Radner equilibrium.

(b) Suppose $\left\{(q,p); \left(z^{i*}, x^{i*}\right)_{i\in I}\right\}$ is a Radner equilibrium.

i. Because in the Radner equilibrium, the asset portfolio is arbitrage free, we have $q = \mu \cdot R$ with $\mu \in R_{++}^S$
and for $\forall s \in S$, $\mu_s p_s = p_s^{AD}$.
For $\forall i \in I$, $m^i = \left\{p_1\left(x_1^{i*} - e_1^i\right), ..., p_S\left(x_S^{i*} - e_S^i\right)\right\}^T$ s.t. $1_s \cdot m^i = 0$ for $\forall i$(budget constraint) and $\sum_i m^i = 0$(feasibility).
By completeness of an asset market structure, rank$R = S$ and
we can find vectors $z^{i*} \in R^K$ s.t. $z^{i*} = R^{-1} m^i$ for $\forall i \in I$ and therefore
$q \cdot z^{i*} = \mu \cdot R \cdot z^{i*} = \mu \cdot R \cdot R^{-1} \cdot m^i = \mu \cdot m^i = \sum_s \mu_s p_s \left(x_s^{i*} - e_s^i\right) \leq 0$ for $\forall i$
$\rightarrow \sum_s p_s^{AD}(x_s^i - e_s^i) \leq 0$
Hence, $x^* \in B_i^R \rightarrow x^* \in B_i^{AD}$
Moreover $p_s\left(x_s^{i*} - e_s^i\right) = \frac{p_s^{AD}}{\mu_s}\left(x_s^{i*} - e_s^i\right) \leq 0$ because $p \in R_{++}^{LS}$, $e \in R_+^{LS}$, and $\sum_{s\in S} p_s^{AD} \cdot x_s^i \leq \sum_{s\in S} p_s^{AD} \cdot e_s^i$
and $q \cdot z^{i*} = \mu \cdot R \cdot z^{i*} = \mu \cdot m^i = \sum_s \mu_s p_s \left(x_s^{i*} - e_s^i\right) \leq 0$ for $\forall i$
Hence, $x^* \in B_i^{AD} \rightarrow x^* \in B_i^R$

ii. feasibility condition is automatically satisfied.

5. $S = 2, L = 2, I = 2$

with $\pi_1^1 = \pi_2^1 = \pi_1^2 = \pi_2^2 = \frac{1}{2}$ and

For both $s = 1, 2$,

$u_1\left(x_{11}^1, x_{21}^1\right) = 4 \ln x_{11}^1 + \ln x_{21}^1$ and $u_1\left(x_{12}^1, x_{22}^1\right) = 4 \ln x_{12}^1 + \ln x_{22}^1$

$u_2\left(x_{11}^2, x_{21}^2\right) = \ln x_{11}^2 + 4 \ln x_{21}^2$ and $u_2\left(x_{11}^2, x_{21}^2\right) = \ln x_{12}^2 + 4 \ln x_{22}^2$

For $s=1$, $e_1^1=(1,1)\,,e_1^2=(2,2)$

For $s=2$, $e_2^1=(2,2)\,,e_2^2=(1,1)$

(a) Find an Arrow-Debreu equilibrium with complete contingent commodity markets

Because there is no aggregation risk and consumers have the same probability assessments,

$\frac{\pi_1}{\pi_2}=\frac{p_{11}}{p_{12}}=\frac{p_{21}}{p_{22}}=1$ which is the price of good 1 at state 1 and 2.

Therefore, $p_{11}=p_{12}=p_{21}=p_{22}$

Then $p_{11}==p_{12}=p_{21}=p_{22}=1$ will support the following allocation as an Arrow-Debreu equilibrium.

$$\frac{\frac{\partial u_1}{\partial x_{11}^1}}{\frac{\partial u_1}{\partial x_{12}^1}}=\frac{\frac{4}{x_{11}^1}}{\frac{4}{x_{12}^1}}=\frac{\frac{\partial u_1}{\partial x_{21}^1}}{\frac{\partial u_1}{\partial x_{22}^1}}=\frac{\frac{1}{x_{21}^1}}{\frac{1}{x_{22}^1}}=\frac{\frac{\partial u_2}{\partial x_{11}^2}}{\frac{\partial u_2}{\partial x_{12}^2}}=\frac{\frac{1}{x_{11}^2}}{\frac{1}{x_{12}^2}}=\frac{\frac{\partial u_2}{\partial x_{21}^2}}{\frac{\partial u_2}{\partial x_{22}^2}}=\frac{\frac{4}{x_{21}^2}}{\frac{4}{x_{22}^2}}=1$$

First, $x_{11}^1=x_{12}^1,x_{21}^1=x_{22}^1$, $x_{11}^2=x_{12}^2,x_{21}^2=x_{22}^2$; that is, the states are symmetric.

Therefore, let' consider state 1's consumption only.

$$\frac{4x_{21}^1}{x_{11}^1}=\frac{x_{21}^2}{4x_{11}^2}=1$$

$4x_{12}^1=x_{11}^1$ and $x_{21}^2=4x_{11}^2$

From the feasibility,

$$\begin{cases} x_{11}^1+x_{11}^2=3\rightarrow 4x_{12}^1+\frac{1}{4}x_{21}^2=3 \\ \qquad\qquad x_{21}^1+x_{21}^2=3 \end{cases}$$

$$\rightarrow\begin{cases} 4x_{12}^1+\frac{1}{4}x_{21}^2=3 \\ 4x_{21}^1+4x_{21}^2=12 \end{cases}$$

$-\frac{15}{4}x_{21}^2=-9\rightarrow x_{21}^2=\frac{12}{5}, x_{11}^2=\frac{3}{5}, x_{21}^1=\frac{3}{5}, x_{11}^1=\frac{12}{5}$

$x^1=\left\{\left(\frac{12}{5},\frac{3}{5}\right),\left(\frac{12}{5},\frac{3}{5}\right)\right\},x^2=\left\{\left(\frac{3}{5},\frac{12}{5}\right),\left(\frac{3}{5},\frac{12}{5}\right)\right\}$

Therefore, an Arrow-Debreu equilibrium is

$\left\{(1,1)_{s=1},(1,1)_{s=2};\left(x^1,x^2\right)_{s=1}=\left\{\left(\frac{12}{5},\frac{3}{5}\right),\left(\frac{3}{5},\frac{12}{5}\right)\right\},\left(x^1,x^2\right)_{s=2}=\left\{\left(\frac{12}{5},\frac{3}{5}\right),\left(\frac{3}{5},\frac{12}{5}\right)\right\}\right\}$

(b) Find a Radner equilibrium with Arrow securities

Define q and z s.t $q_s=p_{1s}$ and $z_s^{i*}=\frac{p_s}{p_{1s}}\left(x_s^{i*}-e_s^i\right)$ for $\forall s\in S$

Then with $p=\{(1,1)_{s=1},(1,1)_{s=2}\}$ and

$x=\left\{\left(x^1,x^2\right)_{s=1},\left(x^1,x^2\right)_{s=2}\right\}=\left\{\left\{\left(\frac{12}{5},\frac{3}{5}\right),\left(\frac{3}{5},\frac{12}{5}\right)\right\},\left\{\left(\frac{12}{5},\frac{3}{5}\right),\left(\frac{3}{5},\frac{12}{5}\right)\right\}\right\}$,

$q=(1,1)$

$z^*=\left\{z^{1*}=\{1,-1\},z^{2*}=\{-1,1\}\right\}$

Feasibility!!

(c) Find a Radner equilibrium with Assets $r_1=(1,1)\,,r_2=(2,1)$

Define q s.t $q_k=\sum_s p_{1s}r_{sk}$ for $\forall k\in K$

Denote by Π the $S\times S$ diagonal matrix whose s diagonal entry is p_{1s}.

Then $q^T = 1_s \cdot \Pi \cdot R$, where 1 is a a column vector with 1.

$$q^T = [1,1] \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = [2,3]$$

For $\forall i \in I$, $m^i = \left\{p_1\left(x_1^i - e_1^i\right), ..., p_S\left(x_S^i - e_S^i\right)\right\}^T$ s.t. $1_s \cdot m^i = 0$ for $\forall i$(budget constraint) and $\sum_i m^i = 0$(feasibility).

$$m^1 = \begin{bmatrix} p_1\left(x_1^1 - e_1^1\right) \\ p_2\left(x_2^1 - e_2^1\right) \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$m^2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

By completeness of the asset market, rank$\Pi \cdot R = S$ and
therefore, we can find vectors $z^{i*} \in R^K$ s.t. $z^{i*} = (\Pi R)^{-1} m^i$ for $\forall i \in I$

$$z^{1*} = (-1) \cdot \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

$$z^{1*} = (-1) \cdot \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

Then with $p = \{(1,1)_{s=1}, (1,1)_{s=2}\}$ and
$x = \left\{\left(x^1, x^2\right)_{s=1}, \left(x^1, x^2\right)_{s=2}\right\} = \left\{\left\{\left(\frac{12}{5}, \frac{3}{5}\right), \left(\frac{3}{5}, \frac{12}{5}\right)\right\}, \left\{\left(\frac{12}{5}, \frac{3}{5}\right), \left(\frac{3}{5}, \frac{12}{5}\right)\right\}\right\}$,
$q = (2,3)$
$z^* = \left\{z^{1*} = \{-3,2\}, z^{2*} = \{3,-2\}\right\}$
Feasibility!!

(d) Radner with Arrow Security⇔Radner with assets

i. →

$p = \{(1,1)_{s=1}, (1,1)_{s=2}\}$
$x = \left\{\left(x^1, x^2\right)_{s=1}, \left(x^1, x^2\right)_{s=2}\right\} = \left\{\left\{\left(\frac{12}{5}, \frac{3}{5}\right), \left(\frac{3}{5}, \frac{12}{5}\right)\right\}, \left\{\left(\frac{12}{5}, \frac{3}{5}\right), \left(\frac{3}{5}, \frac{12}{5}\right)\right\}\right\}$,
$q = (1,1)$
$z^* = \left\{z^{1*} = \{1,-1\}, z^{2*} = \{-1,1\}\right\}$

Then with $R = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$,

$$Rz^1 = Iz^{1*} \rightarrow \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \cdot \left[z_1^1, z_2^1\right]^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\left[z_1^1, z_2^1\right] = (-1) \cdot \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} = [-3,2]$$

$$\left[z_1^2, z_2^2\right] = (-1) \cdot \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} = [3,-2]$$

ii. ←

$p = \{(1,1)_{s=1}, (1,1)_{s=2}\}$
$x = \left\{\left(x^1, x^2\right)_{s=1}, \left(x^1, x^2\right)_{s=2}\right\} = \left\{\left\{\left(\frac{12}{5}, \frac{3}{5}\right), \left(\frac{3}{5}, \frac{12}{5}\right)\right\}, \left\{\left(\frac{12}{5}, \frac{3}{5}\right), \left(\frac{3}{5}, \frac{12}{5}\right)\right\}\right\}$,

$q = (2,3)$
$z^{*} = \left\{z^{1*} = \{-3,2\}, z^{2*} = \{3,-2\}\right\}$
Then with $R = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$,
$Rz^{1} = Iz^{1*} \rightarrow \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \left[z_1^1, z_2^1\right]^T$
$\left[z_1^1, z_2^1\right] = (-1) \cdot \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 2 \end{bmatrix} = [1, -1]$
$\left[z_1^2, z_2^2\right] = (-1) \cdot \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -2 \end{bmatrix} = [-1, 1]$