Microeconomics Theory 8101-8104 Notes

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Dedicated to Nadezda Golyaeva

The author thanks Jan Werner, Beth Allen and Aldo Rustichini for the lectures during 2006-2007 academic year.

Abstract. Replace this text with your own abstract.

Contents

Preface

The following is in a sense condensed collection of lecture notes in 8101-8103 Microeconomic Theory courses given during 2006-07 academic year in Minnesota. A number of textbooks, such as Mas-Colell, Whinston and Green, Stokey and Lucas, Fudenberg and Tirole, etc. were also used in preparation of this manuscript.

All the remaining errors are mine alone.

Part 1

Mathematical Introduction

THEOREM 1. *(Maximum Theorem). Let $X \subseteq \mathbb{R}^L$, $Y \subseteq \mathbb{R}^M$, $f : X \times Y \to \mathbb{R}$ be a continuous function, $\Gamma : X \rightrightarrows X$ be a nonempty, compact valued and continuous correspondence. Define:*

$$\begin{aligned} h(x) &= \max_{y \in \Gamma(x)} f(x,y) \\ G(x) &= \{y \in \Gamma(x) \mid f(x,y) = h(x)\} \end{aligned}$$

Then $h(x)$ is a continuous function, and $G(x)$ is a nonempty, compact valued and upper hemicontinuous correspondence.

THEOREM 2. *(Envelope Theorem). Let $X \subseteq \mathbb{R}^L$, $Y \subseteq \mathbb{R}^M$, $f : X \times Y \to \mathbb{R}$ be some function and denote:*

$$h(x) = \max_{y \in Y} f(x,y)$$

Let also $y^*(x)$ be the solution to the maximization problem as a function of parameters in x. Suppose that $h(x)$ is differentiable in some small neighborhood around x. Then:

$$\frac{dh(x)}{dx} = \left.\frac{\partial f(x,y)}{\partial x}\right|_{y=y^*(x)}$$

PROOF. Observe that $h(x) = f(x, y^*(x))$, then:

$$\frac{dh(x)}{dx} = \frac{\partial f(x,y^*(x))}{\partial x} = \left.\frac{\partial f(x,y)}{\partial x}\right|_{y=y^*(x)} + \frac{\partial f(x,y^*(x))}{\partial y} \cdot \frac{\partial y^*(x)}{\partial x}$$

But now observe that $\frac{\partial f(x,y^*(x))}{\partial y} = 0$ as an immediate consequence of the F.O.C. in the maximization problem. Hence the theorem's claim follows. QED. □

DEFINITION 1. *A function $f(x)$ is **continuous** at $x = x_0$ is $\forall \varepsilon > 0$ $\exists \delta > 0$ such that $\forall x : \|x - x_0\| < \delta$ we have $\|f(x) - f(x_0)\| < \varepsilon$.*

DEFINITION 2. *Let $X \subseteq \mathbb{R}^L$ be a convex set. A function $f : X \to \mathbb{R}$ is **quasi-convex** if $\forall x, y \in X : x \neq y$ and $\forall \lambda \in (0,1)$ we have:*

$$f(\lambda x + (1-\lambda) y) \leq \max\{f(x), f(y)\}$$

*A function is **strictly quasi-convex**, if the above inequality is strict. A function $f(x)$is **quasi-concave** (or **strictly quasi-concave**) is the function $-f(x)$ is **quasi-convex** (or **strictly quasi-convex**).*

EXAMPLE 1. *A quasi-convex function that is not convex is, for example, $y = \sqrt{|x|}$*

DEFINITION 3. *A correspondence $\Gamma : X \rightrightarrows Y$ is **nonempty** at $x \in X$ if the set $\Gamma(x)$ is nonempty.*

DEFINITION 4. *A correspondence $\Gamma : X \rightrightarrows Y$ is **convex valued** at $x \in X$ if the set $\Gamma(x)$ is a convex set.*

DEFINITION 5. *A correspondence $\Gamma : X \rightrightarrows Y$ is **upper hemicontinuous** at $x \in X$ if $\Gamma(x)$ is nonempty and if for any sequence $\{x_n\}_{n=1}^{\infty} : x_n \to x$, and for every sequence $\{y_n\}_{n=1}^{\infty} : y_n \in \Gamma(x_n)$ $\forall n$ and $\lim_{n\to\infty} y_n = y$ we have $y \in \Gamma(x)$.*

DEFINITION 6. *A correspondence $\Gamma : X \rightrightarrows Y$ is **lower hemicontinuous** at $x \in X$ if $\Gamma(x)$ is nonempty and for every sequence $\{x_n\}_{n=1}^{\infty} : x_n \to x$ and every sequence $\{y_n\}_{n=1}^{\infty} : y_n \in \Gamma(x_n)$ $\forall n$ there exists a convergent subsequence $\{y_{n_k}\}_{n_k=1}^{\infty} : y_{n_k} \to y$ and $y \in \Gamma(x)$.*

THEOREM 3. *(Brouwer's Fixed Point Theorem). Let* $A \subset \mathbb{R}^N$ *be a nonempty, compact and convex set, let* $f : A \to A$ *be a continuous function. Then* $\exists x^* \in A : f(x^*) = x^*$.

THEOREM 4. *(Kakutani's Fixed Point Theorem). Let* $A \subset \mathbb{R}^N$ *be a nonempty, compact and convex set, let* $f : A \rightrightarrows A$ *be a nonempty, convex valued and uhc correspondence. Then* $\exists x^* : x^* \in f(x^*)$.

THEOREM 5. *(Supporting Hyperplane Theorem). Let* $X \subseteq \mathbb{R}^L$ *be a nonempty and convex set, and let* $a \in \mathbb{R}^L$ *be a point such that* $a \cap int[X] = \emptyset$. *Then* $\exists p \in \mathbb{R}^L : p \neq 0$ *such that* $\forall x \in X : p \cdot x \geq p \cdot a$.

Part 2

Producer Theory, Consumer Theory and Choice Under Uncertainty

CHAPTER 1

Production Economy

1. Basic Definitions

Denote the production set by Y and let $\pi(p) = \max\{p \cdot y : y \in Y\}$ be the associated profit function and $s(p) = \{y \in Y : p \cdot y = \pi(p)\}$ be the associated supply correspondence. Suppose that Y is a closed set. Then:

(1) $\pi(p)$ is a continuous function
(2) $\pi(p)$ is homogenous of degree 1.

Proof. $\pi(p) = p \cdot y \Rightarrow \pi(\lambda p) = (\lambda p) \cdot y = \lambda \pi(p)$. QED □

(3) $\pi(p)$ is a convex function.

Proof. Pick some $p \subset \mathbb{R}^L$, $p' \subset \mathbb{R}^L$ and $\alpha \in [0,1]$. Denote $p_\alpha = \alpha p + (1-\alpha)p'$ and pick some $\tilde{y} \in s(p_\alpha)$. Then we have $p\tilde{y} \leq \pi(p)$ and $p'\tilde{y} \leq \pi(p')$ by definition of the profit function. Thus:

$$p_\alpha \cdot \tilde{y} = (\alpha p + (1-\alpha)p') \cdot \tilde{y} = \alpha p \cdot \tilde{y} + (1-\alpha)p' \cdot \tilde{y} \leq \alpha\pi(p) + (1-\alpha)\pi(p') \tag{1.1}$$

Since $\tilde{y} \in s(p_\alpha)$, we have $p_\alpha \cdot \tilde{y} = \pi(p_\alpha) = \pi(\alpha p + (1-\alpha)p')$. So by (1.1) $\pi(p)$ is concave. QED □

(4) $s(p)$ is homogenous of degree 0.

Proof. $\forall p \in \mathbb{R}^L$ and $\forall y \in s(p)$ we have $\pi(p) = p \cdot y$ and $\pi(\lambda p) = \lambda\pi(p) \Rightarrow \lambda p \cdot y = \lambda(p \cdot s(p))$. QED □

(5) If Y is a convex set, then $s(p)$ is convex $\forall p$. If Y is strictly convex, then $s(p)$ is either single-valued or empty.

Proof. By definition $s(p) = \{y \in Y \mid p \cdot y = \pi(p)\}$. If Y is convex, then $\forall y, y' \in Y : y \neq y'$ and $\forall \lambda \in [0,1]$ we have $\lambda y + (1-\lambda)y' \in Y$. Fix some $p \in \mathbb{R}^L$ and assume $y, y' \in s(p)$. Since $py = \pi(p)$ and $py' = \pi(p)$ by definition of supply, clearly:

$$p[\lambda y + (1-\lambda)y'] = \lambda py + (1-\lambda)py' = \lambda\pi(p) + (1-\lambda)\pi(p) = \pi(p)$$

Hence $\lambda y + (1-\lambda)y' \in s(p)$, so $s(p)$ is a convex set. QED.

For the second part of the proof it is clear that with strict convexity of the production set the profit maximizing production plan is at most unique. Therefore, supply is at most single-valued. QED. □

(6) If $s(p)$ is single-valued, then $D_p\pi(p) = s(p)$ (Hotelling lemma).

Proof. This is a direct consequence of the Envelope Theorem, that can be found in the first chapter. □

(7) If the profit function $\pi(p)$ is twice differentiable, then $D_p^2\pi(p) = D_p s(p)$, which is a symmetric and prsotive semi-definite matrix.

PROOF. This follows from the fact that the profit function is convex – the Hessian is then necessarily positive semi-definite and symmetric.

For the supply, this implies that:

$$\frac{\partial s_i}{\partial p_i} \geq 0$$

This can be shown directly. Fix some $p, p' \in \mathbb{R}^L$ and pick some $y \in s(p)$ and $y' \in s(p')$. Consider the following inequality:

$$(p - p')(y - y') = (py - py') + (p'y' - p'y) \geq 0$$

This follows from $\pi(p) = py \geq py'$ $\forall y' \in Y$ and similarly for p and p'. QED. □

THEOREM 6. *(Profit Function Rationaloizability). Suppose that the function $\pi(p)$ has the following properties:*

(1) *$\pi(p)$ is homogenous of degree 1*
(2) *$\pi(p)$ is continuous*
(3) *$\pi(p)$ is convex (its Hessian matrix is positive semidefinite).*

If so, there exists a closed and convex production set Y such that $\pi(p)$ is a profit function associated with this set, i.e. $\pi(p)$ is profit-rationalized by Y.

THEOREM 7. *(Supply Function Rationaloizability). Suppose that the function $s(p)$ has the following properties:*

(1) *$s(p)$ is homogenous of degree zero*
(2) *$D_p s(p)$ is a positive semidefinite and symmetric matrix.*

Then $\pi(p) \equiv p \cdot s(p)$ is a profit function that satisfies the above properties (i.e. homogeneity of degree 1, continuity and convexity) and there exists a closed and convex production set Y such that $s(p)$ is a supply function associated with this set.

AXIOM 1. *(Weak Axiom of Profit Maximization (WAPM)). Suppose one observes T pairs of outputs and prices: $(y_1, p_1), \ldots, (y_T, p_T)$. If, $\forall t, j \in 1, \ldots T$ we have $p_t y_t \geq p_t y_j$, then these observations satisfy WAPM. Also we can say that these observations satisfy WAPM iff there exists a closed and convex production set Y that profit-rationalizes these pairs.*

CHAPTER 2

Consumer Theory

1. Preference Relation

Definition 7. *A preference relation $\succeq$ on $X \subseteq \mathbb{R}^L$ is **complete** if $\forall x, y \in X$ we have $x \succeq y$ or $y \succeq x$ or both.*

Definition 8. *A preference relation $\succeq$ on $X \subseteq \mathbb{R}^L$ is **transitive** if $\forall x, y, z \in X$ such that $x \succeq y$ and $y \succeq z$ implies that $x \succeq z$.*

Definition 9. *A preference relation $\succeq$ on $X \subseteq \mathbb{R}^L$ is **weakly convex** if $\forall x, y \in X : y \succeq x$ and $\forall \lambda \in [0,1]$ we have $\lambda y + (1-\lambda) x \succeq x$.*

Definition 10. *A preference relation $\succeq$ on $X \subseteq \mathbb{R}^L$ is **convex** if $\forall x, y \in X$: $y \succeq x$ and $\forall \lambda \in (0,1)$ we have $\lambda y + (1-\lambda) x \succ x$.*

Definition 11. *A preference relation $\succeq$ on $X \subseteq \mathbb{R}^L$ is **strictly convex** if $\forall x, y \in X : y \sim x$ and $\forall \lambda \in [0,1]$ we have $\lambda y + (1-\lambda) x \succ x$.*

Definition 12. *A preference relation $\succeq$ on $X \subseteq \mathbb{R}^L$ is **continous** if $\forall \{x_n\}_{n=1}^{\infty}$: $x_n \to x$ and $\{y_n\}_{n=1}^{\infty} : y_n \to y$ such that $\forall n : x_n \succeq y_n$ we have $y \succeq x$. Alternatively, preference relation $\succeq$ on X is **continous** if $\forall x \in X$ the sets $UCS(x) = \{y \in X \mid y \succeq x\}$ and $LCS(x) = \{y \in X \mid x \succeq y\}$ are closed sets in X.*

Definition 13. *A preference relation $\succeq$ on $X \subseteq \mathbb{R}^L$ is **locally nonsatiated** if $\forall x \in X$ and $\forall \varepsilon > 0$ $\exists y \in X : y \succeq x$ and $\|x - y\| < \varepsilon$.*

Definition 14. *A preference relation $\succeq$ on $X \subseteq \mathbb{R}^L$ is **monotone** if $\forall x, y \in X : y \gg x$ we have $y \succ x$.*

Definition 15. *A preference relation $\succeq$ on $X \subseteq \mathbb{R}^L$ is **strongly monotone** if $\forall x, y \in X : y \geq x$ and $y \neq x$ we have $y \succ x$.*

Corollary 1. *If preference relation $\succeq$ on $X \subseteq \mathbb{R}^L$ is continuous, then strict convexity implies convexity, which, in turn, implies weak convexity. Also if preference relation $\succeq$ on X is weakly convex, the sets $UCS(x)$ and $LCS(x)$ are convex sets.*

Theorem 8. *Let $\succeq$ on X be a continuous complete preorder. Then $\forall e_i \in \mathbb{R}^L_+$, $\forall p \gg 0$ the demand $x_i(p, e_i)$ is well defined, i.e. nonempty.*

Proof. Since $p \gg 0$, the budget set $B(p, e_i) = \left\{x \in \mathbb{R}^L_+ \mid p \cdot x \leq p \cdot e_i\right\}$ is compact. Define $S_x = \{y \in B(p, e_i) \mid y \succeq_i x\}$. Pick some $x_1, \ldots, x_n : x_i \in B(p, e_i)$ $\forall i$. Clearly $S_{x_1} \cap S_{x_2} \cap \cdots \cap S_{x_n} \neq \emptyset$. Then by finite intersection property we have:

$$\bigcap_{x \in B(p, e_i)} S_x \equiv x_i(p, e_i) \neq \emptyset$$

Also observe that since continuous complete preorder preferences can be represented by a continuous utility function, and since $B(p, e_i)$ is a nonempty, compact valued and continuous as a correspondence, the Maximum Theorem guarantees that $x_i(p, e_i)$ is nonempty, compact valued and uhc. □

2. Utility Functions and Demands

Theorem 9. *(Utility Representation Theorem) Let $\succeq$ be a continous complete preorder on $X \subseteq \mathbb{R}^L$. Then there exists a continuous function $u : X \to \mathbb{R}$ that represents $\succeq$, i.e. $\forall x, x' \in X$ we have $x \succeq x' \iff u(x) \geq u(x')$.*

Proof. For this proof we assume that $\succeq$ is also monotone and $X = \mathbb{R}^L_+$. Define $e = (1, \dots, 1) \in \mathbb{R}^L_+$, and denote $Z = \{x \in X \mid x = \alpha e, \alpha \in \mathbb{R}_+\}$, here Z is the diagonal of $\mathbb{R}^L_+$. Pick some $\tilde{x} \in X$, and let $Y(\tilde{x})$ be defined as the set of all points that are indifferent to x under $\succeq$, i.e. $Y(\tilde{x}) = \{y \in X \mid y \sim \tilde{x}\}$. Pick $\alpha = \alpha(\tilde{x})$ such that $\alpha(\tilde{x}) e \in Y(\tilde{x})$. We argue that $\exists! \alpha(\tilde{x})$.

Since $\succeq$ is continuous, the upper and lower contour sets are closed:

$$\begin{aligned} UCS(\tilde{x}) &= \{x \in X \mid x \succeq \tilde{x}\} \\ LCS(\tilde{x}) &= \{x \in X \mid \tilde{x} \succeq x\} \end{aligned}$$

Clearly $Y(\tilde{x}) = UCS(\tilde{x}) \cap LCS(\tilde{x})$. Since $\mathbb{R}^L_+ = UCS(\tilde{x}) \cup LCS(\tilde{x})$ is connected, the intersection $Y(\tilde{x})$ is nonempty, hence existance established. Suppose it is not unique, i.e. $\exists \alpha_1(\tilde{x}), \alpha_2(\tilde{x}) : \alpha_1 \neq \alpha_2$ and $\alpha_1(\tilde{x}) e \sim \alpha_2(\tilde{x}) e \sim \tilde{x}$. But then either $\alpha_1 > \alpha_2$ or $\alpha_1 < \alpha_2$. Suppose $\alpha_1 > \alpha_2$, then by monotonicity of $\succeq$ we have $\alpha_1(\tilde{x}) e \succ \alpha_2(\tilde{x}) e$, and this is a contradiction. The case $\alpha_1 < \alpha_2$ is similar, so uniqueness has been established.

Now denote $u(\tilde{x}) = \alpha(\tilde{x})$, and we want to show that $\alpha(\tilde{x})$ represents $\succeq$ on X. Pick some $x, x' \in X$ such that $x' \succeq x$. Then by above argument $\exists! \alpha(x), \alpha(x')$ such that $\alpha(x) e \sim x$ and $\alpha(x') e \sim x'$. Since $\succeq$ is transitive we have $\alpha(x') e \succeq \alpha(x) e \iff \alpha(x') \geq \alpha(x)$, i.e. x' gives the higher utility than x, or just x' is better than x. Suppose now that we have some $x, x' \in X$ such that $\alpha(x') \geq \alpha(x)$. Then we have that $\alpha(x') e \succeq \alpha(x) e$ by monotonicity of $\succeq$. By construction of $\alpha(\cdot)$ we have $\alpha(x) e \sim x$ and $\alpha(x') e \sim x'$, so by transitivity $x' \succeq x$. Therefore $\alpha(x)$ represents $\succeq$. QED. □

Definition 16. *(Utility Maximization Problem – UMP). The UMP is defined as:*

$$\max_{x \in B(p,w)} u(x)$$

where $B(p, w) = \{x \in X \mid px \leq w\}$ the budget set. The solution to the UMP is $x = x(p, w)$, the ***Walrasian*** *demand function:*

$$x(p, w) = \arg \max_{x \in B(p,w)} u(x)$$

Notice that here w represents the money value of the income of the consumer, i.e. if s/he is endowed with ω goods, then $w = p\omega$. This is important below.

Theorem 10. *Suppose that in UMP we have $u(\cdot)$ being a continuous function representing a LNS preference relation $\succeq$. Then the Walrasian demand $x(p, w)$ has the following properties:*

(1) *$x(p, w)$ is homogenous of degree 0, i.e. $x(\alpha p, \alpha w) = x(p, w)$.*
(2) *$\forall x \in x(p, w)$ Walras' Law holds: $px = w$.*

(3) *If $\succeq$ is convex, i.e. $u(\cdot)$ is quasi-concave then $x(p,w)$ is a convex set. Moreover, if $\succeq$ is strictly convex, i.e. $u(\cdot)$ is strictly quasi-concave, then $x(p,w)$ is single-valued.*

Proof. We establish all three in turn.

(1) By definition, $\forall x \in x(p,w)$ we have $px \leq w$. Clearly, we have then $\forall x \in x(\alpha p, \alpha w)$ satisfies $(\alpha p)x \leq (\alpha w)$, and now cancel α to get $px \leq w$. Thus $x(\alpha p, \alpha w) = x(p,w)$. QED
(2) Suppose not, i.e. $px < w$. Then by LNS $\exists y \in X$ such that $\|x - y\| \leq \varepsilon$ and $y \succ x$. Clearly y is affordable as long as ε is small enough, i.e. $\exists \varepsilon > 0 : py \leq w$. But then $y \succ x$ contradicts $x \in x(p,w)$, so $px = w$. QED.
(3) Suppose that $u(\cdot)$ is quasi-concave. Let $x, x' \in x(p,w)$ for some $p \in \mathbb{R}^L, w \in \mathbb{R}_+$. Notice that $u(x) = u(x') = \bar{u}$. Pick some $\lambda \in [0,1]$, and denote $x_\lambda = \lambda x + (1-\lambda)x'$. Since $u(\cdot)$ is quasi-concave, we have $u(x_\lambda) \geq \bar{u}$. Clearly if $u(x_\lambda) > \bar{u}$, then $px_\lambda > w$, because $x, x' \in x(p,w)$. Hence $x_\lambda = \bar{u}$, and thus:

$$x_\lambda = \lambda x + (1-\lambda)x' \in x(p,w)$$

Notice now that if $u(\cdot)$ is strictly quasi-concave, then $u(x_\lambda) > \bar{u}$, and since this implies $px_\lambda > w$, it means that we cannot pick $x, x' \in x(p,w)$ such that $x \neq x'$. Thus there is at most one $x \in x(p,w)$. QED.

□

Definition 17. *Let $u(\cdot)$ be a continuous utility function representing a LNS preference relation $\succeq$. Let $x(p,w)$ be a Walrasian demand function generated by UMP. Then $u(x(p,w)) \equiv \upsilon(p,w)$ is an **inderect utility function**, i.e. the value function of the UMP as a function of prices and income.*

Theorem 11. *Let $\upsilon(p,w)$ be an indirect utility function as defined above. Then $\upsilon(\cdot)$ has the following properties:*

(1) *$\upsilon(\cdot)$ is homogenous of degree 0, i.e. $\upsilon(\alpha p, \alpha w) = \upsilon(p,w)$.*
(2) *$\upsilon(p,w)$ is continuous in p and v.*
(3) *$\upsilon(p,w)$ is strictly increasing in w and nondecreasing in p.*
(4) *$\upsilon(p,w)$ is a quasi-convex function, i.e. the set $\{(p,w) \mid \upsilon(p,w) \leq \bar{\upsilon}\}$ is convex $\forall \bar{\upsilon}$.*

Proof. Part 1 follows immediately since $\upsilon(p,w) = u(x(p,w)$ and $x(p,w)$ is homogenous of degree 0, as shown above.

Part 2 is an immediate consequence of the Maximum Theorem.

Part 3 is something that I still cannot see how to prove :-(

Part 4 can be shown as follows. Pick some $p, p' \in \mathbb{R}^L$ and $w, w' \in \mathbb{R}_+$, such that $\upsilon(p,w) \leq \bar{\upsilon}$ and $\upsilon(p',w') \leq \bar{\upsilon}$. Pick some $\lambda \in [0,1]$ and define:

$$\begin{aligned} p_\lambda &= \lambda p + (1-\lambda)p' \\ w_\lambda &= \lambda w + (1-\lambda)w' \end{aligned}$$

We argue that $\upsilon(p_\lambda, w_\lambda) \leq \bar{v}$. This means that $\forall x : p_\lambda \cdot x \leq w_\lambda$ we have $u(x) \leq \bar{v}$. Observe that:

$$\begin{aligned} p_\lambda \cdot x &\leq w_\lambda \\ [\lambda p + (1-\lambda) p'] x &\leq \lambda w + (1-\lambda) w' \\ \lambda [px - w] + (1-\lambda)[p'x - w'] &\leq 0 \end{aligned}$$

This means that either $px \leq w$ or $p'x \leq w'$ (or both). The former inequality means that $u(x) \leq \upsilon(p, w) \leq \bar{v}$, and if the latter holds, we have $u(x) \leq \upsilon(p', w') \leq \bar{v}$. Thus anyway $\forall x : p_\lambda \cdot x \leq w_\lambda$ we have $u(x) \leq \bar{v}$, i.e. $\upsilon(p_\lambda, w_\lambda) \leq \bar{v}$. QED. □

Definition 18. *(Expenditure Minimization Problem – EMP). The EMP is defined as:*

$$\min_{u(x) \geq \bar{u}} p \cdot x$$

*where $\bar{u}$ is some positive number, the level of utility that is exogenously given. The solution to the problem is $h(p, \bar{u})$, which is called the **Hicksian** (or compensated) **demand function**. The value function of this problem, which is called the **expenditure function**, is defined as:*

$$e(p, \bar{u}) \equiv p \cdot h(p, \bar{u})$$

Theorem 12. *Let $u(\cdot)$ be a continuous function representing a LNS preference relation $\succeq$. Suppose that EMP is as defined above, and functions $h(p, \bar{u})$ and $e(p, \bar{u})$ are the Hicksian demand function and the expenditure function, respectively. Then they possess the following properties:*

1. *$h(p, \bar{u})$ is homogenous of degree 0 in prices, i.e. $h(\alpha p, \bar{u}) = h(p, \bar{u})$.*
2. *$\forall x \in h(p, \bar{u})$ we have $u(x) = \bar{u}$*
3. *If $\succeq$ is convex, i.e. $u(\cdot)$ is quasi-concave then $h(p, \bar{u})$ is a convex set. Moreover, if $\succeq$ is strictly convex, i.e. $u(\cdot)$ is strictly quasi-concave, then $h(p, \bar{u})$ is single-valued.*

Also the expenditure function has the following properties:

1. *$e(p, \bar{u})$ is homogenous of degree 1 in prices, i.e. $e(\alpha p, \bar{u}) = \alpha \cdot e(p, \bar{u})$.*
2. *$e(p, \bar{u})$ is strictly increasing in $\bar{u}$ and nondecreasing in p.*
3. *$e(p, \bar{u})$ is concave in prices.*
4. *$e(p, \bar{u})$ is continuous in both its arguments.*

Finally, there are some joint properties of $e(p, \bar{u})$ and $h(p, \bar{u})$:

1. *$h(p, \bar{u}) = D_p e(p, \bar{u})$ and hence $D_p h(p, \bar{u}) = D_p^2 e(p, \bar{u})$*
2. *$p \cdot D_p h(p, \bar{u}) = 0$*
3. *$D_p^2 e(p, \bar{u})$ is a negative semidefinite and symmetric matrix*

Proof. We first prove the properties of the compensated demand function.

1. Recall that the objective function in the EMP is $p \cdot x$, which obviously has the same minimum point as the problem $\alpha p \cdot x$, i.e. we can multiply the objective by some positive constant.
2. Suppose not, i.e. $\exists \tilde{x} \in h(p, \bar{u})$ we have $u(\tilde{x}) > \bar{u}$. Then by continuity of $\succeq$ $\exists y \in X$ such that $\|\tilde{x} - y\| \leq \varepsilon$, $u(y) \geq \bar{u}$ and $py < p\tilde{x}$, as long as ε is small enough. But this contradicts $\tilde{x}$ being a solution of EMP, so $\tilde{x}$ does not exist. QED

(3) The logic is similar to the proof for Walrasian demand. Suppose that $u(\cdot)$ is quasi-concave. Let $x, x' \in h(p, \bar{u})$ for some $p \in \mathbb{R}^L, \bar{u} \in \mathbb{R}$. Notice that $p \cdot x = p \cdot x'$ by the definition of $h(p, \bar{u})$. Pick some $\lambda \in [0, 1]$, and denote $x_\lambda = \lambda x + (1 - \lambda) x'$. Sure enough, $p \cdot x_\lambda = p \cdot x = p \cdot x'$. Since by quasi-convexity of $u(\cdot)$ we have $u(x_\lambda) \geq \bar{u}$, we conclude that $x_\lambda \in h(p, \bar{u})$, so it is indeed convex.

Notice now that if $u(\cdot)$ is strictly quasi-concave, then $u(x_\lambda) > \bar{u}$, and hence by continuity of $\succeq$ $\exists y \in X$ such that $\|\tilde{x} - y\| \leq \varepsilon$ and $u(y) \geq \bar{u}$ and $py < px_\lambda$. This contradicts x_λ being a solution to EMP. Thus there is at most one $x \in h(p, \bar{u})$. QED.

We now establish properties of the expenditure function:

(1) This follows immediately from $h(p, \bar{u})$ being homogenous of degree 0 in prices and the definition of $e(p, \bar{u})$:

$$e(p, \bar{u}) = p \cdot h(p, \bar{u}) \Longrightarrow e(\alpha p, \bar{u}) = \alpha p \cdot h(\alpha p, \bar{u}) = \alpha [p \cdot h(p, \bar{u})] = \alpha e(p, \bar{u})$$

(2) Suppose first that $e(p, \bar{u})$ is not strictly increasing in $\bar{u}$. That is, $\exists u_1, u_2 :$ $u_1 < u_2$ and $e(p, u_1) > e(p, u_2)$. This means that $\exists x_1, x_2 \in X : x_1 \neq x_2$ and $x_1 \in h(p, u_1)$, $x_2 \in h(p, u_2)$ with the property that $p \cdot x_1 > p \cdot x_2$. Since $\succeq$ is continuous, $\exists y$ such that $\|x_2 - y\| \leq \varepsilon$ and $u(y) > u_1$ and $py < px_1$. This is possivle as long as ε is small enough. But this constitutes a contradiction with x_1 being a solution to the EMP. Hence $e(p, \bar{u})$ is uncreasing in $\bar{u}$.

To show that $e(p, \bar{u})$ is nondecreasing in prices, let $\bar{u}$ be given and consider $p, p' \in \mathbb{R}^L$ such that $p_i = p'_i \ \forall i \neq k$ and $p_k \geq p'_k$, for all $i = 1, \ldots L$ and for some $k \in [1, L]$. That is, these price vectors are exactly identical for all dimensions except k. Denote by $x \in h(p, \bar{u})$ the optimizing bundle under prices p. Then observe that:

$$e(p, \bar{u}) = p \cdot x \geq p' \cdot x \geq e(p', \bar{u})$$

The first inequality follows from definition of p and p', and the second one follows from the definition of the expenditure function. Since k was arbitrary, the result follows. QED.

(3) Pick $p, p' \in \mathbb{R}^L$, pick $\lambda \in [0, 1]$ and let $p_\lambda \equiv \lambda p + (1 - \lambda) p'$. We want to show that, $\forall \bar{u}$:

$$\lambda e(p, \bar{u}) + (1 - \lambda) e(p, \bar{u}) \leq e(p_\lambda, \bar{u})$$

This is straightforward. Observe that:

$$\lambda e(p, \bar{u}) + (1 - \lambda) e(p, \bar{u}) = \lambda [p \cdot h(p, \bar{u})] + (1 - \lambda) [p' \cdot h(p', \bar{u})]$$

Let $x \in h(p, \bar{u})$ and $x' \in h(p', \bar{u})$. Since $e(p_\lambda, \bar{u}) = p_\lambda \cdot h(p_\lambda, \bar{u})$, denote $x_\lambda \in h(p_\lambda, \bar{u})$. Now rewrite the above equation as follows:

$$\lambda [p \cdot x] + (1 - \lambda) [p' \cdot x'] \leq \lambda [p \cdot x_\lambda] + (1 - \lambda) [p' \cdot x_\lambda] = [\lambda p + (1 - \lambda) p'] \cdot x_\lambda = p_\lambda \cdot x_\lambda = e(p_\lambda, \bar{u})$$

where the inequality sign follows from the fact that x is the EMP solution under prices p and x' is the EMP solution under the prices p', and hence under both these prices costs of bundle x_λ can only be greater than or equal to the costs of the optimal bundles. QED.

(4) This is again an immediate consequence of the Maximum Theorem.

Finally, we establish the joint properties.

Observe that the objective function of the EMP is $e(p, \bar{u}) = p \cdot x$. The constraint does not depend on p, and thus we can apply the Envelope Theorem:

$$D_p e(p, \bar{u}) = D_p (p \cdot x)|_{x=h(p,\bar{u})} = h(p, \bar{u})$$

Notice that this immediately implies the other two joint properties. In particular:

$$D_p e(p, \bar{u}) = D_p (p \cdot h(p, \bar{u})) = h(p, \bar{u}) + p \cdot D_p h(p, \bar{u}) = h(p, \bar{u}) \iff p \cdot D_p h(p, \bar{u}) = 0$$

And differentiating the envelope condition again yields the second derivative result. Notice that since $e(p, \bar{u})$ is concave in p, its Hessian is necessarily negative semi-definite and symmetric. QED. □

We now state a key theorem that relates all the concepts introduced.

THEOREM 13. *Let $u(\cdot)$ be a continuous utility function representing a LNS preference relation $\succeq$. Then the following holds:*

$$\begin{aligned} x(p, w) &= h(p, v(p, w)) \\ h(p, \bar{u}) &= x(p, e(p, \bar{u})) \end{aligned}$$

Or, stated in words:

1. *If a consumption bundle $\tilde{x}$ is optimal in the UMP with prices p and wealth w, then $\tilde{x}$ is also optimal in the EMP with the prices p and $\bar{u} = u(\tilde{x})$. Moreover, $e(p, u(\tilde{x})) = w$.*
2. *If a consumption bundle $\tilde{x}$ is optimal in the EMP with prices p and the utility level fixed at $\bar{u}$, then $\tilde{x}$ is also optimal in the UMP with prices p and wealth given by $e(p, \bar{u})$. Moreover, $u(\tilde{x}) = \bar{u}$.*

PROOF. First, suppose that $\tilde{x}$ is optimal in the UMP but not in the EMP with $\bar{u} = u(\tilde{x})$. Then $\exists x' \in X : u(x') \succeq u(\tilde{x})$ and $p \cdot x' < p \cdot \tilde{x} \leq w$. Then by continuity and LNS of $\succeq$ $\exists x'' \in X$ such that $\|x'' - x'\| \leq \varepsilon$ and $u(x'') > u(x')$ and $p \cdot x'' < w$. But this implies immediately that $x'' \in B(p, w)$ and $u(x'') > u(\tilde{x})$, which contradicts $\tilde{x}$ being a solution to the UMP. Hence $\tilde{x}$ has to be optimal in the EMP, too. The minimized expenditures are then $p \cdot \tilde{x}$ and by Walras' Law we have $e(p, u(\tilde{x})) \equiv p \cdot \tilde{x} = w$.

Now, suppose that $\tilde{x}$ is optimal in the EMP but not in the UMP with wealth given by $e(p, \bar{u})$. Then $\exists x' \in X : u(x') \succeq u(\tilde{x})$ and $p \cdot x' \leq p \cdot \tilde{x}$. Then by continuity and LNS of $\succeq$ $\exists x'' \in X$ such that $\|x'' - x'\| \leq \varepsilon$ and $u(x'') > u(x')$ and $p \cdot x'' < p \cdot \tilde{x}$. But this contradicts $\tilde{x}$ being a solution to the EMP. Hence $\tilde{x}$ must also solve UMP. The utility conjecture follows from the properties of Hicksian demand. □

We now introduce the concept of a Slutsky equation, which is tightly connected with the income and substitution effect decomposition.

DEFINITION 19. *(Slutsky equation). Consider the following equation:*

$$\begin{aligned} D_p h(p, \bar{u}) &= D_p x(p, e(p, \bar{u})) = D_p x(p, w) + D_w x(p, w) D_p e(p, \bar{u}) = \\ &= D_p x(p, w) + D_w x(p, w) \cdot h(p, \bar{u}) = \\ &= D_p x(p, w) + D_w x(p, w) \cdot x(p, w) \end{aligned}$$

The last line is called the ***Slutsky Equation****, that decomposes the change in demand due to price change into income effect and substitution effect. The derivation uses the facts that* $x(p, e(p, \bar{u})) = h(p, \bar{u})$ *and that* $e(p, \bar{u}) = w$.

The Slutsky Matrix is an $L \times L$ *matrix* S*, where the* s_{ij} *element is given by:*

$$s_{ij} = \frac{\partial x_i(p, w)}{\partial p_j} + \frac{\partial x_i(p, w)}{\partial w} x_j(p, w)$$

Observe that $S = D_p h(p, \bar{u})$*,* $S \cdot p = 0$ *and* S *is negative semidefinite and symmetric, by the properties of* $D_p h(p, \bar{u})$.

THEOREM 14. *(Integrability). Let* $x : \mathbb{R}^L_{++} \times \mathbb{R}_+ \to \mathbb{R}^L_+$ *be a continuously differentiable function such that:*

(1) *It satisfies Walras' Law:* $p \cdot x = w$.
(2) *It is homogenous of degree* 0.
(3) *The Slutsky Matrix associated with* x *is negative semi-definite and symmetric matrix.*

If so, $\exists u : \mathbb{R}^L_+ \to \mathbb{R}$ *– a strictly increasing and strictly quasi-concave utility function such that* x *is the Walrasian demand function generated by* u.

3. Revealed Preference

THEOREM 15.

AXIOM 2. *(Weak Axiom of Revealed Preference (WARP)). Suppose one observes* T *pairs of prices and consumption bundles:* $(x_1, p_1), \ldots, (x_T, p_T)$*. Suppose that the consumer has a complete continuous and LNS preference relation* $\succeq$*. We say that the observations* $\{(x_i, p_i)\}_{i=1}^T$ *satisfy WARP iff* $\forall i, j \in 1, \ldots, T$ *we have:*

$$p_i \cdot x_j \leq p_i \cdot x_i \iff p_j \cdot x_i \geq p_j \cdot x_j$$

In other words, should the bundle x_j *be affordable under prices* p_i*, it must immediately imply that bundle* x_i *is at least as expensive as bundle* x_j *when prices are* p_j*. The idea is that the observed* T *pairs are the choices made by the consumer, and we would like to see whether one could argue that the choice is "rational" in some sense.*

DEFINITION 20. *We say that an utility function* $u(\cdot)$ *rationalizes observations* $\{(x_i, p_i)\}_{i=1}^T$ *iff* $\forall x \in X$ *and* $\forall i = 1, \ldots, T$*:*

$$p_i \cdot x \leq p_i \cdot x_i \Longrightarrow u(x_i) \geq u(x)$$

i.e. for every bundle affordable under prices p_i *the bundle* x_i *yields the highest utility possible.*

Unfortunately, WARP is only necessary, but not sufficient for rationalizabiltiy, so we have to introduce SARP.

DEFINITION 21. *Let* $\{(x_i, p_i)\}_{i=1}^T$ *be given.* $\forall x \in X$ *and* $\forall i = 1, \ldots, T$*, define the following relations on* $\mathbb{R}^L_+ \times \mathbb{R}^L_{++}$*:*

$$x_i \; R \; x \iff p_i \cdot x \leq p_i \cdot x_i$$
$$x_i \; P \; x \iff p_i \cdot x < p_i \cdot x_i$$

If $x_i \; R \; x$ *we say that* x_i *is* ***directly weakly revealed preferred*** *to* x*. If* $x_i \; P \; x$ *we say that* x_i *is* ***directly strongly revealed preferred*** *to* x.

AXIOM 3. *(Strong Axiom of Revealed Preference). Let* $\{(x_i, p_i)\}_{i=1}^{T}$ *be given. We say that SARP holds iff* $\forall n \leq T$*:*

$$x_1 \; R \; x_2 \; R \; x_3 \; R \; \ldots \; R \; x_n \iff \textbf{not} \, (x_n \; P \; x_1)$$

Of course, we assume $x_n \neq x_1$ *here. Also this must be true regardless of the indexing of the observations.*

CHAPTER 3

Choice Under Uncertainty

1. Lotteries

A simple lottery is a probability distribution over a set of outcomes, which is assumed to be finite: $L = \{p_1, \dots, p_N\}$, where p_i is the probability of the outcome i. A compound lottery is:

$$\begin{aligned} \{(L_1, \dots, L_k), (\pi_1, \dots, \pi_k)\} &= \left\{\left(\left\{p_1^1, \dots, p_N^1\right\}, \dots \left\{p_1^k, \dots, p_N^k\right\}\right), (\pi_1, \dots, \pi_k)\right\} = \\ &= \left\{p_1^1\pi_1, \dots p_N^1\pi_1, \dots, p_1^k\pi_k, \dots p_N^k\pi_k\right\} \end{aligned}$$

Observe that the probabilities of the compound lotteries sum up to 1, as expected. We define a preference relation over the reduced lotteries only (what comes after the second $=$ sign is the reduced version of the original compund lottery). Fix the set of outcomes and define by L the set of all possible lotteries over these outcomes.

Definition 22. *A preference relation $\succeq$ on L is **complete** if $\forall L_1, L_2 \in L$ we have $L_1 \succeq L_2$ or $L_2 \succeq L_1$ or both.*

Definition 23. *A preference relation $\succeq$ on L is **transitive** if $\forall L_1, L_2, L_3 \in L$ if $L_1 \succeq L_2$ and $L_2 \succeq L_3$ we have $L_1 \succeq L_3$.*

Definition 24. *A preference relation $\succeq$ on L is **continuous** if $\forall L_1, L_2, L_3 \in L$ the sets:*

$$\begin{aligned} UCS(L_3) &= \{\alpha \in [0,1] \mid \alpha L_1 + (1-\alpha) L_2 \succeq L_3\} \\ LCS(L_3) &= \{\alpha \in [0,1] \mid L_3 \succeq \alpha L_1 + (1-\alpha) L_2\} \end{aligned}$$

are both closed.

Definition 25. *A preference relation $\succeq$ on L satisfies the **independence axiom** if $\forall L_1, L_2, L_3 \in L$ and $\forall \alpha \in [0,1]$ we have:*

$$L_1 \succeq L_2 \iff \alpha L_1 + (1-\alpha) L_3 \succeq \alpha L_2 + (1-\alpha) L_3$$

i.e. the preference over any two lotteries do not depend on any other lottery.

There is a subtle distinction between an expected utility form and the expected utility representation of a utility function.

Definition 26. *A utility function has an **expected utility form** if $\exists N$ numbers $u_1, \dots, u_N$ for each outcome of the lottery $L = \{p_1, \dots, p_N\}$ such that $U(L) = \sum_{i=1}^N u_i p_i$.*

Definition 27. *A utility function has an **expected utility representation** if there exists a transformation $\widetilde{U}$ of the original function U such that $\widetilde{U}$ preserves the ordering of the lotteries and is linear in probabilities.*

THEOREM 16. *A utility function has an expected utility form iff it is linear in probabilities, i.e.:*

$$U\left(\sum_{k=1}^{K}\alpha_k L_k\right)=\sum_{k=1}^{K}\alpha_k U\left(L_k\right) \tag{1.1}$$

PROOF. First, we show that the linearity implies the expected utility form. Suppose (1.1) is true. Define $L^i = \{p_1, \ldots, p_N \mid p_i = 1\}$, a degenerate lottery. Then $\forall L \in L$ we can write $L = \sum_{i=1}^{N} p_i L^i$, i.e. we can represent any lottery as a sum of degenerate lotteries. Then:

$$U\left(L\right)=U\left(\sum_{i=1}^{N}p_i L^i\right)=\sum_{i=1}^{N}p_i U\left(L^i\right)=\sum_{i=1}^{N}p_i u_i$$

where the second equality uses (1.1) and the last one is just the definition, since L^i is a degenerate lottery, i.e. a number. This proves the expected utility form.

Next, suppose that a function has the expected utility form, i.e. $U(L) = \sum_{i=1}^{N} p_i u_i$. Consider a compound lottery $(L_1, \ldots, L_k; \alpha_1, \ldots, \alpha_k)$, where $L_k = \left(p_1^k, \ldots, p_N^k\right)$. Then:

$$U\left(L\right)=U\left(\sum_{k=1}^{K}\alpha_k L_k\right)=\sum_{i=1}^{N}u_i\left(\sum_{k=1}^{K}\alpha_k p_i^k\right)=\sum_{k=1}^{K}\alpha_k\sum_{i=1}^{N}u_i p_i^k=\sum_{k=1}^{K}\alpha_k U\left(L_k\right)$$

so (1.1) holds and hence QED. □

THEOREM 17. *Suppose a preference relation $\succeq$ is represented by an expected utility function. Then $\succeq$ satisfies the independence axiom.*

PROOF. Pick $L = (p_1, \ldots, p_N)$, then $U(L) = \sum_{i=1}^{N} p_i u_i$. Define $L' = (p'_1, \ldots, p'_N)$ and $L'' = (p''_1, \ldots, p''_N)$ and fix some $\alpha \in (0, 1)$. Clearly if $L \succeq L'$ then $\sum_{i=1}^{N} p_i u_i \geq \sum_{i=1}^{N} p'_i u_i$. Now observe that:

$$\begin{aligned}\sum_{i=1}^{N}p_i u_i &\geq \sum_{i=1}^{N}p'_i u_i\\ \alpha\sum_{i=1}^{N}p_i u_i+(1-\alpha)\sum_{i=1}^{N}p''_i u_i &\geq \alpha\sum_{i=1}^{N}p'_i u_i+(1-\alpha)\sum_{i=1}^{N}p''_i u_i\\ \alpha L+(1-\alpha)L'' &\succeq \alpha L'+(1-\alpha)L''\end{aligned}$$

where the second line follows from the fact that we just add the same value to both sides of the first line, and the last one uses the definition of the expected utility function. But since the last line is the definition of the independence axiom, so QED. □

THEOREM 18. *(Expected Utility Representation). Suppose $\succeq$ is rational, continuous and satisfies the independence axiom. Then this preference relation has an expected utility representation, i.e. $\exists u_i : i = 1, \ldots, N$ such that if $L \succeq L'$ then $\sum_{i=1}^{N} p_i u_i \geq \sum_{i=1}^{n} p'_i u_i$.*

PROOF. PROOF. Assume that the set of outcomes is finite, i.e. $N < \infty$. Then there exist best and worst lotteries, i.e. $\exists \overline{L}, \underline{L} : \overline{L} \succeq L \succeq \underline{L}$ for all L. This is straightforward: since the set of outcomes is finite, there exist the best and the worst outcome. Define $\underline{L}$ as the lottery that gives the worst outcome with probability 1, and similarly define $\overline{L}$ as the lottery that gives the best outcome with probability 1. We now claim that $\overline{L} \succeq L \succeq \underline{L}$ for all L. To show this, we prove the following lemma:

Lemma 1. *Let $L_0, L_1, \ldots L_k$ be $k+1$ lotteries and $\alpha_1, \ldots, \alpha_k : \sum_i \alpha_i = 1$ be some probabilities, i.e. $\alpha_i \geq 0\ \forall i$. If $\forall i : L_i \succeq L_0$ then $\sum_{i=1}^k \alpha_i L_i \succeq L_0$, and similarly if $\forall i : L_0 \succeq L_i$ then $L_0 \succeq \sum_{i=1}^k \alpha_i L_i$.*

Proof. We prove by induction. If $k=1$, the lemma is trivial. Suppose it is true for some $k-1>1$, and let $\forall i : L_i \succeq L_0$. By the definition of the compound lottery:

$$\sum_{i=1}^{k} \alpha_i L_i = (1-\alpha_k) \sum_{i=1}^{k-1} \frac{\alpha_i}{1-\alpha_k} L_i + \alpha_k L_k$$

By the induction hypothesis we have $\hat{L}_{k-1} \equiv \sum_{i=1}^{k-1} \frac{\alpha_i}{1-\alpha_k} L_i \succeq L_0$. Now use the independence axiom:

$$(1-\alpha_k)\,\hat{L}_{k-1} + \alpha_k L_k \succeq (1-\alpha_k)\,L_0 + \alpha_k L_k \succeq (1-\alpha_k)\,L_0 + \alpha_k L_0 = L_0$$

Therefore $\hat{L}_k = (1-\alpha_k)\,\hat{L}_{k-1} + \alpha_k L_k \succeq L_0$ and induction done. QED. □

Now define by L^n the lottery that gives outcome n with probability 1. Then by definition $\overline{L} \geq L^n \geq \underline{L}$, since these are just numbers. But notice that any arbitrary L can we written as $L = \sum_{i=1}^N p_i L^i$, so $\overline{L} \geq L$ by the lemma above. We thus proved the existence of the best lottery. Proving the second part of the lemma allows to show that the worst lottery also exists. Therefore the worst and the best lottery exist. QED. □

Claim 1. *If $L \succ L'$ and $\alpha \in (0,1)$, then $L \succ \alpha L + (1-\alpha)\,L' \succ L'$.*

Proof. Notice that $L = \alpha L + (1-\alpha)\,L$. By independence axiom we have:

$$L \succ L' \iff \alpha L + (1-\alpha)\,L'' \succ \alpha L' + (1-\alpha)\,L''$$

Therefore:

$$L = \alpha L + (1-\alpha)\,L \succ \alpha L + (1-\alpha)\,L' \succ \alpha L' + (1-\alpha)\,L' = L'$$

which proves the claim. QED. □

Claim 2. *Let $\alpha, \beta \in [0,1]$. Then $\beta\overline{L} + (1-\beta)\,\underline{L} \succ \alpha\overline{L} + (1-\alpha)\,\underline{L} \iff \beta > \alpha$.*

Proof. Let $\beta > \alpha$. Then:

$$\beta\overline{L} + (1-\beta)\,\underline{L} = \gamma\overline{L} + (1-\gamma)\left[\alpha\overline{L} + (1-\alpha)\,\underline{L}\right]$$

where $\gamma = \frac{\beta-\alpha}{1-\alpha}$, $1-\gamma = \frac{1-\beta}{1-\alpha}$. Notice that by the previous claim, we have:

$$\overline{L} \succ \alpha\overline{L} + (1-\alpha)\,\underline{L} \Longrightarrow \gamma\overline{L} + (1-\gamma)\left[\alpha\overline{L} + (1-\alpha)\,\underline{L}\right] \succ \alpha\overline{L} + (1-\alpha)\,\underline{L}$$

Therefore:

$$\beta\overline{L} + (1-\beta)\,\underline{L} \succ \alpha\overline{L} + (1-\alpha)\,\underline{L}$$

and we proved the first part of the claim. Now let $\beta \leq \alpha$. Clearly if $\beta = \alpha$, we have:

$$\beta\overline{L} + (1-\beta)\,\underline{L} \sim \alpha\overline{L} + (1-\alpha)\,\underline{L}$$

so assume $\beta < \alpha$. By the same argument then we can argue that in this case:

$$\alpha\overline{L} + (1-\alpha)\,\underline{L} \succ \beta\overline{L} + (1-\beta)\,\underline{L}$$

and thus claim is proved. QED. □

CLAIM 3. *$\forall L : \exists! \alpha_L : \alpha_L \overline{L} + (1-\alpha_L)\,\underline{L} \sim L$. That is, any lottery can be represented by the weighted combination of the best and the worst lotteries.*

PROOF. By the continuity of $\succeq$, the sets:

$$\left\{\alpha \in [0,1] \mid \alpha\overline{L} + (1-\alpha)\,\underline{L} \succ L\right\}$$
$$\left\{\alpha \in [0,1] \mid L \succ \alpha\overline{L} + (1-\alpha)\,\underline{L}\right\}$$

are closed, so any $\alpha \in [0,1]$ belongs at least to the one of these sets. And since $[0,1]$ is a connected set, $\exists\hat{\alpha}$ that belongs to both of the above sets. Define it as α_L, and thus existence is proved. Uniqueness is obvious: suppose $\exists\alpha_L^1, \alpha_L^2 \in [0,1] : \alpha_L^1 \neq \alpha_L^2$ and they both satisfy this criterion. But then either $\alpha_L^1 > \alpha_L^2$ or $\alpha_L^1 < \alpha_L^2$ and by one of the above claims (3) we will have that $\alpha_L^1\overline{L} + \left(1-\alpha_L^1\right)\underline{L} \succ \alpha_L^2\overline{L} + \left(1-\alpha_L^2\right)\underline{L}$ or the other way, so we get a contradiction, and hence uniqueness is verified. QED. □

CLAIM 4. *The function $u : \pounds \to \mathbb{R}$ that assigns $u(L) = \alpha_L$, represents $\succeq$.*

PROOF. $\forall L, L'$ we have:

$$L \succeq L' \iff \alpha_L\overline{L} + (1-\alpha_L)\,\underline{L} \succeq \alpha_L'\overline{L} + (1-\alpha_L')\,\underline{L}$$

by one of the previous claims (4). So by yet another previous claim (3) we have:

$$L \succeq L' \iff \alpha_L \geq \alpha_L'$$

and this proves the claim. QED. □

CLAIM 5. *$\forall L$, the function $u(L) = \alpha_L$ is linear in probabilities, i.e. has the expected utility form.*

PROOF. We want to show that $\forall L, L'$ and $\forall\beta \in [0,1]$, we have:

$$u(\beta L + (1-\beta)L') = \beta u(L) + (1-\beta)\,u(L')$$

By definition we have:

$$\begin{aligned} L &\sim u(L)\,\overline{L} + (1-u(L))\,\underline{L} \\ L' &\sim u(L')\,\overline{L} + (1-u(L'))\,\underline{L} \end{aligned}$$

Then by the independence axiom:

$$\begin{aligned} \beta L + (1-\beta)L' &\sim \beta\left[u(L)\,\overline{L} + (1-u(L))\,\underline{L}\right] + (1-\beta)\left[u(L')\,\overline{L} + (1-u(L'))\,\underline{L}\right] \sim \\ &\sim [\beta u(L) + (1-\beta)\,u(L')]\,\overline{L} + [1-\beta u(L) - (1-\beta)\,u(L')]\,\underline{L} \end{aligned}$$

So $u(\beta L + (1-\beta)L') = \beta u(L) + (1-\beta)\,u(L')$, and this proves the claim and the theorem. Done. QED. □

□

2. States

We assume that consumption can take finitely many values: $c = (c_1, \ldots, c_s) \in \mathbb{R}^s$. The expected utility function is then $u : \mathbb{R}^s \to \mathbb{R}$. We assume that it is strictly increasing and continuous.

Axiom 4. *(**Independence axiom** a.k.a. **sure-thing principle**). Pick some $c, d \in \mathbb{R}^s$. Define $\forall y \in \mathbb{R}$:*

$$c_{-k}y = (c_1, \ldots, c_{k-1}, y, c_{k+1}, \ldots, c_s)$$

Definition 28. *Then the independence axiom states that $\forall w \in \mathbb{R}$:*

$$u(c_{-k}y) \geq u(d_{-k}y) \iff u(c_{-k}w) \geq u(d_{-k}w)$$

In other words, if random variables c and d have the same outcome in some state, then the preference relation over these two variables do not depend on the value of this outcome at that particular state.

Definition 29. *A utility function $u : \mathbb{R}^s \to \mathbb{R}$ has a **state-separable representation**, if $\exists$ functions $v_k : \mathbb{R} \to \mathbb{R}$ $\forall k = 1, \ldots, s.$, such that:*

$$u(c_1, \ldots, c_s) \geq u(c_1', \ldots, c_s') \iff \sum_{k=1}^{s} v_k(c_k) \geq \sum_{k=1}^{s} v_k(c_k')$$

Theorem 19. *Let $s \geq 3$, $u(\cdot)$ be strictly increasing and continuous. Then $u(\cdot)$ has a state-separable representation iff it obeys a sure-thing principle.*

If there are 2 states only – every strictly increasing function suffices. The case of 1 state is trivial.

Definition 30. *A utility function $u : \mathbb{R}^s \to \mathbb{R}$ has an expected utility representation with respect to the probabilities $\{\pi_1, \ldots, \pi_s\}$, if $\exists v : \mathbb{R} \to \mathbb{R}$ such that:*

$$u(c_1, \ldots, c_s) \geq u(c_1', \ldots, c_s') \iff \sum_{k=1}^{s} \pi_k v(c_k) \geq \sum_{k=1}^{s} \pi_k v(c_k')$$

(Observe that here v is the same for all states, while for the state-separable representation we can possibly have different v_k for every state).

Definition 31. *(Risk Aversion). Let $\{\pi_1, \ldots, \pi_s\}$ be the probabilities of states. Let $E[c] = \sum_{k=1}^{s} \pi_k c_k$ be the expected value of vector $c = (c_1, \ldots, c_s)$. Define by $\widehat{Ec}$ th deterministic vector $(E[c], \ldots, E[c])$. Then $u(\cdot)$ says a consumer is risk averse if $u(c) \leq u\left(\widehat{Ec}\right)$.*

Theorem 20. *Let $s \geq 3$, $u(\cdot)$ be stictly increasing and continuous. Then $u(\cdot)$ satisfies sure-thing principle and is risk averse iff it has a concave expected utility representation (given $\{\pi_1, \ldots, \pi_k\}$).*

Proof. Since $u(\cdot)$ satisfies sure-thing principle, it has a state-separable representation $u(\cdot) = \sum_{k=1}^{s} v_k(c_k)$. For this proof we assume v_k is differentiable. Consider the problem:

$$\max_c \sum_{k=1}^{s} v_k(c_k) \text{ s.t. } E[c] = x, \text{ for some } x \in \mathbb{R}$$

By risk aversion the vector $c_x \equiv (x, \ldots, x)$ must necessarily be a solution to this problem. Evaluate the F.O.C. at this solution ($\forall k = 1, \ldots, s$):

$$v_k'(x) = \lambda \pi_k$$

Now divide this by the F.O.C. for $k = 1$ to get:

$$v_k'(x) = \frac{\pi_k}{\pi_1} v_1'(x)$$

Since x was arbitrary, we can integrate x out:

$$v_k(x) = \frac{\pi_k}{\pi_1} v_1(x) + \Delta_k$$

Therefore all v_k's are just the affine transformations of v_1. Therefore:

$$\hat{u} = \sum_{k=1}^{s} \pi_k v(c_k) = \sum_{k=1}^{s} \frac{\pi_k}{\pi_1} v_1(c_k)$$

And since the original $u(\cdot)$ was risk averse, $v(\cdot)$ is concave. □

3. Risk Aversion

We now consider only random variables that have some distribution over the interval $[a, b]$. We denote with x the random variable in question and by $F_x(t)$ and $f_x(t)$ its c.d.f. and p.d.f., respectively, $\forall t \in [a, b]$.

Definition 32. *(First Order Stochastic Dominance). A random variable z first order stochastically dominates random variable y if $F_z(t) \leq F_y(t)$ $\forall t \in [a, b]$. This definition works for scalars only. though.*

Definition 33. *(First Order Stochastic Dominance). A random variable z first order stochastically dominates random variable y iff $E[v(z)] \geq E[v(y)]$ $\forall v(\cdot)$ such that $v(\cdot)$ is a continuous and nondecreasing function.*

Corollary 2. *z FOSD y iff every expected-utility maximizing agent with non-decreasing utility prefers z to y.*

We now introduce the concept of second order dominance, which is usually much more useful.

Definition 34. *(Second Order Stochastic Dominance). A random variable z second order stochastically dominates random variable y if $\forall \omega \in [a, b]$ we have:*

$$\int_a^\omega F_z(t)\,dt \leq \int_a^\omega F_y(t)\,dt$$

Again, this definition works only for scalar random variables.

Definition 35. *(Second Order Stochastic Dominance). A random variable z second order stochastically dominates random variable y iff $E[v(z)] \geq E[v(y)]$ $\forall v(\cdot)$ such that $v(\cdot)$ is a continuous, nondecreasing and* concave *function.*

Corollary 3. *z SOSD y iff every expected-utility maximizing* risk-averse *agent with nondecreasing utility prefers z to y.*

Clearly since:

$$E[z] = \int_a^b z dF_z(t) = b - \int_a^b F_z(t)\, dt$$

we have:

$$z\ SOSD\ y \Longrightarrow E[z] \geq E[y]$$

and also:

$$z\ FOSD\ y \Longrightarrow z\ SOSD\ y$$

DEFINITION 36. *Let z and y be random variables such that $E[z] = E[y]$. Suppose that z SOSD y. We then say that y is* ***more risky*** *than z, since every risk-averse agent prefers z to y.*

THEOREM 21. *If $E[z] = 0$, then kz is more risky than z, $\forall k > 1$.*

PROOF. Obviously since k is a constant, $E[z] = E[kz] = 0$. We therefore must demonstrate that $E[v(z)] \geq E[v(kz)]$ for every concave nondecreasing continuous $v(\cdot)$. Observe that:

$$z = \frac{1}{k}kz + \left(1 - \frac{1}{k}\right)0$$

Use this and the concavity of $v(\cdot)$ to show that:

$$v(z) \geq \frac{1}{k}v(kz) + \left(1 - \frac{1}{k}\right)v(0)$$

Take expectations:

$$E[v(z)] \geq \frac{1}{k}E[v(kz)] + \left(1 - \frac{1}{k}\right)v(0)$$

Now use the Jensen's inequality: $E[v(z)] \leq v(E(z)) = v(0)$. Then obviously we can replace $v(0)$ with $E[v(kz)]$ and preserve the inequality:

$$\begin{aligned} E[v(z)] &\geq \frac{1}{k}E[v(kz)] + \left(1 - \frac{1}{k}\right)E[v(kz)] \\ E[v(z)] &\geq E[v(kz)] \end{aligned}$$

This concludes the proof. QED. □

THEOREM 22. *Let z and y be random variables such that $E[z] = E[y]$. Suppose that y is more risky than z. Then $Var[y] \geq Var[z]$.*

PROOF. We know that:

$$E[v(z)] \geq E[v(y)]$$

for every concave nondecreasing continuous $v(\cdot)$. Pick $v(x) = -(\alpha - x)^2$, then:

$$-\alpha^2 + 2\alpha E[z] - E\left[z^2\right] \geq -\alpha^2 + 2\alpha E[y] - E\left[y^2\right]$$

Since $E[z] = E[y]$, we're left with:

$$E\left[z^2\right] \leq E\left[y^2\right]$$

Now use the fact that $E\left[x^2\right] = (E[x])^2 + Var(x)$:

$$(E[z])^2 + Var(z) \leq (E[y])^2 + Var(y)$$

Again by the equality of means we cancel those and are left with the conclusion of the theorem:

$$Var\left(z\right) \leq Var\left(y\right)$$

QED. □

Part 3

General Equilibrium

CHAPTER 4

Basic Definitions

Definition 37. *A competitive equilibrium is a set of allocations* $\{x_i\}_{i=1}^n : x_i \in \mathbb{R}_+^L$ *and a price vector* $p \in \mathbb{R}^L : p \neq 0$ *such that* $\forall i : x_i \in x_i(p, e_i)$ *and* $\sum_{i=1}^n x_i = \sum_{i=1}^n e_i$. *Alternatively, at the competitive equilibrium allocations the aggregate excess demand* $z(p) \equiv \sum_{i=1}^n z_i(p) = \sum_{i=1}^n (x_i(p, e_i) - e_i)$ *must be zero (or contain it).*

Definition 38. *An allocation* $x = (x_1, \ldots, x_n) : x_i \in \mathbb{R}_+^L$ *is* ***weakly Pareto optimal*** *if it is feasible, i.e.* $\sum_{i=1}^n x_i = \sum_{i=1}^n e_i$ *and there does not exist any other feasible allocation* $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_n) : \hat{x}_i \in \mathbb{R}_+^L$ *such that* $\sum_{i=1}^n \hat{x}_i = \sum_{i=1}^n e_i$ *and* $\hat{x}_i \succ_i x_i \; \forall i \in I$.

Definition 39. *An allocation* $x = (x_1, \ldots, x_n) : x_i \in \mathbb{R}_+^L$ *is* ***strongly Pareto optimal*** *if it is feasible, i.e.* $\sum_{i=1}^n x_i = \sum_{i=1}^n e_i$ *and there does not exist any other feasible allocation* $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_n) : \hat{x}_i \in \mathbb{R}_+^L$ *such that* $\sum_{i=1}^n \hat{x}_i = \sum_{i=1}^n e_i$ *and* $\hat{x}_i \succeq_i x_i \; \forall i \in I$ *and* $\exists j \in I : \hat{x}_j \succ_j x_j$.

CHAPTER 5

Existence Theorems

THEOREM 23. *(Very Easy). Let* $z : \bar{\Delta} \to \mathbb{R}^L$ *be a continuous function that satisfies Walras' Law:* $p \cdot z(p) = 0$. *Then* $\exists p^* \in \bar{\Delta} : z(p^*) \leq 0$.

PROOF. Define $G : \bar{\Delta} \to \mathbb{R}^L$ as follows: $G_i(p) = p_i + \max\{0; z_i(p)\}$ for $i = 1, \ldots, L$. Now define $f : \bar{\Delta} \to \bar{\Delta}$ as:

$$f_i(p) = \frac{G_i(p)}{1 + \sum_{i=1}^{L} G_i(p)}$$

Again, $i = 1, \ldots, L$. Clearly $f(p)$ is a continuous function, and $\bar{\Delta}$ is a nonempty convex and compact set. Thus by Brouwer's Fixed Point Theorem, $\exists p^* : f(p^*) = p^*$. Clearly $z(p^*) \leq 0$:

$$0 = p^* \cdot z(p^*) = f(p^*) \cdot z(p^*) = \frac{G(p^*)}{1 + \sum_{i=1}^{L} G_i(p^*)} \cdot z(p^*) = \frac{0 + \max\{0; z(p^*) \cdot z(p^*)\}}{1 + \sum_{i=1}^{L} G_i(p^*)}$$

where the first equality uses Walras' Law, second – the definition of the fixed point, third – the definition of $f(p)$ and the last – Walras' Law again. Hence:

$$\max\{0; z(p^*) \cdot z(p^*)\} = 0 \iff z(p^*) \leq 0$$

□

THEOREM 24. *(Easy). Let* $z : \Delta \to \mathbb{R}^L$ *be a continuous function that satisfies:*

- *Walras' Law:* $p \cdot z(p) = 0 \ \forall p \in \Delta$
- *Boundedness from below:* $\forall i = 1, \ldots, L : z_i(p) \geq -B$
- *If* $\{p_n\}_{n=1}^{\infty} : p_n \to p \in \partial\Delta$, *then the sequence* $\{\|z(p_n)\|\}_{n=1}^{\infty}$ *is unbounded.*

Then $\exists p^* \in \Delta : z(p^*) = 0$.

PROOF. The proof is organized in five steps.

Step 1. Define $f(p)$ for $p \in \Delta$ as:

$$f(p) = \left\{q \in \bar{\Delta} \mid q \cdot z(p) \geq q' \cdot z(p) \ \ \forall q' \in \bar{\Delta}\right\}$$

That is, we maximize the value of excess demand for good that is most demanded. This means $q_i = 1$ for $z_i(p) = \max_i\{z_i(p)\}$ and $q_i = 0$ for other goods.

Step 2. Define $f(p)$ for $p \in \partial\Delta$ as follows:

$$f(p) = \left\{q \in \bar{\Delta} \mid q_i = 0 \text{ for } p_i > 0\right\}$$

We put zero prices for all those that were previously nonzero, and whatever else for zeros. Notice that since $p \neq 0$, there is no fixed point on the boundary:

$$\forall p \in \partial\Delta : p \cdot q = 0 \Longrightarrow p \in f(p) \iff p \cdot p = 0 \iff p = 0$$

which is a contradiction.

Step 3. Let $p^* \in f(p^*)$. By step 2, $p^* \gg 0$, i.e. $p^* \notin \partial\Delta$. Suppose $z(p^*) \neq 0$, then, since by Walras' Law $p^* \cdot z(p^*) = 0$, $\exists i_1 : z_{i_1}(p^*) > 0$ and $\exists i_2 : z_{i_2}(p^*) < 0$.

Therefore, by construction of $f(p)$ in step 1, $f(p^*) = \{q \in \bar{\Delta} \mid q_{i_2} = 0\}$ because of maximization of value of excess demand. Thus $f(p^*) \in \partial\Delta$, which contradicts $p^* \gg 0$, and hence $z(p^*) = 0$.

Step 4. Obviously $f(p)$ is nonempty. We argue that it is convex valued. Pick some $p \in \bar{\Delta}$, and some $q', q'' \in f(p)$, and also some $\lambda \in [0,1]$. If $p \in \Delta$, then:

$$z(p)\cdot[\lambda q' + (1-\lambda) q''] = \lambda z(p)\cdot q' + (1-\lambda) z(p)\cdot q'' \geq \lambda z(p)\cdot q + (1-\lambda) z(p)\cdot q = z(p)\cdot q$$

which follows from definition of $f(p)$ in step 1, and hence $f(p)$ is convex valued. If, on the other hand, $p \in \partial\Delta$, we fix some $i : p_i > 0$. Then $q_i' = q_i'' = 0$, and thus $\lambda q_i' + (1-\lambda) q_i'' = 0$, which establishes convex valuedness of $f(p)$.

We also want to show that $f(p)$ is upper hemicontinuous. Since it is compact valued, it suffices to establish that $f(p)$ has a closed graph. We want to show that $\forall \{p^n\}_{n=1}^{\infty} : p^n \to p$, and $\forall \{q^n\}_{n=1}^{\infty} : q^n \to q$, such that $\forall n : q^n \in f(p^n)$, it is true that $q \in f(p)$.

(1) Suppose $p \in \Delta$. Then $p^n \in \Delta$ for large n. Since $q^n \cdot z(p^n) \geq q' \cdot z(p^n)$ $\forall q' \in \bar{\Delta}$ and since $z(p)$ is continuous, we have that

$$q \cdot z(p) \geq q' \cdot z(p) \Longrightarrow q \in f(p)$$

(2) Let $p \in \partial\Delta$. Fix some $i : p_i > 0$. We want to show that $\exists \hat{N} : \forall n > \hat{N} :$ $q_i^n = 0$, and thus $q_i = 0$, so $q \in f(p)$. Since $p_i > 0$, $\exists \varepsilon > 0 : p_i^n > \varepsilon$ for large n.

- If $p^n \in \partial\Delta$, then $q_i^n = 0$ by construction of $f(p)$ in step 2, so $q_i^n = 0$ for large n, what we needed
- If $p^n \in \Delta$, then $\exists M : \forall n > M$ we have $z_i(p^n) < \max_{i=1,\ldots,L}\{z_i(p^n)\}$, so $q_i^n = 0$ by construction in step 1. This is true since $\max_{i=1,\ldots,L}\{z_i(p^n)\} \to \infty$, because $p^n \to p \in \partial\Delta$, but $z_i(p^n)$ is bounded:

$$p_i^n > \varepsilon \iff \frac{1}{\varepsilon} p_i^n > 1 \Longrightarrow z_i(p^n) \leq z_i(p^n) \frac{1}{\varepsilon} p_i^n$$

By Walras' Law:

$$z_i(p^n) \frac{1}{\varepsilon} p_i^n = -\frac{1}{\varepsilon} \sum_{j \neq i}^{L} p_j^n z_j(p^n)$$

Since $z_i(p) \geq -B$, $-z_i(p) \leq B$, so:

$$-\frac{1}{\varepsilon} \sum_{j \neq i}^{L} p_j^n z_j(p^n) \leq \frac{1}{\varepsilon} \sum_{j \neq i}^{L} p_j^n B \leq \frac{1}{\varepsilon} \sum_{j=1}^{L} p_j^n B = \frac{B}{\varepsilon}$$

Thus $z_i(p^n) \leq \frac{B}{\varepsilon}$, so it is bounded.

Therefore if $p^n \in \Delta$ we have that $z_i(p^n) < \max_{i=1,\ldots,L}\{z_i(p^n)\}$ for large n, thus $f(p^n) = \{q^n \mid q_i^n = 0\}$. So if $p \in \partial\Delta$, if $p_i > 0$ for some $i = 1, \ldots, L$, $q_i^n = 0$ for large n, so $q_i = 0$, and hence $q \in f(p)$. This establishes that the correspondence has closed graph (and hence, is uhc).

Step 5 Since $f(p)$ is nonempty, convex valued and uhc correspondence defined on a compact and convex and nonempty set, by Kakutani's Fixed Point Theorem, $\exists p^* \in \Delta : p^* \in f(p^*)$. By step 3, it is a CE, and thus the proof is complete. □

CHAPTER 6

Welfare Theorems

THEOREM 25. *(First Welfare Theorem). Let* $\succeq$ *on* X *be a complete, continuous preorder that is locally nonsatiated. Then, if* x *is a competitive equilibrium allocation, it is weakly Pareto optimal.*

PROOF. Suppose not. Then there exists $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_n) : \hat{x}_i \in \mathbb{R}^L_+$ such that $\sum_{i=1}^n \hat{x}_i = \sum_{i=1}^n e_i$ and $\hat{x}_i \succeq_i x_i \ \forall i \in I$ and $\exists j \in I : \hat{x}_j \succ_j x_j$. By construction of the competitive equilibrium allocation it must be the case that $p \cdot x_i \geq p \cdot e_i \ \forall i$ and $p \cdot x_j > p \cdot e_j$ for guy j. Then when one adds up the budget constraints across agents:

$$p \cdot \sum_{i=1}^n e_i = p \cdot \sum_{i=1}^n \hat{x}_i > p \cdot \sum_{i=1}^n x_i = p \cdot \sum_{i=1}^n e_i$$

But this is a contradiction, QED. □

THEOREM 26. *(Second Welfare Theorem). Let* $\succeq$ *on* X *be a complete, continuous preorder that is strictly monotone and convex. Let* $x = (x_1, \ldots, x_n) : x_i \in \mathbb{R}^L_{++}$ *be a weakly Pareto optimal allocation. Then* $\exists p \in \mathbb{R}^L : p \neq 0$*, such that* (p, x) *is a competitive equilibrium for* $e_i = x_i \ \forall i$*.*

PROOF. We know that $x \in \mathbb{R}^{Ln}_{++}$ is PO. Since $\succeq$ are continuous and convex, they are weakly convex and hence the sets $U_i(x_i) = \left\{y_i \in \mathbb{R}^L_+ \mid y_i \succ_i x_i\right\}$ are convex $\forall i$. Define

$$U = \sum_{i=1}^n U_i(x_i)$$

We know that $U \in \mathbb{R}^{Ln}_{++}$ and that U is a convex set. Denote $E = \sum_{i=1}^n x_i$, $E \in \mathbb{R}^L_{++}$. Clearly $E \notin U$ or else E is not a PO point. By the supporting hyperplane theorem, $\exists p \neq 0 : \forall z \in U$ we have $p \cdot z \geq p \cdot E$.

Step 1. We claim that $p \geq 0$. Denote by i_k the k-th unit ort in $\mathbb{R}^L_+$, and fix some $k \in 1, \ldots, L$. By strong monotonicity of preferences we know that $E + e_k \succ_i E \ \forall i$, and hence the point $E + e_k \in U$. Then by supporting hyperplane theorem:

$$p \cdot (E + e_k) \geq p \cdot E \iff p \cdot e_k \geq 0 \iff p_k \geq 0$$

But since k was arbitrary, it follows that $p \geq 0$.

Step 2. We claim that $\forall i$ if $\hat{x}_i > x_i$ then $p \cdot \hat{x}_i > p \cdot x_i$. Clearly if $\exists i : \hat{x}_i \succ_i x_i$, then $\sum_{i=1}^n \hat{x}_i \in U$ and thus by supporting hyperplane theorem $p \cdot \sum_{i=1}^n \hat{x}_i \geq p \cdot \sum_{i=1}^n x_i$. Let us denote by j the guy such that $\hat{x}_j \succ_j x_j$. Pick some $\varepsilon > 0$, and let $i = (1, \ldots, 1) \in \mathbb{R}^L_{++}$. Denote $\tilde{x}_j = \hat{x}_j - \varepsilon i$, clearly by continuity of $\succeq$ we have $\tilde{x}_j \succ x_j$ if ε is small enough. We also define $\tilde{x}_i = x_i + \frac{\varepsilon}{n-1} i$, $\forall i \neq j$. By strict monotonicity of $\succeq$ we have $\tilde{x}_i \succ x_i$, and hence $\tilde{x} \succ x$. Therefore $\sum_{i=1}^n \tilde{x}_i \in U(x)$, and thus

$$
\begin{aligned}
p \cdot \sum_{i=1}^{n} \tilde{x}_i &\geq p \cdot \sum_{i=1}^{n} x_i \\
p \cdot \left[\hat{x}_j - \varepsilon i + \sum_{i \neq j}^{n} x_i + \frac{\varepsilon}{n-1} i \sum_{i \neq j}^{n} \right] &\geq p \cdot \left[x_j + \sum_{i \neq j}^{n} x_i \right] \\
p \cdot \hat{x}_j &\geq p \cdot x_j
\end{aligned}
$$

But since j was arbitrary chosen, claim follows.

Now, since $\succeq$ are continuous, if $\hat{x}_i \succ_i x_i$, then $(1-\varepsilon)\,\hat{x}_i \succ_i x_i$ for ε small enough. By step 2, $p \cdot \hat{x}_i \geq p \cdot x_i \; \forall \hat{x}_i \succeq_i x_i$. This means that either $p \cdot \hat{x}_i = p \cdot x_i$, or $p \cdot \hat{x}_i > p \cdot x_i$. Suppose $p \cdot \hat{x}_i = p \cdot x_i$, then $p \cdot (1-\varepsilon)\,\hat{x}_i < p \cdot x_i$, but this contradicts step 2 and $(1-\varepsilon)\,\hat{x}_i \succ_i x_i$. Therefore, $p \cdot \hat{x}_i > p \cdot x_i$, which is precisely the definition of CE, and this completes the proof. □

CHAPTER 7

Cooperative Games

1. Concepts

Definition 40. *An allocation* $x \in \mathbb{R}_+^L$ *is individually rational* (IR_i) *if* $x_i \succeq_i e_i$.

Definition 41. *A coalition is a subset of a set of consumers. If* $\#\{I\} = n$, *then total possible number of coalitions is* $2^n - 1$.

Definition 42. *A feasible allocation* $x \in \mathbb{R}_+^{Ln}$ *is blocked by coalition* $S \subseteq I$, *if* $\exists y \in \mathbb{R}_+^{Ln}$ *such that* $\sum_{i \in S} y_i = \sum_{i \in S} e_i$ *(i.e.* y *is feasible for coalition* S*) and* $\forall i \in S : y_i \succ_i x_i$.

Definition 43. *A core of an economy* $Core(\varepsilon)$ *is a set of all feasible allocations that cannot be blocked by any coalition.*

Definition 44. *A replica economy is:*

$$\varepsilon^r = \{I = \{11, \ldots, 1r, \ldots, n1, \ldots nr\}, \{e_{11}, \ldots, e_{1r}, \ldots, e_{n1}, \ldots e_{nr}\}, \{\succeq_{11}, \ldots, \succeq_{1r}, \ldots, \succeq_{n1}, \ldots \succeq_{nr}\}\}$$

where $e_{ij} = e_{ik}$ *and* $\succeq_{ij} = \succeq_{ik}$, $\forall i \in [1, n], j, k \in [1, r]$. *Essentially we have* r *agents of each type.*

Theorem 27. *A competitive equilibrium allocation belongs to the core of the economy.*

Proof. Suppose not, i.e. x^* is a CE allocation which is not in the core. Then $\exists S \subseteq I$ – a blocking coalition that can improve upon x^*, i.e. $\exists \hat{x} \in \mathbb{R}_{++}^{ln}$ such that $\hat{x}$ is feasible for $S : \sum_{i \in S} \hat{x}_i = \sum_{i \in S} e_i$, and $\forall i \in S$ we have $\hat{x}_i \succ_i x_i^*$. Let p^* be a price vector associated with the CE allocation x^*, i.e. (p^*, x^*) constitute a CE. We must necessarily have $p^* \cdot \hat{x}_i > p^* \cdot e_i$ $\forall i \in S$, or else x_i^* would not be a CE allocation. But then:

$$p \cdot \sum_{i \in S} e_i = p \cdot \sum_{i \in S} \hat{x}_i > p \cdot \sum_{i \in S} x_i^* = p \cdot \sum_{i \in S} e_i$$

which is a contradiction. Thus CE allocation belongs to the core. QED. □

2. Equal Treatment Property and Debreu-Scarf Theorem

Theorem 28. *(Equal Treatment Property). Let* $\hat{x} \in Core(\varepsilon^r)$, *where* $\succeq_i$ *are strictly monotone, strictly convex, continuous and complete preorders. Then* $\hat{x}_{ik} = \hat{x}_{ij}$, *for* $\forall i \in [1, n], j, k \in [1, r]$.

Proof. Suppose not, i.e. $\hat{x} \in Core(\varepsilon^r)$, but $\hat{x}$ does not have the ETP. Then $\exists \hat{\imath}, j, k : \hat{x}_{\hat{\imath}k} \neq \hat{x}_{\hat{\imath}j}$, and without loss of generality we assume $\hat{\imath} = 1$. Then we argue that $\exists S \subseteq I$ that blocks $\hat{x}$. We can reorder consumers across replicas, so that consumers of type $i1$ would be the worst treated consumers of type i, i.e. $x_{ij} \succeq_i x_{i1}$ $\forall i \in [1, n], j \in [1, r]$.

Define $x_i = \frac{1}{r}\sum_{j=1}^{r}\hat{x}_{ij}$, the average consumption of each consumer type. By strict convexity of $\succeq$ we have $x_i \succeq_i \hat{x}_{i1}$ and $x_1 \succ_1 \hat{x}_{11}$, $\forall i = 1,\dots n$. Let $S = \{11, 21, \dots, n1\}$ be a coalition of n members. Note that $x = (x_{11}, \dots, x_{n1}) \in \mathbb{R}_+^{Ln}$ is feasible for S, because:

$$\sum_{i=1}^{n} x_i = \frac{1}{r}\sum_{i=1}^{n}\sum_{j=1}^{r}\hat{x}_{ij} = \sum_{i=1}^{n} e_i$$

Therefore, there is a feasible allocation that can be blocked by coalition S, because it makes one of its members strictly better off and does not make any other members worse off. To make every member of S better off, define $\bar{x}_{11} = x_{11} - \epsilon i_L$, and $\bar{x}_{i1} = x_{i1} + \frac{1}{n-1}\epsilon i_L$, where $\epsilon > 0$ is small enough, and i_L is an L-dimensional vector of ones. By continuity $\bar{x}$ is feasible for S, and by strict monotonicity $\bar{x}_i \succ_i x_i$ $\forall i \in S$. Hence every member of S is better off.

So the allocation $\hat{x}$ can be blocked by some coalition. But this contradicts $\hat{x}$ being a core allocation, so this is a contradiction. QED. □

THEOREM 29. *(Debreu-Scarf Theorem). Consider a sequence $\{\varepsilon^r\}_{r=1}^{\infty}$ of replica economies, where $e_i \gg 0$, $\forall i \in I$, $\succeq_i$ are monotone, strictly convex, continuous and complete preorders. Then $CE = \cap_{r=1}^{\infty} Core\,(\varepsilon^r)$, or, in other words, $\forall r_1, r_2 : r_1 > r_2$ we have $Core\,(\varepsilon^{r_1}) \supseteq Core\,(\varepsilon^{r_2})$.*

PROOF. We showed previously that CE allocations are in the core $\forall r \geq 1$. We also know that CE allocations are the same $\forall r$, according to the Equal Treatment Property. Thus we argue that:

$$CE \subseteq \cap_{r=1}^{\infty} Core\,(\varepsilon^r) \text{ and } \cap_{r=1}^{\infty} Core\,(\varepsilon^r) \supseteq CE$$

We organize the proof in several steps.

First, since CE allocations are in the core $\forall r \geq 1$, clearly $CE \subseteq \cap_{r=1}^{\infty} Core\,(\varepsilon^r)$, so the first inclusion is proved.

Second. denote $\hat{x} \in \mathbb{R}_+^{ln} : \hat{x} \in Core\,(\varepsilon^r)\ \forall r$. Here $\hat{x} = (\hat{x}_1, \dots, \hat{x}_n)$ where $\forall i : \hat{x}_i \in \mathbb{R}_+^l$. We argue that $\exists p \in \Delta$ such that $(p, \hat{x})$ is a CE for $\varepsilon^r, \forall r$. A CE is defined as:

$$\begin{aligned} \forall i &= 1, \dots, n : p \cdot \hat{x}_i = p \cdot e_i \\ \forall y_i &\in \mathbb{R}_+^l : y_i \succ_i \hat{x}_i \Longrightarrow p \cdot y_i > p \cdot \hat{x}_i \end{aligned}$$

i.e. a CE allocation is feasible, and any better point is not feasible. Define also:

$$U_i\,(\hat{x}_i) = \left\{ z_i \in \mathbb{R}_+^l \mid z_i + e_i \succ_i \hat{x}_i \right\}$$

CLAIM 6. *A convex hull of the union of these U_i sets contains no points from $\mathbb{R}_{--}^l$. That is, define:*

$$\bar{A} = CH\left[\bigcup_{i=1}^{n} U_i\,(\hat{x}_i)\right]$$

and we argue that $\bar{A} \cap \mathbb{R}_{--}^l = \emptyset$.

PROOF. Suppose not, i.e. $\exists \bar{z} \ll 0 : \bar{z} \in \bar{A}$. Then by definition of a convex hull $\exists \alpha_1, \dots, \alpha_n -$ scalars, such that $\alpha_i \geq 0$ and $\sum_{i=1}^{n}\alpha_i = 1$ and also $\exists \bar{z}_i \in U_i\,(\hat{x}_i)$ such that $\sum_{i=1}^{n}\alpha_i \bar{z}_i = \bar{z} \in \mathbb{R}_{--}^l$. Since $\bar{z} \ll 0$, there exists an open set D around it such that $\bar{z} \in D \subset \mathbb{R}_{--}^l$ (that is, since $\bar{z}$ is separated from 0, there is a neighborhood around it that is contained in $\mathbb{R}_{--}^l$). But then $\exists \tilde{z} \in \mathbb{R}_{--}^l$ and $\exists \tilde{\alpha}_1, \dots, \tilde{\alpha}_n -$ scalars, such that $\tilde{\alpha}_i \geq 0$ and $\sum_{i=1}^{n} \tilde{\alpha}_i = 1$ and also $\exists \tilde{z}_i \in U_i(\hat{x}_i)$ such that $\sum_{i=1}^{n} \tilde{\alpha}_i \tilde{z}_i = \tilde{z} \in \mathbb{R}^l_{--}$. (This basically says that the set D contains at least one more point.)

Now let us note that $\exists \bar{r}$ – an integer number, and also some integers $\beta_1, \ldots, \beta_n$ such that $\forall i : \tilde{\alpha}_i = \frac{\beta_i}{\bar{r}}$ (say, $\bar{r}$ be the LCM of $\tilde{\alpha}_i$'s). Consider $\bar{r}$-th replica of this economy and construct a coalition S such that it has exactly β_i agents of type i. We argue that S can block $\hat{x}$.

Consider $\tilde{z}_i + e_i \succ_i \hat{x}_i \ \forall i \in S$ by construction. It is feasible for S:

$$\sum_{i \in S} \beta_i (\tilde{z}_i + e_i) = \bar{r} \sum_{i \in S} \tilde{\alpha}_i (\tilde{z}_i + e_i) = \bar{r}\tilde{z} + \bar{r} \sum_{i \in S} \tilde{\alpha}_i e_i = \bar{r}\tilde{z} + \sum_{i \in S} \beta_i e_i \ll \sum_{i \in S} \beta_i e_i$$

We can use the usual continuity-monotonicity trick to show that everyone in S is better-off with this allocation (i.e. we take away a little from one agent and split across all others). But this says $\hat{x}$ is not a core allocation, which is a contradiction. Hence $\bar{A} \cap \mathbb{R}^l_{--} = \emptyset$. QED □

Because of that, we can use the supporting hyperplane theorem. Then $\exists p \neq 0 : p \cdot z_i \geq 0, \forall z_i \in U_i(\hat{x}_i) \ \forall i$.

Claim 7. *The above p is a CE price vector, i.e.:*

$$\begin{aligned} \forall i &= 1, \ldots, n : p \cdot \hat{x}_i = p \cdot e_i \\ \forall y_i &\in \mathbb{R}^l_+ : y_i \succ_i \hat{x}_i \Longrightarrow p \cdot y_i > p \cdot \hat{x}_i \end{aligned}$$

Proof. Pick $y_i : y_i \succ_i \hat{x}_i$. Then $(y_i - e_i) \in U_i(\hat{x}_i)$ by definition of U_i, and hence $p \cdot (y_i - e_i) \geq 0$ or just $p \cdot y_i \geq p \cdot e_i$. We first argue that $p \cdot \hat{x}_i = p \cdot e_i$.

Let $\varepsilon > 0$ be small enough. By monotonicity, $\hat{x}_i + \varepsilon \cdot i_L \succ_i \hat{x}_i$, where i_L is a L-dimensional vector of ones. By the supporting hyperplane theorem, $p \cdot (\hat{x}_i + \varepsilon \cdot i_L) \geq p \cdot e_i$. Now let $\varepsilon \to 0$ from above, then we get $p \cdot \hat{x}_i \geq p \cdot e_i \ \forall i$, but since $\sum_{i=1}^{n} \hat{x}_i = \sum_{i=1}^{n} e_i$, we get the equality sign here.

We now argue that $\forall y_i \in \mathbb{R}^l_+ : y_i \succ_i \hat{x}_i$ implies $p \cdot y_i > p \cdot \hat{x}_i$. Since by above argument $p \cdot \hat{x}_i = p \cdot e_i$, and since $e_i \gg 0$, we have $p \cdot e_i > 0$. Hence $\exists \mathring{x} \in \mathbb{R}^l_{++} : \mathring{x} \in B(p, e)$. If, for some $x'' \in B(p, e)$ we have both $x''_i \succ_i x_i$ and $p \cdot x''_i = p \cdot e_i$, then the points on the line segment $[\mathring{x}, x'']$ that are close enough to x'' are strictly preferred to x_i, and cost less. But this contradicts the feasiblity and hence x'' does not exist. So $\forall y_i \in \mathbb{R}^l_+ : y_i \succ_i \hat{x}_i \Longrightarrow p \cdot y_i > p \cdot \hat{x}_i$ established. QED □

Since $CE \subseteq \cap_{r=1}^{\infty} Core(\varepsilon^r)$ and $\cap_{r=1}^{\infty} Core(\varepsilon^r) \supseteq CE$ we conclude that $CE = \cap_{r=1}^{\infty} Core(\varepsilon^r)$. QED. □

CHAPTER 8

Nonconvexities

We use convexity of $\succeq$ in the proof of existence of CE – one requires convex-valued correspondence for Kakutani's FPT. We also use convexity in the proof of Second Welfare Theorem to apply a supporting hyperplane theorem.

For nonconvex $\succeq$, demand may fail to be single valued. Example: $u(x, y) = x^2 + y^2$. Then there could be no equilibrium.

But suppose there are many consumers with preferences as given above. Then if we can force them to consume at different points, we can approximate the equilibrium – i.e. get arbitrary close to supply. This is then called an approximate equilibrium (and approximate market clearing). This was an idea of Ross and Starr. In particular, Starr showed if there are many small consumers, one could approximate the equilibrium arbitrary well.

For firms, nonconvexities mean nonconvex production sets. This is usually true for big firms, which are then usually not price takers. This is troublesome.

If there is a continuum of nonconvex consumers, Aumann showed that one can integrate over them and get a convex aggregate excess demand, and hence apply existence theorems. But this is technically challenging.

CHAPTER 9

Generic Approach

There is a somewhat negative result that we start with.

THEOREM 30. *(Sonnenschein Conjecture). Let $z : \Delta \to \mathbb{R}^L$ be a continuous function that satisfies the assumptions of the Easy Existence Theorem (namely, Walras' Law, boundedness from below, boundary condition). Then z is an aggregate excess demand function for some economy with L consumers with monotone, continuous complete strictly convex preorders $\succeq_i$ on $\mathbb{R}^L_+$, endowments $e_i \in \mathbb{R}^L_{++}$, and z is defined on $\Delta(\varepsilon) = \{p \in \Delta \mid p_j \geq \varepsilon > 0\}$.*

This says that one cannot justify any additional assumptions on the economy – everything else is actually possible. Obviously this result is not very satisfactory – therefore we would like to see how "likely" are all the bad cases to happen. For this we have to define a bunch of additional concepts.

DEFINITION 45. *A smooth economy is an economy where $z(p)$ is a smooth function, i.e. continuously differentiable everywhere.*

THEOREM 31. *Suppose that preferences $\succeq_i$ in the economy are such that they can be represented by utility function $u : \mathbb{R}^L_{++} \to \mathbb{R}$ such that $u(\cdot)$ is strictly differentiably concave (i.e. $D^2 u \leq 0$), strictly differentiably monotone (i.e. $Du \gg 0$) and satisfies the boundary condition:*

$$\forall \bar{x} \in \mathbb{R}^L_{++} : d_{\mathbb{R}^L} \cap \partial \mathbb{R}^L_{++} = \emptyset$$

where $d_{\mathbb{R}^L} = \left\{ y \in \mathbb{R}^L_{++} \mid u(y) \geq u(\bar{x}) \right\}$. If so, then the aggregate excess demand function in this economy is smooth.

DEFINITION 46. *A manifold is some smooth surface. Locally it is equivalent to Euclidean space. For instance, an empty set $\emptyset$ is a manifold of* $\dim = -1$*, a point x in some Euclidean space is a manifold of* $\dim = 0$*, and so on.*

DEFINITION 47. *Let $f : \mathbb{R}^N \to \mathbb{R}^M$ be such that $f \in C^2$, i.e. is smooth. A **critical point** of f is any point $\hat{x} \in \mathbb{R}^N$ such that $Df(\hat{x})$ is not of full rank m.*

DEFINITION 48. *Let $f : \mathbb{R}^N \to \mathbb{R}^M$ be such that $f \in C^2$, i.e. is smooth. A **regular point** of f is any point $\bar{x} \in \mathbb{R}^N$ such that $Df(\bar{x})$ has full rank m.*

DEFINITION 49. *Let $f : \mathbb{R}^N \to \mathbb{R}^M$ be such that $f \in C^2$, i.e. is smooth. A **critical value** of f is an image of any critical point $\hat{x}$.*

DEFINITION 50. *Let $f : \mathbb{R}^N \to \mathbb{R}^M$ be such that $f \in C^2$, i.e. is smooth. A **regular value** of f is any point that is not a critical point (i.e. an image of a point at which function is not defined is still a regular value).*

The important result here is the Regular Value Theorem.

THEOREM 32. *(Regular Value Theorem). The inverse image of a regular value is a smooth manifold of* $\dim = N - M$*, or empty.*

THEOREM 33. *(Sard's Theorem). Let* $F : U \to \mathbb{R}^M$ *be* C^1*, where* U *is an open subset of* $\mathbb{R}^N$*. Define:*

$$J \equiv \{x \in U \mid \det [DF (x)] = 0\}$$

Then $F (J)$ *has measure zero in* $\mathbb{R}^N$*.*

For the picture to be complete, we provide two more definitions of related concepts.

DEFINITION 51. *A set* S *has measure zero if* $\forall\varepsilon > 0$ *there exists a sequence of open balls in* $\mathbb{R}^N$ *such that* $\cup_{i=1}^{\infty} V_i \supseteq F (S)$ *and* $\sum_{i=1}^{\infty} vol (V_i) < \varepsilon$*.*

THEOREM 34. *Let* $F : \mathbb{R}^N \to \mathbb{R}^M$ *be in* C^1 *and* $\det [F (\bar{x})] \neq 0$*, then* $F (\cdot)$ *has local well-defined inverse at* $F (\bar{x})$*, and this inverse is a* C^1 *manifold.*

Based on all this math above, we now employ a generic approach to the analysis of the economy - we look at a typical case in a set of full measure. For now we fix $\{\succeq_i\}_{i=1}^n$ and let endowments $\{e_i\}_{i=1}^n$ vary. Every set of endowments define a new economy. We introduce some new notation:

(1) Let $\hat{z} (p, e_1, \ldots, e_n)$ be the aggregate excess demand function with the last commodity dropped. (This is done because of Walras' Law, to avoid rank issues).
(2) Let $\hat{z}_e (p)$ be $\hat{z} (p, e_1, \ldots, e_n)$ with a fixed set of endowments.
(3) Finally, let $\hat{z}_e^{-1} (0) = \{p \in \Delta \mid z (p, e_1, \ldots, e_n) = 0\}$.

This all means that 0 is a regular value of $\hat{z}_e (p)$. Then, by the regular value theorem, inverse image of the zero point is a smooth manifold of $\dim = l - 1 + ln - (l - 1) = ln$.

Define now $\pi : \hat{z}_e^{-1} (0) \to \mathbb{R}_{++}^{ln}$ – a projection of $\hat{z}_e^{-1} (0)$ on the space of endowments (this basically ignores the prices). Clearly π is also a smooth manifold, and by Sard's theorem, $\pi (C)$ is a set of measure zero, where C is a set of all critical equilibria. Then:

- $\pi (C)$ – a set of all critical economies;
- $\mathbb{R}_{++}^{ln} \backslash \pi (C)$ – a set of all regular economies.

The bottom line here is that all the problems with multiple equilibria only arise in critical economies, and the set of critical economies has measure zero, i.e. most of the economies are regular.

Conclusions:

(1) We can define all economies into regular and critical economies.
(2) Regular economies are generic – i.e. the set of all critical economies has measure zero.
(3) Regular economies have very nice properties:
    - all equilibria are locally isolated within some neighborhood
    - there is a finite and odd number of equilibria, they are in some compact subset of Δ,
    - the number of equilibria is locally constant (i.e. if one wiggles the endowments a bit, the number of equilibria doesn't change)
    - all equilibria smoothly depend on the parameters of the model

Part 4

Game Theory

CHAPTER 10

Basic Definitions and Existence

DEFINITION 52. *A normal form game (NFG) is the following object:*

(1) *A set of players* $I = 1, 2, \ldots, n$*, i.e.* $\#I = n$*.*
(2) *A set of actions for each player:* $\left(A^i\right)_{i\in I} : A_i = \left\{a_1^i, \ldots a_{n^i}^i\right\}$*, with a possibly different number of actions for each player.*
(3) *A collection of payoff functions* $\left(u^i\right)_{i\in I} : u^i = u^i\left(a^i, a^{-i}\right)$*. This can be treated as the von Neumann-Morgenstern utility function over actions of all players.*

In a concise way, $G = \left\{I, \left(A^i\right)_{i\in I}, \left(u^i\left(a^i, a^{-i}\right)\right)_{i\in I}\right\}$*. We also define* $A = \underset{i\in I}{\times} A^i$

DEFINITION 53. *A mixed extension of the normal form game* G *is* $\left\{I, \left(S^i\right)_{i\in I}, \left(u^i\left(s^i, s^{-i}\right)\right)_{i\in I}\right\}$*, where* $S^i = \Delta\left(A^i\right)$*, or, more formally:*

$$S^i = \Delta\left(A^i\right) = \left\{s^i \in \mathbb{R}^{\#A^i} \mid \forall a^i \in A^i : s^i\left(a^i\right) \geq 0, \sum_{a^i\in A^i} s^i\left(a^i\right) = 1\right\}$$

Observe a slight abuse of notation, since we use the same $u(\cdot)$ *for NFG and its mixed extension. It usually does not cause any confusion, though. Similarly* $S = \underset{i\in I}{\times} S^i$

DEFINITION 54. *A best response correspondence is* $\forall i \in I$*,* $\forall s \in S$*:*

$$BR^i(s) = \left\{t^i \in S^i \mid u^i\left(t^i, s^{-i}\right) = \max_{r^i\in S^i} u^i\left(r^i, s^{-i}\right)\right\}$$

The following theorem is due to Nash. It establishes the existence of the Nash Equilibrium by showing that evey player's Best Response satisfies the Kakutani's FPT assumptions. A NE is then defined as a fixed point of the Cartesian product of all Best Responses.

THEOREM 35. $\forall i \in I$*,* $\forall s \in S$ $BR^i(s)$ *is nonempty valued, closed and convex-valued and u.h.c.*

PROOF. We establish each claim.

For nonemptyness, fix s^{-i}. Then $u\left(\cdot, s^{-i}\right) : S^i \to \mathbb{R}$ is defined as $\sum_{a^i\in A^i} t^i\left(a^i\right) \cdot u^i\left(a^i, s^{-i}\right)$ which is a linear function, and thus is continuous. Since S^i is a compact set by definition, $u\left(\cdot, s^{-i}\right)$ attains a maximum, and necessarily has a maximizer, so $BR^i(s)$ is nonempty.

For closed valuedness, let $t_n^i \in BR^i(s) : t_n^i \to t^i$ as $n \to \infty$. By definition, $u^i\left(t^i, s^{-i}\right) = \max_{r^i\in S^i} u^i\left(r^i, s^{-i}\right)$, since $u(\cdot)$ is continous and $t_n^i \to t^i$, we necessarily have $t^i \in BR^i(s)$, so $BR^i(s)$ is closed valued.

For convex-valuedness, let $t^i, \tilde{t}^i \in BR^i(s)$, i.e.:

$$u^i\left(t^i, s^{-i}\right) = u^i\left(\tilde{t}^i, s^{-i}\right) = \max_{r^i \in S^i} u^i\left(r^i, s^{-i}\right)$$

Pick some $\lambda \in [0,1]$ and consider $u^i\left(\lambda t^i + (1-\lambda)\tilde{t}^i, s^{-i}\right)$. By definition of von Neumann-Morgenstern utility:

$$u^i\left(\lambda t^i + (1-\lambda)\tilde{t}^i, s^{-i}\right) = \lambda u^i\left(t^i, s^{-i}\right) + (1-\lambda) u^i\left(\tilde{t}^i, s^{-i}\right) = \max_{r^i \in S^i} u^i\left(r^i, s^{-i}\right)$$

Hence $BR^i(s)$ is convex valued.

To show $BR^i(s)$ is uhc, since it is closed valued, and since boundedness of values is obvious – i.e. $BR^i(s)$ is compact-valued, it suffices to show that $BR^i(s)$ has a closed graph. That is, we want to show that if $\left(s_n^i, s_n^{-i}\right) \to \left(s^i, s^{-i}\right)$ and $\forall n : s_n^i \in BR^i(s_n)$ then $s^i \in BR^i(s)$.

Suppose not, i.e. $\exists \left\{\left(s_n^i, s_n^{-i}\right)\right\}_{n=1}^{\infty}$ such that $\left(s_n^i, s_n^{-i}\right) \to \left(s^i, s^{-i}\right)$ and $\forall n : s_n^i \in BR^i(s_n)$ but $s^i \notin BR^i(s)$. Then $\exists \varepsilon > 0$ and $\tilde{s}^i : u^i\left(\tilde{s}^i, s^{-i}\right) > u^i\left(s^i, s^{-i}\right) + 3\varepsilon$ (because s^i is not in the best response, there are better options available, and we make them at least 3ε better). We want to get a contradiction that $\tilde{s}^i \in BR^i(s_n)$ for some n.

By continuity of $u^i(\cdot)$ and since $\left(\tilde{s}_n^i, s_n^{-i}\right) \to \left(s^i, s^{-i}\right)$ for n large enough we have:

$$\left|u^i\left(\tilde{s}^i, s_n^{-i}\right) - u^i\left(\tilde{s}^i, s^{-i}\right)\right| < \varepsilon$$

Rewrite as:

$$-\varepsilon < u^i\left(\tilde{s}_n^i, s_n^{-i}\right) - u^i\left(\tilde{s}^i, s^{-i}\right) < \varepsilon$$

And this, in turn, as:

$$u^i\left(\tilde{s}^i, s_n^{-i}\right) > u^i\left(\tilde{s}^i, s^{-i}\right) - \varepsilon$$

Now, from the above we know that:

$$\begin{aligned} u^i\left(\tilde{s}^i, s^{-i}\right) &> u^i\left(s^i, s^{-i}\right) + 3\varepsilon \\ u^i\left(\tilde{s}^i, s^{-i}\right) - \varepsilon &> u^i\left(s^i, s^{-i}\right) + 2\varepsilon \end{aligned}$$

Finally, since $\left(s_n^i, s_n^{-i}\right) \to \left(s^i, s^{-i}\right)$, we know that for n large enough, we have:

$$\left|u^i\left(s^i, s_n^{-i}\right) - u^i\left(s^i, s^{-i}\right)\right| < \varepsilon$$

This can be rewritten as:

$$-\varepsilon < u^i\left(s^i, s_n^{-i}\right) - u^i\left(s^i, s^{-i}\right) < \varepsilon$$

And this – as:

$$u^i\left(s^i, s^{-i}\right) > u^i\left(s^i, s_n^{-i}\right) - \varepsilon$$

Now we combine everything together to get:

$$u^i\left(\tilde{s}^i, s_n^{-i}\right) > u^i\left(\tilde{s}^i, s^{-i}\right) - \varepsilon > u^i\left(s^i, s^{-i}\right) + 2\varepsilon > u^i\left(s^i, s_n^{-i}\right) + \varepsilon$$

But this means $\tilde{s}^i \in BR^i(s_n)$, which is a contradiction. Hence closed graph is established. QED. □

Corollary 4. *A Cartesian product* $BR(s) = \times_{i \in I} BR^i(s)$ *has the same properties. Hence by Kakutani's FPT it has a fixed point. This proves the existence of NE in a NFG, possibly in mixed strategies.*

CHAPTER 11

Minimax and Zero-Sum Games

A zero sum game is a special case of a constant sum game. A constant sum game is a special case of a strictly competitive game.

Definition 55. *A NFG* $G = \left\{I, \left(S^i\right)_{i\in I}, \left(u^i\left(s^i, s^{-i}\right)\right)_{i\in I}\right\}$*, where* $\#I = 2$ *is strictly competitive in mixed strategies iff:*

$$\forall s_1, s_2 \in S : u^i\left(s_1^i, s_1^{-i}\right) > u^i\left(s_2^i, s_2^{-i}\right) \iff u^{-i}\left(s_1^i, s_1^{-i}\right) < u^{-i}\left(s_2^i, s_2^{-i}\right)$$

Assume a zero sum game $G = \left(I, \left\{S^i\right\}_{i\in I}, \left\{u^i\left(s^i, s^{-i}\right)\right\}_{i\in I}\right)$ is given and $\#I = 2$.

Define:

$$\begin{aligned}\overline{v^i} &= \min_{s^j}\max_{s^i} u^i\left(s^i, s^j\right)\\ \underline{v^i} &= \max_{s^i}\min_{s^j} u^i\left(s^i, s^j\right)\end{aligned}$$

Theorem 36. *It is always true that* $\underline{v^i} \leq \overline{v^i}$.

Proof. Consider the following claim:

$$\min_{s^j} u^i\left(s^i, s^j\right) \leq u^i\left(s^i, s^j\right)$$

Obviously:

$$\min_{s^j} u^i\left(s^i, s^j\right) \leq \max_{s^i} u^i\left(s^i, s^j\right)$$

Next, we preserve the inequality by:

$$\max_{s^i}\min_{s^j} u^i\left(s^i, s^j\right) \leq \max_{s^i} u^i\left(s^i, s^j\right)$$

And finally the last step:

$$\begin{aligned}\max_{s^i}\min_{s^j} u^i\left(s^i, s^j\right) &\leq \min_{s^j}\max_{s^i} u^i\left(s^i, s^j\right)\\ \underline{v^i} &\leq \overline{v^i}\end{aligned}$$

Done. □

Definition 56. *Let G be a strictly competitive game in mixed strategies. Let* $\left(\hat{s}^i, \hat{s}^j\right)$ *be a Nash Equilibrium strategy profile for G. Then* $\left(\hat{s}^i, \hat{s}^j\right)$ *is a saddle point, i.e.* $\forall s^i \in S^i$ *and* $\forall s^j \in S^j$*:*

$$u^i\left(\hat{s}^i, s^j\right) \geq u^i\left(\hat{s}^i, \hat{s}^j\right) \geq u^i\left(s^i, \hat{s}^j\right)$$

Theorem 37. *Let G be a strictly competitive game in mixed strategies. Let* $\left(\hat{s}^i, \hat{s}^j\right)$ *be a Nash Equilibrium strategy profile for G. Then* $\underline{v^i} = u^i\left(\hat{s}^i, \hat{s}^j\right) = \overline{v^i}$.

Proof. We know that $\underline{v^i} \leq \overline{v^i}$, so we want to show $\underline{v^i} \geq \overline{v^i}$ to prove the claim. Consider:

$$\underline{v^i} = \max_{s^i} \min_{s^j} u^i\left(s^i, s^j\right) \geq \min_{s^j} u^i\left(s^i, s^j\right) \ \ \forall s^i$$

We let $s^i = \hat{s}^i$ and then:

$$\min_{s^j} u^i\left(\hat{s}^i, s^j\right) = u^i\left(\hat{s}^i, \arg\min_{r^j}\left[u^i\left(\hat{s}^i, r^j\right)\right]\right)$$

But since the game is strictly competitive:

$$\arg\min_{r^j}\left[u^i\left(\hat{s}^i, r^j\right)\right] = \arg\max_{r^j}\left[u^j\left(\hat{s}^i, r^j\right)\right] = \hat{s}^j$$

Therefore:

$$\underline{v^i} \geq \min_{s^j} u^i\left(\hat{s}^i, s^j\right) = u^i\left(\hat{s}^i, \arg\min_{r^j}\left[u^i\left(\hat{s}^i, r^j\right)\right]\right) = u^i\left(\hat{s}^i, \hat{s}^j\right)$$

So we're half way done. Next:

$$\overline{v^i} = \min_{s^j} \max_{s^i} u^i\left(s^i, s^j\right) \leq \max_{s^i} u^i\left(s^i, s^j\right) \ \ \forall s^j$$

Put $s^j = \hat{s}^j$, then by definition:

$$\max_{s^i} u^i\left(s^i, \hat{s}^j\right) = u^i\left(\hat{s}^i, \hat{s}^j\right)$$

So we have:

$$\underline{v^i} \geq u^i\left(\hat{s}^i, \hat{s}^j\right) \geq \overline{v^i}$$

And by minimax theorem, we have $\underline{v^i} \leq \overline{v^i}$, all these together imply $\underline{v^i} = u^i\left(\hat{s}^i, \hat{s}^j\right) = \overline{v^i}$. QED. □

CHAPTER 12

Perfect Equilibrium

Definition 57. *A perturbation of the game is* $\varepsilon = \left\{\varepsilon^i\right\}_{i=1}^n$*, so that* $\varepsilon^i = \left\{\varepsilon^i\left(a^i\right)\right\}_{a^i \in A^i}$.

Definition 58. *A perturbed mixed strategy, given a perturbation* ε*, is*

$$S_\varepsilon^i = \left\{s^i \in S^i \mid \forall a^i \in A^i : s^i\left(a^i\right) \geq \varepsilon^i\left(a^i\right)\right\}$$

Notice that $\sum_{a^i \in A^i} \varepsilon^i\left(a^i\right) < 1$, so $\nexists s^i\left(a^i\right) = 0$. Now denote the restricted mixed strategies space as:

$$S_\varepsilon = \underset{i=1}{\overset{n}{\times}} S_\varepsilon^i$$

Since S^i is compact and convex, S_ε^i is also such, and hence so is S_ε. Let us call an ε-game G_ε the following object:

$$G_\varepsilon = \left(I, \left\{S_\varepsilon^i\right\}_{i=1}^n, \left\{u^i\left(s^i, s^{-i}\right)\right\}_{i=1}^n\right)$$

Definition 59. *A Nash Equilibrium of an* ε*-game is a strategy profile* $\hat{s}_\varepsilon \in S_\varepsilon$ *such that* $\forall i \in I : \hat{s}_\varepsilon \in BR^i\left(\hat{s}_\varepsilon\right)$*, where:*

$$BR^i\left(\hat{s}_\varepsilon\right) = \left\{r^i \in S_\varepsilon^i \mid \forall t^i \in S_\varepsilon^i : u^i\left(r^i, \hat{s}_\varepsilon^{-i}\right) \geq u^i\left(t^i, \hat{s}_\varepsilon^{-i}\right)\right\} \tag{0.1}$$

Theorem 38. *Given a perturbation* ε*, a constrained Best Response correspondence defined in (0.1) is nonempty valued, convex valued and compact valued and upper hemicontinuous. Also the set* S_ε^i *is convex, nonempty and compact.*

Proof. Consider the set first. It is convex because $\forall \alpha, \beta \in [\varepsilon, 1]$ we have $\forall \lambda \in [0,1]$ that $\lambda\alpha + (1-\lambda)\beta \in [\varepsilon, 1]$, so the set is convex. Also it is trivially compact since the limit points are contained in the set. Finally, point $(1, 0, \ldots, 0)$, is in the set so it is nonempty.

Now consider the correspondence. Since $u^i(\cdot)$ is linear and is defined on a nonempty compact, it has a maximizer by the Weierstrass' theorem, so it's nonempty valued. Next, consider $s_\varepsilon^i(1), s_\varepsilon^i(2) \in BR^i\left(\hat{s}_\varepsilon\right)$ and pick some $\lambda \in [0,1]$. Obviously:

$$u^i\left[\lambda s_\varepsilon^i(1) + (1-\lambda) s_\varepsilon^i(2), \hat{s}_\varepsilon^{-i}\right] = \lambda u^i\left[s_\varepsilon^i(1), \hat{s}_\varepsilon^{-i}\right] + (1-\lambda)^i\left[s_\varepsilon^i(2), \hat{s}_\varepsilon^{-i}\right]$$

Since $s_\varepsilon^i(j) \in BR^i\left(\hat{s}_\varepsilon\right)$ means that $\forall t^i \in S_\varepsilon^i$ we have $u^i\left(s_\varepsilon^i(j), \hat{s}_\varepsilon^{-i}\right) \geq u^i\left(t^i, \hat{s}_\varepsilon^{-i}\right)$ and $j = 1, 2$, then

$$\lambda u^i\left[s_\varepsilon^i(1), \hat{s}_\varepsilon^{-i}\right] + (1-\lambda)^i\left[s_\varepsilon^i(2), \hat{s}_\varepsilon^{-i}\right] = u^i\left[s_\varepsilon^i(j), \hat{s}_\varepsilon^{-i}\right]$$

where j is either 1 or 2, so convex-valuedness established.

Then, take a sequence $\left\{s_\varepsilon^i(n)\right\}_{n=1}^\infty$ such that $s_\varepsilon^i(n) \to s_\varepsilon^i$. By continuity of $u^i(\cdot)$, we have $\forall t^i \in S_\varepsilon^i$:

$$u^i\left[s_\varepsilon^i(n), \hat{s}_\varepsilon^{-i}\right] \geq u^i\left(t^i, \hat{s}_\varepsilon^{-i}\right) \to u^i\left[s_\varepsilon^i, \hat{s}_\varepsilon^{-i}\right] \geq u^i\left(t^i, \hat{s}_\varepsilon^{-i}\right)$$

and so closedness established, and boundedness is obvious, so by Heine-Borel theorem, the correspondence is compact valued. Finally we want to show the closed graph property, i.e. $\forall\left(s_{\varepsilon}^{i}(n), \hat{s}_{\varepsilon}^{-i}(n)\right)$ such that $\left(s_{\varepsilon}^{i}(n), \hat{s}_{\varepsilon}^{-i}(n)\right) \rightarrow\left(s_{\varepsilon}^{i}, \hat{s}_{\varepsilon}^{-i}\right)$, and $s_{\varepsilon}^{i}(n) \in BR^{i}\left(s_{\varepsilon}(n)\right) \forall n$, we have $s_{\varepsilon}^{i} \in BR^{i}\left(s_{\varepsilon}\right)$. Suppose not, then $\exists \tilde{s}_{\varepsilon}^{i} \in S_{\varepsilon}^{i}$ such that:

$$u^{i}\left(\tilde{s}_{\varepsilon}^{i}, s_{\varepsilon}^{-i}\right)>u^{i}\left(s_{\varepsilon}^{i}, s_{\varepsilon}^{-i}\right)+3 \varepsilon \tag{0.2}$$

where $\varepsilon>0$ is small enough. We know that since $s_{\varepsilon}^{-i}(n) \rightarrow s_{\varepsilon}^{-i}$:

$$\begin{aligned}
\left|u^{i}\left(\tilde{s}_{\varepsilon}^{i}, s_{\varepsilon}^{-i}(n)\right)-u^{i}\left(\tilde{s}_{\varepsilon}^{i}, s_{\varepsilon}^{-i}\right)\right| &< \varepsilon \\
u^{i}\left(\tilde{s}_{\varepsilon}^{i}, s_{\varepsilon}^{-i}(n)\right)-u^{i}\left(\tilde{s}_{\varepsilon}^{i}, s_{\varepsilon}^{-i}\right) &> -\varepsilon \\
u^{i}\left(\tilde{s}_{\varepsilon}^{i}, s_{\varepsilon}^{-i}(n)\right) &> u^{i}\left(\tilde{s}_{\varepsilon}^{i}, s_{\varepsilon}^{-i}\right)-\varepsilon
\end{aligned}$$

Combine it with (0.2) to have:

$$u^{i}\left(\tilde{s}_{\varepsilon}^{i}, s_{\varepsilon}^{-i}(n)\right)>u^{i}\left(\tilde{s}_{\varepsilon}^{i}, s_{\varepsilon}^{-i}\right)-\varepsilon>u^{i}\left(s_{\varepsilon}^{i}, s_{\varepsilon}^{-i}\right)+2 \varepsilon \tag{0.3}$$

Finally, we know that since $s_{\varepsilon}^{-i}(n) \rightarrow s_{\varepsilon}^{-i}$:

$$\begin{aligned}
\left|u^{i}\left(s_{\varepsilon}^{i}, s_{\varepsilon}^{-i}(n)\right)-u^{i}\left(s_{\varepsilon}^{i}, s_{\varepsilon}^{-i}\right)\right| &< \varepsilon \\
u^{i}\left(s_{\varepsilon}^{i}, s_{\varepsilon}^{-i}(n)\right)-u^{i}\left(s_{\varepsilon}^{i}, s_{\varepsilon}^{-i}\right) &< \varepsilon \\
u^{i}\left(s_{\varepsilon}^{i}, s_{\varepsilon}^{-i}\right) &> u^{i}\left(s_{\varepsilon}^{i}, s_{\varepsilon}^{-i}(n)\right)-\varepsilon
\end{aligned}$$

Once you add 2ε to both sides of the last equation and combine it with (0.3) you get:

$$u^{i}\left(\tilde{s}_{\varepsilon}^{i}, s_{\varepsilon}^{-i}(n)\right)>u^{i}\left(\tilde{s}_{\varepsilon}^{i}, s_{\varepsilon}^{-i}\right)-\varepsilon>u^{i}\left(s_{\varepsilon}^{i}, s_{\varepsilon}^{-i}\right)+2 \varepsilon>u^{i}\left(s_{\varepsilon}^{i}, s_{\varepsilon}^{-i}(n)\right)+\varepsilon$$

Therefore:

$$u^{i}\left(\tilde{s}_{\varepsilon}^{i}, s_{\varepsilon}^{-i}(n)\right)>u^{i}\left(s_{\varepsilon}^{i}, s_{\varepsilon}^{-i}(n)\right)$$

But this says $\tilde{s}_{\varepsilon}^{i} \in BR^{i}\left(s_{\varepsilon}(n)\right)$, and this together with (0.2) contradicts $s_{\varepsilon}^{i} \in BR^{i}\left(s_{\varepsilon}(n)\right)$, so $\tilde{s}_{\varepsilon}^{i}$ actually does not exist, and we're done. □

COROLLARY 5. *By the theorem above, for a given perturbation ε, a NE in a ε-perturbed game always exists.*

DEFINITION 60. *A strategy profile $\hat{s}$ is a perfect equilibrium if $\exists\left\{\varepsilon_{n}\right\}_{n=1}^{\infty}$ such that $\lim_{n \rightarrow \infty} \varepsilon_{n}=0$ and $\exists s\left(\varepsilon_{n}\right) \in NE\left[G\left(\varepsilon_{n}\right)\right]$ such that $\lim_{n \rightarrow \infty} s\left(\varepsilon_{n}\right)=\hat{s}$.*

THEOREM 39. *A set of perfect equilibria of a NFG is nonempty.*

PROOF. Take a sequence $\left\{\varepsilon_{n}\right\}_{n=1}^{\infty}: \varepsilon_{n} \rightarrow 0$. Then $s\left(\varepsilon_{n}\right) \rightarrow \hat{s}$. Clearly $s\left(\varepsilon_{n}\right) \in NE\left[G\left(\varepsilon_{n}\right)\right]$, so $\hat{s} \in NE[G(0)]$. Since $\left\{s\left(\varepsilon_{n}\right)\right\}$ is a sequence in a compact space $S_{\varepsilon_{n}}$, it has a convergent subsequence. □

THEOREM 40. *A perfect equilibrium is a Nash Equilibium.*

PROOF. Suppose not. Let $\hat{s}$ be a perfect equilibrium strategy profile. Then $\forall i \in I$ and $\forall t^{i} \in S^{i}$:

$$u^{i}\left(\hat{s}^{i}, \hat{s}^{-i}\right) \geq u^{i}\left(t^{i}, \hat{s}^{-i}\right)$$

Clearly$\forall i \in I$ and $\forall\left\{t_{\varepsilon_{n}}^{i}\right\}_{k=1}^{\infty}$ such that $t_{\varepsilon_{n}}^{i} \rightarrow t^{i}$

$$u^{i}\left(\hat{s}_{\varepsilon_{n}}^{i}, \hat{s}_{\varepsilon_{n}}^{-i}\right) \geq u^{i}\left(t_{\varepsilon_{n}}^{i}, \hat{s}_{\varepsilon_{n}}^{-i}\right)$$

So, by continuity of $u^{i}(\cdot)$, we have the first inequality. □

DEFINITION 61. *Let ε be any perturbation. A strategy profile is called ε-perfect if $\forall i \in I$ and $\forall b^i \in A^i$ there is a strategy profile s^i such that $\forall a^i \in A^i$ such that if $u^i\left(b^i, s^{-i}\right) < u^i\left(a^i, s^{-i}\right)$, we have $s^i\left(b^i\right) < \varepsilon$.*

DEFINITION 62. *Let ε be any perturbation. A strategy profile is called ε-proper if $\forall i \in I$ and $\forall b^i \in A^i$ there is a strategy profile s^i such that $\forall a^i \in A^i$ such that if $u^i\left(b^i, s^{-i}\right) < u^i\left(a^i, s^{-i}\right)$, we have $s^i\left(b^i\right) < \varepsilon s^i\left(a^i\right)$.*

DEFINITION 63. *A strategy profile $\hat{s}$ is called a proper equilibrium is $\exists s\left(\varepsilon_n\right) : \varepsilon_n \to 0$, $s\left(\varepsilon_n\right)$ is a sequence of ε-proper equilibrium profiles and $s\left(\varepsilon_n\right) \to \hat{s}$.*

THEOREM 41. *The three following conditions are equilivalent:*

(1) (a) $\hat{s}$ is a perfect equilibrium strategy profile;

DEFINITION 64.

THEOREM 42. (1) *There exists a sequence $s\left(\varepsilon_n\right)$ of ε_n-perfect equilibria and $s\left(\varepsilon_n\right) \to \hat{s}$;*
(2) *$\hat{s}$ is the limit of sequence $t_n \in S$ such that $\forall n \geq 1$, t_n are fully mixed and $\forall n, \forall i \in I$ we have $\hat{s}^i \in BR_U^i\left(t_n\right)$, where BR_U indicates unconstrained best response.*

PROOF. We prove (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (1) stepwise. First, (1) $\Longrightarrow$ (2):
Take $s\left(\eta_n\right)$ – a sequence of NE such that $s\left(\eta_n\right) \to \hat{s}$ as $\eta_n \to 0$. Define:

$$\varepsilon_n \equiv \max_{i \in I} \max_{a^i \in A^i} \eta_n^i\left(a^i\right)$$

Clearly, $\forall n > 0$ we have $\varepsilon_n > 0$ and $\lim_{n\to\infty} \varepsilon_n = 0$.

Now, put $s\left(\varepsilon_n\right) = s\left(\eta_n\right)$, then we have a sequence $\{s\left(\varepsilon_n\right)\} : s\left(\varepsilon_n\right) \to \hat{s}$. We argue that $s\left(\varepsilon_n\right)$ is ε_n-perfect. By definition, $\forall i \in I$ and $\forall a^i \in A^i$ we have $s\left(\varepsilon_n\right)^i\left(a^i\right) \geq \varepsilon_n > 0$, so $s\left(\varepsilon_n\right)$ is fully mixed. If so, then $\exists a^i, b^i \in A^i$ for some $i \in I$, such that:

$$u^i\left(a^i, s\left(\varepsilon_n\right)^{-i}\right) < u^i\left(b^i, s\left(\varepsilon_n\right)^{-i}\right)$$

But since $s\left(\varepsilon_n\right)$ (which is equal to $s\left(\eta_n\right)$) is a NE in G_{ε_n}, we have that $s\left(\varepsilon_n\right)^i\left(a^i\right) = \eta_n^i\left(a^i\right) \leq \varepsilon_n^i\left(a^i\right)$, and so $s\left(\varepsilon_n\right)$ is ε_n-perfect, so (2) follows.

Consider (2) $\Longrightarrow$ (3) now:

□

CHAPTER 13

Extensive Form Games

Also definitions

APPENDIX A

The First Appendix

The appendix fragment is used only once. Subsequent appendices can be created using the Chapter Section/Body Tag.