On Efficient Distribution with Private Information, by Atkeson and Lucas

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1-Introduction

- This paper characterizes efficient allocations under asymmetric information

- Dynamics of the distribution of consumption will be examined from a normative perspective

1-Introduction

- There is a large literature prior to this paper on partial equilibrium analysis of incentive problem

- In all these papers there is single principal that minimizes the resources needed to provide given utility level to privately informed agents

- There is no period by period resource constraint in these papers

- Principal is faced with fixed prices

1-Introduction

- This paper develops a methodology to solve the problem of the planner subject to a resource constraint in each period

- Efficient allocation is characterized for certain parameterization of preferences

- It is shown that, for the cases considered, efficient allocation induces a cross sectional distribution of consumption that is spreading over time

2-Problem Statement

- There is continuum of agents with preferences

$$E\left\{\sum_{t=0}^{\infty}(1-\beta)\beta^t V(c_t)\theta_t\right\}$$

  each agent has a constant endowment of y every period.

- Idiosyncratic taste shocks $\theta \in \Theta = \{\theta^1, \ldots, \theta^n\}$ where $\theta^1 > \cdots > \theta^n > 0$.

- Probability distribution over Θ is μ and $E(\theta) = 1$. The shocks are i.i.d.

2-Problem Statement

- Current period utility function $V : \mathbb{R}_+ \to D \subseteq \mathbb{R}$ is continuous, strictly increasing and strictly concave

- Let inverse of V be $C : D \to \mathbb{R}_+$. $C(x)$ is the resources needed to deliver utility $(1 - \beta)x\theta$.

- Consumers are identified with expected utility promised to them, w.

2-Problem Statement

- μ^{t+1} is the product measure over history $\theta^t \in \Theta^{t+1}$

- Individual report at date t is $z_t : \Theta^{t+1} \rightarrow \Theta$. a reporting strategy is $z = \{z_t(\theta^t)\}_{t=0}^{\infty} \in Z$.

- Truthful reporting strategy is $z^* = \{z_t^*(\theta^t)\}_{t=0}^{\infty}$. With $z_t^*(\theta^t) = \theta_t$ for all t and all $\theta^t \in \Theta^{t+1}$

2-Problem Statement

- Planner assigns sequence $u = \{u_t(w, z^t)\}_{t=0}^{\infty}$ of utility to consumers.

- A **plan** is a sequence u (defined above) such that for all $t \geq 0$ and all $z^t \in \Theta^{t+1}$, $u_t(., z^t)$ is a Borel-measurable function on D and

$$\lim_{t\to\infty} \beta^t \sum_{t=s}^{\infty} \beta^s u_{t+s}(w, \theta^{t+s})\theta_{t+s} = 0 \qquad (1)$$

for all $w \in D$ and $\{\theta_t\} \in \Theta^{\infty}$. Set all plans is S.

- Set all plans is S.

2-Problem Statement

- Define total expected utility function $U : D \times S \times Z \to D$

$$U(w,u,z) = (1-\beta)\sum_{t=0}^{\infty} \beta^t \int_{\Theta^{t+1}} u_t[w, z^t(\theta^t)]\theta_t d\mu^{t+1}$$

- An **allocation** is a plan $u \in S$ such that

$$w = U(w,u,z^*) \tag{2}$$

for all $w \in D$ and

$$U(w,u,z^*) \geq U(w,u,z) \tag{3}$$

for all $w \in D$ and all $z \in Z$.

2-Problem Statement

- Let $\mathbf{D}$ be the set of all Borel subsets of D. Let M be set of all probability measures on $(D, \mathbf{D})$. Then any element $\psi \in M$ is a **utility distribution**.

- We say an allocation u *attains* ψ *with resources* y if

$$\int_{D\times\Theta^{t+1}} C[u_t(w,\theta^t)]d\mu^{t+1}d\psi \leq y \tag{4}$$

for all t.

3- A Recursive formulation

- Planner is choosing functions $f_t(w, \theta)$ (current utility) and $g_t(w, \theta)$ (future utility) for agent w who reports θ.

- At each t, $(f_t, g_t) : D \times \Theta \to D \times D$, and $f_t(., \theta)$ and $g_t(., \theta)$ are Borel-measurable. Let $\sigma = \{f_t, g_t\}_{t=0}^{\infty}$.

3- A Recursive formulation

- σ defines a plan as follows.
  - $\{w_t\}$ is the solution to $w_{t+1} = q_t(w_t, z_t)$ with initial value w_0.
  - So, $w_t : D \times \Theta^t \to D$.
  - $u_t(w_0, z^t) = f_t[w_t(w_0, z^{t-1}), z^t]$

  Then we say u is **generated** by σ

3- A Recursive formulation

- σ is an **allocation rule** if

1. Plans generated by σ belong to S.

2. For all $w_0 \in D$ and all $\{\theta_t\} \in \Theta^\infty$

$$\lim_{t\to\infty} \beta^t w_t(w_0, \theta^{t-1}) = 0 \tag{5}$$

3. For all $t \geq 0$ and all $w \in D$,

$$w = \int [(1-\beta) f_t(w,\theta)\theta + \beta g_t(w,\theta)] d\mu \tag{6}$$

and for all $t \geq 0$, all $w \in D$ and all $\theta, z \in \Theta$

$$(1-\beta) f_t(w,\theta)\theta + \beta g_t(w,\theta) \geq (1-\beta) f_t(w,z)\theta + \beta g_t(w,z) \tag{7}$$

3- A Recursive formulation

- Given an initial distribution ψ, $g(w,\theta)$ defines next period distribution map $S_g\psi$.

$$(S_g\psi)(D_0) = \int 1_{\{g(w,\theta)\in D_0\}} d\mu d\psi$$

- Given ψ_0 the sequence $\{\psi_t\}_{t=0}^{\infty}$ is defines by $\psi_{t+1} = S_{g_t}\psi_t$

- The allocation rule σ *attains distribution* ψ *with resources y* if

$$\int_{D\times\Theta} C[f_t(w,\theta)]d\mu d\psi_t \leq y \tag{8}$$

for all $t \geq 0$

3- A Recursive formulation

Lemma 1 *Let* $\psi \in M$ *and suppose the allocation* u *attains* ψ *with resources* y*. Then there is an allocation rule* σ *that attains* ψ *with resources* y*.*

Lemma 2 *Let* $\psi \in M$*. suppose allocation rule* σ *attains* ψ *with resources* y *and* u *is the plan generated by* σ*. Then* u *is an allocation, and* u *attains* ψ *with resources* y*.*

4- A Bellman Equation

- Let

  $$\varphi^*(\psi) = \inf_u \{y | \exists \text{ allocation } u \text{ that attains } \psi \text{ with resources } y\}$$

- An allocation is **efficient** if it attains ψ with resources $\varphi^*(\psi)$.

- Our next goal is to define $\varphi^*(\psi)$ in a recursive fashion

4- A Bellman Equation

- Consider allocation rule $\sigma = ((f_0, g_0), \sigma')$ that attains ψ with resources y.

- Then it must be true that

$$\int_{D\times\Theta} C[f_0(w,\theta)] d\mu a\psi \leq y$$

and σ attains $S_{g_0}\psi$ with resources y.

- Hence, the minimum resources needed to attain ψ is

$$\max\left\{\int_{D\times\Theta} C[f_0(w,\theta)] d\mu d\psi, \varphi^*(S_{g_0}\psi)\right\}$$

4- A Bellman Equation

- This suggest that φ^* should satisfy

$$\varphi(\psi) = \inf_{f,g} \max \left\{ \int_{D\times\Theta} C[f_0(w,\theta)] d\mu d\psi, \varphi(S_{g_0}\psi) \right\}$$

subject to

$$w = \int [(1-\beta) f(w,\theta)\theta + \beta g(w,\theta)] d\mu$$

$$(1-\beta) f(w,\theta)\theta + \beta g(w,\theta) \geq (1-\beta) f(w,z)\theta + \beta g(w,z)$$

$\forall w \in D, \quad \forall \theta, z \in \Theta$

- Let $B = \{(f,g) | (f,g)$ satisfy the above $\}$

4- A Bellman Equation

- Define operator T

$$(T\varphi)(\psi) = \inf_{(f,g)\in B} \max \left\{ \int_{D\times\Theta} C[f_0(w,\theta)] d\mu d\psi, \varphi(S_{g_0}\psi) \right\}$$

- Claim: φ^* is a fix point of T

- Proof is easy. By ruling out $\varphi^* > T\varphi^*$ and $\varphi^* < T\varphi^*$

4- A Bellman Equation

Now we want to find a strategy to construct φ^*.

Lemma 3 *If there exist functions* φ_a, phi_c *and* φ *such that for all* $\varphi \in M$

(i) $\varphi_c \leq \varphi \leq \varphi_a$

(ii) $\lim_{n\to\infty} T^n \varphi_a = \lim_{n\to\infty} T^n \varphi_c = \varphi$

Then $\varphi^* = \varphi$

Proof. Immediate by monotonicity of T and the fact that φ^* is a fix point of T ■

4- A Bellman Equation

- Candidate for φ_a: Consider autarkic allocation that assigns constant utility w at each period.

$$\varphi_a = \int_D C(x) d\psi$$

it is obvious that this is an allocation, therefore $\varphi_a(\psi) \geq \varphi^*(\psi)$

4- A Bellman Equation

- Candidate for φ_c: Consider solution to the following problem

$$\varphi_c = \min_u \int_D C[u(w,\theta)]d\psi d\mu$$

subject to

$$\int_\Theta \theta u(w,\theta)d\mu = w \quad \forall w \in D$$

Since any allocation satisfy the constraint of this problem we have $\varphi_c(\psi) \leq \varphi^*(\psi)$.

5- Examples

- Proving the existence of φ^* in the general case is very difficult.

- From now on assume $V(c) = \log(c)$ (hence $C(u) = \exp(u)$)

- Then

$$\varphi_a(\psi) = \int_D \exp(w) d\psi$$

$$\varphi_c(\psi) = \exp\{-E[\theta \log(\theta)]\} \int_D \exp(w) d\psi$$

5- An example

- φ_a and φ_c have similar form of $\alpha \int_D \exp(w) d\psi$

- Both depend only on one particular moment of ψ

- The same is true for the case of CRRA and CARA utility function

5- An example

- Our plan is to find $(T\varphi_a)(\psi)$ and $(T\varphi_c)(\psi)$

- Then $\varphi^* = \lim_{n\to\infty} T^n\varphi_a = \lim_{n\to\infty} T^n\varphi_c$

- We solve $T\varphi$ for general form of $\varphi = \alpha \int_D \exp(w)d\psi$

- In this case

$$\begin{aligned}\varphi(S_g\psi) &= \alpha \int_D \exp(w')dS_g\psi = \alpha \int_{D\times\Theta} \exp(w')1_{\{g(w,\theta)=w'\}} d\psi d\mu \\ &= \alpha \int_{D\times\Theta} \exp(g(w,\theta))d\psi d\mu\end{aligned}$$

5- An example

- Consider this problem

$$\min_{f,g} \max \left\{ \int_{D\times\Theta} \text{exp}(f(w,\theta))d\psi d\mu, \alpha \int_{D\times\Theta} \text{exp}(g(w,\theta))d\psi d\mu \right\}$$

subject to

$$w = \int [(1-\beta)f(w,\theta)\theta + \beta g(w,\theta)]d\mu$$

$$(1-\beta)f(w,\theta)\theta + \beta g(w,\theta) \geq (1-\beta)f(w,z)\theta + \beta g(w,z)$$

$\forall w \in D, \quad \forall \theta, z \in \Theta$

5- An example

- It turns out that the solution to this problem has the form

$$f(w,\theta) = w + r(\theta), \quad g(w,\theta) = w + h(\theta)$$

- Therefore we can reduce it to the following problem

$$\phi(\alpha) = \min_{r,h} \max \left\{ \int_\Theta \exp(r(\theta)) d\mu, \alpha \int_\Theta \exp(h(\theta)) d\mu \right\}$$

subject to

$$\int [(1-\beta) r(\theta)\theta + \beta h(\theta)] d\mu = 0$$

$$(1-\beta) r(\theta)\theta + \beta h(\theta) \geq (1-\beta) r(z)\theta + \beta h(z)$$

$\forall \theta, z \in \Theta$

5- An Example

- Claim: The function $\phi(\alpha)$ has a unique fixed point $\alpha^* \in [\alpha_c, a\alpha_a]$ and $\lim_{n\to\infty} \phi^n(\alpha_i) = \alpha^*$ for $i = a, c$

- Therefore

$$\varphi^*(\psi) = \alpha^* \int_D \exp(w) d\psi$$

- The results depend crucially on separability of function $C(.)$ (which is the result of nice utility functions)

6- Efficient Allocation

- Continue to assume the suitable form for utility (i.e. log utility)

- recall that

$$f(w,\theta) = w + r(\theta), \quad g(w,\theta) = w + h(\theta)$$

6- Efficient Allocation

- So we can generate an allocation u given w_0

$$w_t(w_0, \theta^{t-1}) = w_0 + \sum_{s=0}^{t-1} h(\theta_s)$$

  and

$$u_t(w_0, \theta^t) = f(w_t(w_0, \theta^{t-1}), \theta_t) = w_0 + \sum_{s=0}^{t-1} h(\theta_s) + r(\theta_t)$$

- Expected utility follows a random walk

6- Efficient Allocation

- Which implies consumption

$$c_t(w_0, \theta^t) = C[u_t(w_0, \theta^t)] = \exp(w_0) \prod_{s=0}^{t-1} \exp(h(\theta_s)) \exp(r(\theta_t))$$

- And variance of the cross-sectional distribution of utility

$$Var(u_t(w_0, \theta^t)) = Var(w_0) + (t-1)Var(h(\theta)) + Var(r(\theta))$$

6- Efficient Allocation

- Note that from definition of $\phi(\alpha)$ and the fact that $\alpha^* = \phi(\alpha^*)$ we have

$$\int_{\Theta} \exp(h(\theta)) d\mu = 1$$

This implies

$$\int_{\Theta} h(\theta) d\mu < 0$$

- $\{w_t\}$ drifts toward $-\infty$.

6- Efficient Allocation

- Go back to the problem of finding $r(\theta)$ and $h(\theta)$
- Not that at the optimal solution

$$\int_{\Theta} \exp(r(\theta))d\mu = \alpha \int_{\Theta} \exp(h(\theta))d\mu$$

- This fact help us to write down the problem in a simpler form
- Assume $\Theta = \{\theta_1, \theta_2\}$, $\theta_1 > \theta_2$

6- Efficient Allocation

$$\phi(\alpha) = \min_{r,h} \sum_{i=1,2} \mu_i C(r_i)$$

subject to

$$\sum_{i=1,2} \mu_i [C(r_i) - \alpha C(h_i)] = 0, \quad \lambda$$

$$\sum_{i=1,2} \mu_i [(1-\beta)\theta_i r_i + \beta h_i] = 0, \quad \xi$$

$$\begin{aligned} (1-\beta)\theta_1 r_1 + \beta h_1 &\geq (1-\beta)\theta_1 r_2 + \beta h_2, \quad \delta \\ (1-\beta)\theta_2 r_2 + \beta h_2 &\geq (1-\beta)\theta_2 r_1 + \beta h_1, \quad \nu \end{aligned}$$

6- Efficient Allocation

Lemma 4 $r_1 > r_2$, $h_1 < h_2$, $\delta = 0$ *and* $\nu > 0$

Proof. Add incentive constraints we get

$$(\theta_1 - \theta_2)(r_1 - r_2) \geq 0$$

with equality *iff* both constraints binding.
Then, $\theta_1 > \theta_2$ implies

$$r_1 \geq r_2, \quad h_1 \leq h_2$$

with equality *iff* both constraints binding.

6- Efficient Allocation

Suppose both constraints bind. Then $r_1 = r_2$, $h_1 = h_2$ and $\nu = \delta > 0$. But from FOC we have

$$\begin{aligned}(1-\lambda)\mu_1 C'(r_1) &= \xi(1-\beta)\mu_1\theta_1 + \delta(1-\beta)\theta_1 - \nu(1-\beta)\theta_2 \\ (1-\lambda)\mu_2 C'(r_2) &= \xi(1-\beta)\mu_2\theta_2 - \delta(1-\beta)\theta_1 + \nu(1-\beta)\theta_2\end{aligned}$$

this implies $C'(r_1) < C'(r_2)$, contradiction.
Now $h_1 > h_2$ together with the FOC w.r.t h_i implies

$$\delta < \nu$$

but only one of them can be positive ■

6- Efficient Allocation

- In efficient allocation, incentive constraint for the agent with highest shock binds

- And does not bind for agents with lower shock

7- Decentralization

- Consider problem of a planner (read principal) that is dealing only with agent w_0

- Assume this planner can borrow and lend at prices determined by a sequence q=$\{q_t\}_{t=0}^{\infty}$, $q{\in}(0,1)$

- Each of these planners minimize the present value of the resources they allocate

7- Decentralization

- The objective is

$$
\begin{aligned}
v(w_0) &= \min_{\{u_t(w_0,.)\}} (1-q_0) \int_{\Theta} C[u_0(w_0,\theta)]d\mu \\
&\quad + \sum_{t=1}^{\infty} (1-q_t) \prod_{s=0}^{t-1} q_s \int_{\Theta^{t+1}} C[u_1(w_0,\theta^t)]d\mu^{t+1}
\end{aligned}
$$

- Which is maximized subject to incentive constraint and promise keeping (Delivering w_0)

7- Decentralization

Theorem 5 *Suppose there exists allocations u, prices q, distribution ψ and resources y such that*

(i) given q, u solves the planner problem for all w_0

(ii) $\int_{D\times\Theta^{t+1}} C[u_t(w_0,\theta^t)]d\mu^{t+1}d\psi = y$

(iii) $\sum_{t=1}^{\infty}(1-q_t)\prod_{s=0}^{t-1} q_s$

Then allocation u attains ψ with resources y and $y = \varphi^*(\psi)$

7- Decentralization

- With constant q this setup is similar to Green(1987) and Phelan and Townsend(1991).

- It turns out that for some utility functions (e.g. log) $q = \beta$ actually clears the market

- Next we see if decentralization can proceed further

7- Decentralization

- Suppose agents can trade one-period real bond at the price $Q(\psi)$

- For (f, g) to be consistent with un-monitored trading they must satisfy

  $$Q(\psi)V'[C(f(w,\theta,\psi))]\theta = \beta \int_{\Theta} V'[C(f(g(w,\theta,\psi),\theta',S_g\psi))]\theta' d\mu'$$

- It can be shown that for log utility this holds only when there is full information

8- Conclusions

- The authors developed a recursive formulation to solve efficient allocation of resources under private information in a closed economy

- Derivation of the Bellman equation rests on the fact that taste shocks are multiplicative

- For specific parameterizations, only a certain moment of distribution is sufficient to solve for efficient allocations

- In efficient allocation consumer wealth follows a random walk with negative drift

8- Conclusions

- It is argued that these efficient allocation can not be decentralized

- What would be the results if the efficiency is examined in an overlapping generation context?