

A Recursive Formulation for Repeated Agency with History Dependence

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Introduction

- The dynamic incentive problems we studied so far do not consider exogenous links between periods of privately observed variables (for example Atkeson, Lucas presented last time)
- Links between periods in private observed variables, would require history dependence of agent choices, thus complicating the recursive formulation
- Constraints have to ensure that no gains are made by agent that deviated in the previous period

Introduction

- In many situations the evidence suggest the contrary (for example in income correlation)
- In the paper, a method to study such intertemporal links in a dynamic incentive problem is studied
- In particular, the authors consider a model with endowment shocks (extension to a model with taste shocks follows easily)

Outline

- Model
- Planner and auxiliary planner problems
- The recursive formulation
- The case of taste shocks

Model

- Time is discrete
- Endowment process: $h_t \in H = \{h_H, h_L\}$ follows a first order Markov process; assume: $h_H > h_L \geq c \in \mathbb{R}$
- $\pi(h_{t-1})$ probability that $h_t = h_H$ given h_{t-1} ;
assume: $0 < \pi(h_j) < 1, j = H, L$
- $\Pi(h^{t+j}|h_t)$ probability of subsequent history $\{h_{t+1}, \dots, h_{t+j}\} \in H^{j-1}$, given h_t

- Society can linearly transfer resources over time;
 $B = [\underline{b}, \bar{b}] \subset \mathbb{R}$ set of feasible consumption values
- $q \in (0, 1)$ price of consumption good in $t + 1$, in terms of time t consumption
- Consumers derive utility from consumption sequences:
 $\bar{U}(c) = \sum_{t=0}^{\infty} \beta^t U(c_t)$, where $U : B \rightarrow \mathbb{R}$;
- Assume: U strictly increasing and strictly concave; $\beta \in (0, 1)$

- A **reporting strategy** is a sequence $\{\tilde{h}_t(h^t)\}_{t=0}^{\infty}$ mapping histories $h^t = (h_0, h_1, \dots, h_t)$ to a report of current endowment
- A **transfer system** is a sequence of functions $\{\tau_t\}_{t=0}^{\infty}$; assume: $\tau(h^t) \geq -h_t$ for all t and h_t
- We will consider problem with planner assigning utils instead of consumption goods; let $C(\tilde{u}) \equiv U^{-1}(\tilde{u})$
- An **allocation** is a series of functions $u = \{u_t\}_{t=0}^{\infty}$ such that

$$C(u_t(\tilde{h}^t)) = \tilde{h}_t + \tau_t(\tilde{h}_t)$$

Given (h_{-1}, w_0) , a **planner problem** is:

$$V(h_{-1}, w_0) \equiv \min_u \left\{ \sum_{t=0}^{\infty} \sum_{h_t \in H^t} q^t [C(u_t(h^t)) - h_t] \Pi(h^t | h_{-1}) \right\}$$

s.to

$$\sum_{t=0}^{\infty} \sum_{h_t \in H^t} \beta^t u_t(h^t) \Pi(h^t | h_{-1}) \tag{1}$$

$$\geq \sum_{t=0}^{\infty} \sum_{h_t \in H^t} \beta^t U(h_t + C(u_t(\tilde{h}^t(h^t))) - \tilde{h}_t(h^t)) \Pi(h^t | h_{t-1})$$

$$w_0 = \sum_{t=0}^{\infty} \sum_{h_t \in H^t} \beta^t u_t(h^t) \Pi(h^t | h_{-1}) \tag{2}$$

We will characterize solutions to the planner problem using an **auxiliary planner problem**, so that given (h_{-1}, w_0, \hat{w}_0) , the planner now chooses u to solve

$$V_A(h_{-1}, w_0, \hat{w}_0) \equiv \min_u \left\{ \sum_{t=0}^{\infty} \sum_{h_t \in H^t} q^t [C(u_t(h^t)) - h_t] \Pi(h^t | h_{-1}) \right\}$$

s.to

(1), (2) and

$$\hat{w}_0 = \sum_{t=0}^{\infty} \sum_{h_t \in H^t} \beta^t u_t(h^t) \Pi(h^t | h_{-1}^c) \quad (3)$$

Let $W^*(h_{-1})$ be the set of (w_0, \hat{w}_0) that satisfy (1), (2) and (3); we then have:

$$V(h_{-1}, w_0) = \min_{\hat{w}_0} V_A(h_1, w_0, \hat{w}_0)$$

s.to

$$(w_0, \hat{w}_0) \in W^*(h_{-1})$$

Note: in the auxiliary planner problem we are also constrained to deliver utility \hat{w}_0 to agent with initial seed given by h_{-1}^c .

The recursive formulation

- We want to reduce the problem in the form of *SLP*; we need to determine: state variable s_t , outcome function $F(h_t, s_t, s_{t+1})$ and constraint set $s_{t+1}(h^t) \in \Gamma(s_t(h^{t-1}), h_t)$
- Consider a realization h^{t-1} and an announcement \hat{h}^{t-1} . An allocation u is incentive compatible after (h^{t-1}, \hat{h}^{t-1}) , if for all $\{\tilde{h}_\tau\}_{\tau=0}^\infty$:

$$\begin{aligned}
 & \sum_{\tau=0}^{\infty} \sum_{h_\tau \in H^\tau} \beta^\tau u_{t+\tau}(\hat{h}^{t-1}, h^\tau) \Pi(h^\tau | h_{t-1}) \\
 \geq & \sum_{\tau=0}^{\infty} \sum_{h_\tau \in H^\tau} \beta^\tau U(h_\tau + C(u_{t+\tau}(\hat{h}^{t-1}, \tilde{h}^\tau(h^\tau | h^{t-1}, \hat{h}^{t-1}))) \\
 & \quad - \tilde{h}_\tau(h^\tau | h^{t-1}, \hat{h}^{t-1})) \Pi(h^\tau | h_{t-1})
 \end{aligned}$$

Lemma 2.1 *Let u satisfy the time zero incentive constraint (1). Then, for any time period t , past history of reports \hat{h}^{t-1} and true realizations h^{t-1} , u is incentive compatible after (h^{t-1}, \hat{h}^{t-1}) .*

Note: (1) requires that u is incentive compatible after truth telling, here it's prove the stronger result that u is incentive compatible after any previous deviation.

Theorem 2.1 Let $u = \{u_t\}_{t=0}^{\infty}$ be an allocation. The sequence u is incentive compatible if and only if, for all t and $h^t \in H^t$, satisfies:

$$u_t(h^t) + \beta w_{t+1}(h^t) \geq U(h_t + C(u_t(h^{t-1}, h_t^c)) - h_t^c) + \beta \hat{w}_{t+1}(h^{t-1}, h_t^c) \quad (4)$$

where

$$w_{t+1}(h^t) \equiv \sum_{\tau=0}^{\infty} \sum_{h^\tau \in H^\tau} \beta^\tau u_{t+1+\tau}(h^t, h^\tau) \Pi(h^\tau | h_t),$$

$$\hat{w}_{t+1}(h^t) \equiv \sum_{\tau=0}^{\infty} \sum_{h^\tau \in H^\tau} \beta^\tau u_{t+1+\tau}(h^t, h^\tau) \Pi(h^\tau | h_t^c).$$

(4) is a **temporary incentive constraint**.

Proof:

- Suppose u satisfies (1) but violates (4) for some t and h^t . Consider a reporting strategy \tilde{h} that requires lying only at t after h^t . Given that (4) is violated and $\Pi(h^t|h_{-1}) > 0$, (1) is also violated
- Suppose u satisfies (4) but violates (1), then exists \tilde{h} , such that $W_0(\tilde{h}) - w_0 > a$, where:

$$W_0(\tilde{h}) \equiv \sum_{t=0}^{\infty} \sum_{h^t \in H^t} \beta^t U(h_t + C(u_t(\tilde{h}^t(h^t))) - \tilde{h}_t(h^t)) \Pi(h^t|h_{-1})$$

Define \tilde{h}^τ prescribing truth telling for $t \geq \tau$ and following \tilde{h} before:

$$\begin{aligned}
W_0(\tilde{h}^\tau) = & \sum_{t=0}^{\tau} \sum_{h^t \in H^t} \beta^t U(h_t + C(u_t(\tilde{h}^t(h^t))) - \tilde{h}_t) \Pi(h^t | h_{-1}) \\
& + \beta^{\tau+1} \left\{ \sum_h^\tau \in H_{true}^\tau w_{\tau+1}(\tilde{h}^\tau(h^\tau)) \Pi(h^\tau | h_{-1}) \right. \\
& \left. + \sum_h^\tau \in H_{false}^\tau \hat{w}_{\tau+1}(\tilde{h}^\tau(h^\tau)) \Pi(h^\tau | h_{-1}) \right\}
\end{aligned}$$

Since the set of feasible utilities is bounded, and $\beta < 1$, exists τ such that $|W_0(\tilde{h}) - W_0(\tilde{h}^\tau)| < \frac{a}{2}$.

Suppose $\tilde{h}^\tau(h^{\tau-1}) = h_{\tau-1}$, (4) requires that \tilde{h} does not improve on truth-telling from τ on; if $\tilde{h}^\tau(h^{\tau-1}) = h_{\tau-1}^c$, Lemma 2.1 requires that u does not improve on truth-telling from τ on. Using (4) for all $t < \tau$ requires $W_0(\tilde{h}^\tau) \leq w_0$; contradiction.

We can now write the problem recursively. Write (w_t, \hat{w}_t) as

$$w_t(h^{t-1}) = \sum_{h_t} \{u_t(h^t) + \beta w_{t+1}(h^t)\} \Pi(h_t | h_{t-1}) \quad (5)$$

$$\hat{w}_t(h^{t-1}) = \sum_{h_t} \{u_t(h^t) + \beta w_{t+1}(h^t)\} \Pi(h_t | h_{t-1}^c) \quad (6)$$

From (5) and (6), given $s_0 = \{h_{-1}, w_0, \hat{w}_0\}$ we can solve u as function of $w = \{w_t(h^{t-1}), \hat{w}_t(h^{t-1})\}_{t=1}^{\infty}$.

Given s_0 , for all t , h^t , $u_t(h^t)$ is uniquely implied by:

$$s_t(h^{t-1}) \equiv (h_{t-1}, w_t(h^{t-1}), \hat{w}_t(h^{t-1})) \text{ and } s_{t+1}(h^t)$$

The return function is now $F(h_t, s_t, s_{t+1}) = C(u_t(h^t)) - h_t$.

If (4) holds by Theorem 2.1, (1) holds. So if w satisfies (4), (5) and (6) all constraints of the auxiliary planner problem are satisfied.

As in SLP, we now define operator T in the space of bounded continuous function as:

$$\min_{u(h), w'(h), \hat{w}'(h)} T(v_A)(h_-, w, \hat{w}) \equiv \sum_{h \in H} \{C(u(h)) - h + qv_A(h, w'(h), \hat{w}'(h))\} \Pi(h|h_-) \quad (7)$$

s.to

$$w = \sum_{h \in H} \{u(h) + \beta w'(h)\} \Pi(h|h_{-1})$$

$$\hat{w} = \sum_{h \in H} \{u(h) + \beta w'(h)\} \Pi(h|h_{-1}^c)$$

$$u(h) + \beta w'(h) \geq U(h + C(u(h^c)) - h^c) + \beta \hat{w}'(h^c), \quad \forall h \in H$$

$$(w'(h), \hat{w}'(h)) \in W^*(h) \text{ and } u(h) \in [U(\underline{b}), U(\bar{b})].$$

The results from SLP follow once we show that:

Lemma 2.2 *The set $W^*(h_{-1})$ is non-empty and compact for each h_{-1} .*

Note: computationally to determine $W^*(h_{-1})$, we proceed as follows:

1. Determine $W(h_{-1})$ by two function that give highest and lowest \hat{w} that can be achieved by initial (h_{-1}, w) ;
2. Iterate using APS operator.

Lemma 2.3 *The function V_A is the unique fixed point of T . The policies $u(h|h_{-1}, w, \hat{w})$, $w'(h|h_{-1}, w, \hat{w})$, $\hat{w}'(h|h_{-1}, w, \hat{w})$ that solve (7) solve the infinite sequence auxiliary planning problem. And a solution $\{u_t(h^t)\}_{t=0}^{\infty}$ to the infinite sequence auxiliary problem solves*

$$V_A(h_{t-1}, w_t(h^{t-1}), \hat{w}_t(h^{t-1})) = \sum_{h_t \in H} \{C(u_t(h^t)) - h_t + qV_A(h_t, w'_{t+1}(h^t), \hat{w}'_{t+1}(h^t))\} \Pi(h_t|h_{t-1})$$

for all $t - 1, h^{t-1}$

The case of private taste shocks

Take previous framework, but now consider:

- Endowments fixed at \bar{e}
- $h_t \in H$ is a Markov process that determines a shock to utility, so that:

$$\bar{U}(c, h) = \sum_{t=0}^{\infty} \beta^t U(c_t, h_t)$$

Proceed as before, operator T is now given by

$$\begin{aligned}
& \min_{u(h), w'(h), \hat{w}'(h)} && T(v_A)(h_-, w, \hat{w}) \equiv && (8) \\
& && \sum_{h \in H} \{C(u(h), h) - \bar{e} + qv_A(h, w'(h), \hat{w}'(h))\} \Pi(h|h_-) \\
& \text{s.to} &&
\end{aligned}$$

$$w = \sum_{h \in H} \{u(h) + \beta w'(h)\} \Pi(h|h_{-1})$$

$$\hat{w} = \sum_{h \in H} \{u(h) + \beta w'(h)\} \Pi(h|h_{-1}^c)$$

$$u(h) + \beta w'(h) \geq U(h + C(u(h^c), h^c, h) + \beta \hat{w}'(h^c)), \quad \forall h \in H$$

$(w'(h), \hat{w}'(h)) \in W^*(h)$ and $u(h) \in D_h$ set of feasible utilities.

Conclusions

- In the paper, a method to study such intertemporal links in a dynamic incentive problem is studied
- The general procedure applies also to the case with Markov process of higher order and with realization taking more than two values
- Examples of privately observed endowments, taste shocks and effort are also studied