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On the computation of optimal monotone mean–variance portfolios via truncated quadratic utility☆

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ABSTRACT

We report a surprising link between optimal portfolios generated by a special type of variational preferences called divergence preferences (see Maccheroni et al., 2006) and optimal portfolios generated by classical expected utility. As a special case, we connect optimization of truncated quadratic utility (see Černý, 2003) to the optimal monotone mean–variance portfolios (see Maccheroni et al., 2009), thus simplifying the computation of the latter.

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1. Introduction

This paper is motivated by two alternative recent attempts to deal with the non-monotonicity (in the sense of first-order stochastic dominance) of quadratic utilities. The said non-monotonicity is a major drawback of these classical utility functions since the preference for more to less is a basic tenet of economic rationality, which no-arbitrage principles rely upon.

The first approach, Černý (2003), uses expected truncated quadratic utility and leads to the so-called arbitrage-adjusted Sharpe ratio. The second, formulated in Maccheroni et al. (2009), modifies the variational form of mean–variance preferences. The two approaches are *prima facie* altogether different, with fundamental differences in the structure of preferences: in the first case we deal with expected utility, while in the second case we contend with divergence preferences, a special case of the variational preferences of Maccheroni et al. (2006). Nevertheless, in this paper we show that there is an important and useful link between the optimal portfolios that the two approaches generate. This link is all the more interesting because variational preferences are closely related to convex risk measures (see Föllmer and Schied, 2002 and Frittelli and Rosazza Gianin, 2002), the optimization of which has been studied widely and fruitfully in recent years. A detailed discussion of the relationship between utility functionals, risk measures and robust control, together with relevant references, can be found in Föllmer et al. (2009).

To illustrate this surprising link and its significance, let us step back and discuss the relation between the classical (non-monotone) expected quadratic utility and mean–variance preferences. Fix a probability space $(\Omega, \mathcal{F}, P)$ and define the quadratic utility $f_q : \mathbb{R} \to \mathbb{R}$ by

$$f_q(x) = x - \frac{x^2}{2}. \tag{1.1}$$

If $Y \in L^2(P)$ is the value of an investment with zero initial outlay, its expected quadratic utility $F_q(Y) = E\left(f_q(Y)\right)$ corresponds to

$$F_q(Y) = E(Y) - \frac{1}{2}E(Y^2), \tag{1.2}$$

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while its mean–variance utility is

$$\Phi_q(Y) = E(Y) - \frac{1}{2}\mathrm{Var}(Y). \tag{1.3}$$

Formally, the link between the two utility functions is provided by the (not widely known) variational formula

$$\Phi_q(Y) = \inf_{Z\in L^2(P):E(Z)=1} E\left(ZY - f_q^*(Z)\right), \tag{1.4}$$

where $f_q^*(z) = -(1-z)^2/2$ is the Fenchel conjugate of f_q.

Regardless of whether one is aware of the link in (1.4), it is very well understood that optimal portfolios generated by the two criteria are closely related. This is easily checked by direct calculation. For an expected quadratic utility maximizer, the portfolio selection problem $\max_{\alpha\in\mathbb{R}} F_q(\alpha X)$ has a unique optimal portfolio $\hat\alpha = E(X)/E(X^2)$, with value

$$F_q(\hat\alpha X) = \frac{1}{2}\frac{\mathrm{SR}^2}{1+\mathrm{SR}^2}, \tag{1.5}$$

where $\mathrm{SR} = E(X)/\sqrt{\mathrm{Var}(X)}$ is the so-called Sharpe ratio of the excess return X. The optimal portfolio $\hat\alpha$ and value $F_q(\hat\alpha X)$ are related to the optimal portfolio $\hat\beta$ and value $\Phi_q(\hat\beta X)$ of the corresponding mean–variance problem through the following relations:

$$\hat\beta = \hat\alpha(1+\mathrm{SR}^2) = \frac{\hat\alpha}{1-2F_q(\hat\alpha X)}, \tag{1.6}$$

$$\Phi_q(\hat\beta X) = \frac{1}{2}\mathrm{SR}^2 = \frac{F_q(\hat\alpha X)}{1-2F_q(\hat\alpha X)}. \tag{1.7}$$

This is, in a nutshell, the classical relation between expected quadratic utility and mean–variance portfolio analysis. We will revisit these results at the end of Section 2. It is worth pointing out that, when the random variable $\hat\alpha X$ is generalized to a stochastic integral $\hat\alpha \cdot X_T = \int_0^T \hat\alpha_t dX_t$, for suitably defined processes $\hat\alpha$ and X, then (1.7) represents a link between mean-square hedging and the dynamically efficient mean–variance portfolio (see Černý and Kallsen, 2008).

The monotonization of quadratic preferences in Černý (2003) is based on a modification of the expected quadratic utility F_q. Specifically, Černý (2003) replaces the quadratic utility f_q in (1.1) with its monotone truncated version:

$$f(x) = \begin{cases} x - \frac{1}{2}x^2 & \text{for } x \le 1, \\ \frac{1}{2} & \text{for } x > 1. \end{cases} \tag{1.8}$$

Though in this case one no longer obtains simple expressions for the preference functional

$$F(Y) = E\left(f(Y)\right) \tag{1.9}$$

in terms of the first two moments of Y, Černý (2003) shows that solving the portfolio selection problem $\max_{\alpha\in\mathbb{R}^n} F(\alpha X)$ numerically is straightforward due to its low dimensionality and globally concave nature.[1]

The monotonization of mean–variance preferences proposed by Maccheroni et al. (2009) considers the mean–variance utility Φ_q. In particular, Maccheroni et al. (2009) replace Φ_q in (1.4) with its monotonization

$$\Phi(Y) = \inf_{Z\in L^2_+(P):E(Z)=1} E(ZY - f_q^*(Z)). \tag{1.10}$$

This is a simple specification of divergence preferences, which in general evaluate an investment Y through

$$\inf_{Z\in L^1_+(P):E(Z)=1} E(ZY - \phi(Z)),$$

where ϕ is a divergence kernel. Loosely speaking, divergence preferences consider (cautiously because of the inf) all possible probabilistic models Z for Y, penalized through their statistical distance $E(\phi(Z))$ from the reference model.[2] The monotone mean–variance preferences studied by Maccheroni et al. (2009) are the special case where $\phi = f_q^*$ is the Fenchel conjugate of the quadratic utility (1.1).

As argued in detail by Maccheroni et al. (2009), monotone mean–variance preferences are the monotone preferences that best approximate the original mean–variance preference functional (1.3). This gives monotone mean–variance preferences a solid decision-theoretic underpinning as the proper monotone "correction" of mean–variance preferences. The portfolio selection problem $\max_{\beta\in\mathbb{R}^n} \Phi(\beta X)$ is, however, somewhat more complicated to solve than Černý (2003)'s problem $\max_{\alpha\in\mathbb{R}^n} F(\alpha X)$, though Maccheroni et al. (2009) manage to reformulate the optimization of Φ with n risky assets as a system of $n+1$ non-linear equations.

At first sight it is not at all obvious that there should be *any* correspondence between optimal portfolios generated by the monotone criteria (1.9) and (1.10). Nonetheless, it seems natural to enquire whether a result similar to (1.6) and (1.7) might hold for the optimal monotone portfolios. Exploratory numerical results based on Černý (2003) and Maccheroni et al. (2009) have shown that (1.7) indeed holds for the monotone quadratic portfolios provided that the Sharpe ratio in (1.7) is replaced by the so-called arbitrage-adjusted Sharpe ratio introduced in Černý (2003). This, of course, brings us no closer to understanding why such a relationship should exist in the first place.

A first hint on the functional relation between the two approaches is the observation that, when the Fenchel conjugate

$$f^*(z) = \begin{cases} -\frac{(1-z)^2}{2} & \text{for } z \ge 0 \\ -\infty & \text{for } z < 0 \end{cases}$$

of the truncated quadratic utility f in (1.8) is considered, then the preference functional Φ in (1.10) can be rewritten as

$$\Phi(Y) = \inf_{Z\in L^2_+(P):E(Z)=1} E(ZY - f^*(Z)). \tag{1.11}$$

The "monotonized" preference functionals F and Φ thus stand in a similar conjugate relation as the original F_q and Φ_q; that is, $F(Y) = E\left(f(Y)\right)$ and $\Phi(Y) = \inf_{Z\in L^2_+(P):E(Z)=1} E(ZY - f^*(Z))$.

Our analysis builds on this intuition. Drawing on the recent work of Ben-Tal and Teboulle (2007), we show that the different approaches of Černý (2003) and Maccheroni et al. (2009) lead to closely related optimal portfolios. Our main result, Theorem 1, shows that for an important class of objective functions there is an explicit one-to-one relation among the solutions of the portfolio problems $\max_{\alpha\in\mathbb{R}^n} F(\alpha X)$ and $\max_{\beta\in\mathbb{R}^n} \Phi(\beta X)$, which allows one to move back and forth between them. In particular, to find the solutions of $\max_{\beta\in\mathbb{R}^n} \Phi(\beta X)$ it is enough to solve the somewhat simpler problem $\max_{\alpha\in\mathbb{R}^n} F(\alpha X)$.

This surprising connection between the two portfolio problems allows one to combine the decision-theoretic appeal of the monotonization of Maccheroni et al. (2009) with the computational simplicity of that of Černý (2003). We thus have a coherent picture, where the two different corrections of mean–variance preferences nicely complement each other.

[1] Here, and in what follows, X is a vector of excess returns, and so α and β range over $\mathbb{R}^n$.

[2] See Maccheroni et al. (2006) for details. $E\left(\phi(Z)\right)$ is the divergence of the probability ZdP from dP (see Liese and Vajda, 1987).

2. Main result

Consider the following family of utility functions, which we call *monotone HARA*:

$$f_\gamma(x) = \begin{cases} \dfrac{(1+x/\gamma)^{1-\gamma}-1}{1/\gamma-1} & \text{for } x \le -\gamma \\ \dfrac{\gamma}{\gamma-1} & \text{for } x > -\gamma, \end{cases} \qquad \gamma < 0, \quad \text{and} \tag{2.1}$$

$$f_\gamma(x) = \begin{cases} \dfrac{(1+x/\gamma)^{1-\gamma}-1}{1/\gamma-1} & \text{for } x > -\gamma \\ -\infty & \text{for } x < -\gamma, \end{cases} \qquad 0 < \gamma \neq 1. \tag{2.2}$$

Where not specified the value of f_γ at $-\gamma$ is defined as the unique value making f_γ upper semicontinuous.

Observe that in (2.1) and (2.2) one may compute pointwise limits as $\gamma \to 1$ and $\gamma \to \pm\infty$. We therefore define

$$f_1(x) = \begin{cases} \ln(1+x) & \text{for } x > -1 \\ -\infty & \text{for } x < -1, \end{cases} \tag{2.3}$$

$$f_{\pm\infty}(x) = 1 - e^{-x}. \tag{2.4}$$

By $f^* : \mathbb{R} \to [-\infty, \infty)$ we denote the Fenchel concave conjugate of f; that is, $f^*(z) = \inf_{x\in\mathbb{R}}\{xz - f(x)\}$. Simple calculations show that the Fenchel conjugate of f_γ is

$$f^*_\gamma(z) = \begin{cases} \dfrac{-\gamma z^{1-1/\gamma} + (\gamma-1)z + 1}{1/\gamma - 1} & \text{for } z > 0 \\ -\infty & \text{for } z < 0, \end{cases}$$
$$\gamma \in \mathbb{R} \setminus \{0, 1\}, \tag{2.5}$$

with special cases

$$f^*_1(z) = \begin{cases} 1 - z + \ln z & \text{for } z > 0 \\ -\infty & \text{for } z < 0, \end{cases} \tag{2.6}$$

$$f^*_{\pm\infty}(z) = \begin{cases} -z\ln z + z - 1 & \text{for } z > 0 \\ -\infty & \text{for } z < 0. \end{cases} \tag{2.7}$$

The value of f^*_γ at 0 is the unique value that makes f^*_γ upper semicontinuous.

It is important to observe that the Fenchel conjugates of monotone HARA utilities correspond to the most important family of divergence kernels. Specifically, $\{f^*_\gamma\}_{\gamma\neq 0}$ is the family of kernels of *power divergences* (see Liese and Vajda, 1987), widely used in applications as they include the χ^2-divergence, the relative entropy, and the Hellinger distance.

The preference functionals induced by f_γ are denoted by F_γ, $\Phi_\gamma : L^\infty(P) \to \mathbb{R}$:

$$F_\gamma(Y) = E\left(f_\gamma(Y)\right), \tag{2.8}$$

$$\Phi_\gamma(Y) = \inf_{Z\in L^1_+(P):E(Z)=1} E(ZY - f^*_\gamma(Z)). \tag{2.9}$$

For $\gamma = -1$, they coincide with the monotone quadratic preferences (1.9) and monotone mean–variance preferences (1.10) discussed in Section 1. More generally, for $\gamma \in (-\infty, 0)$, the preference functional F_γ can be seen as a monotone hull of the standard non-monotone expected HARA utility, and Φ_γ can be seen as a translation-invariant hull of F_γ; see Filipović and Kupper (2007).

Denoting by $\hat\alpha_{\gamma,X}$, $\hat F_{\gamma,X}$, $\hat\beta_{\gamma,X}$, and $\hat\Phi_{\gamma,X}$ the optimal portfolios and values corresponding to preferences F_γ and Φ_γ,

$$\hat F_{\gamma,X} := \max_{\alpha\in\mathbb{R}^n} F_\gamma(\alpha X), \qquad \hat\Phi_{\gamma,X} := \max_{\beta\in\mathbb{R}^n} \Phi_\gamma(\beta X), \tag{2.10}$$

$$\hat\alpha_{\gamma,X} \in \arg\max_{\alpha\in\mathbb{R}^n} F_\gamma(\alpha X), \qquad \hat\beta_{\gamma,X} \in \arg\max_{\beta\in\mathbb{R}^n} \Phi_\gamma(\beta X), \tag{2.11}$$

we can now state the main result of the paper. It shows that portfolio optimization with power divergence preferences can be solved in two stages, one of which involves solving the monotone optimal portfolio problem for expected HARA utility. Moreover, point (iii) establishes an explicit relationship between the optimal portfolios $\hat\alpha_{\gamma,X}$ and $\hat\beta_{\gamma,X}$, so that the knowledge of $\hat\alpha_{\gamma,X}$ is enough to determine $\hat\beta_{\gamma,X}$. As discussed in Section 1, this is the main finding of the paper. Remarkably, point (iii) shows that $\hat\alpha_{\gamma,X}$ and $\hat\beta_{\gamma,X}$ feature the same mix of risky assets, though the leverage is different (and so the two portfolios can be very different, as illustrated in Section 3).

Theorem 1. *Suppose $X \in L^\infty(P)^n$ is arbitrage free (i.e., $\alpha \in \mathbb{R}^n$ and $\alpha X \ge 0$ imply $\alpha X = 0$). Then, for each $\gamma \in \bar{\mathbb{R}} \setminus \{0\}$, the maximizers $\hat\alpha_{\gamma,X}$ and $\hat\beta_{\gamma,X}$ exist, and*

$$\hat\Phi_{\gamma,X} = \max_{\eta\in\mathbb{R},\beta\in\mathbb{R}^n}\{\eta + F(\beta X - \eta)\}$$
$$= \max_{\eta\in\mathbb{R}}\{\eta + F(\hat\beta_{\gamma,X}X - \eta)\}. \tag{2.12}$$

Moreover, we have the following.

(i) *The maximizer $\hat\eta_{\gamma,X}$ in (2.12) is uniquely determined (i.e., it does not depend on $\hat\beta_{\gamma,X}$), and it satisfies*

$$\hat\eta_{\gamma,X} = \begin{cases} \gamma\left(1 - \left(\left(\dfrac{1}{\gamma} - 1\right)\hat F_{\gamma,X} + 1\right)^{1/\gamma}\right) \\ \quad \text{for } \gamma \in \mathbb{R}\setminus\{0,1\} \\ 0 \\ \quad \text{for } \gamma = 1 \\ -\ln(1 - \hat F_{\gamma,X}) \\ \quad \text{for } \gamma = \pm\infty \end{cases}$$

(ii) *the optimal values $\hat F_{\gamma,X}$ and $\hat\Phi_{\gamma,X}$ are in a one-to-one correspondence, as follows:*

$$\hat\Phi_{\gamma,X} = \begin{cases} \dfrac{\gamma^2}{1-\gamma}\left(\left(\hat F_{\gamma,X}\left(\dfrac{1}{\gamma} - 1\right) + 1\right)^{1/\gamma} - 1\right) \\ \quad \text{for } \gamma \in \mathbb{R}\setminus\{0,1\} \\ \hat F_{\gamma,X} \\ \quad \text{for } \gamma = 1 \\ -\ln(1 - \hat F_{\gamma,X}) \\ \quad \text{for } \gamma = \pm\infty \end{cases}$$

(iii) *the optimal portfolios for the two criteria are related as follows:*

$$\hat\beta_{\gamma,X} = \begin{cases} \hat\alpha_{\gamma,X}\left(\hat F_{\gamma,X}(1/\gamma - 1) + 1\right)^{1/\gamma} & \text{for } \gamma \in \mathbb{R}\setminus\{0,1\} \\ \hat\alpha_{\gamma,X} & \text{for } \gamma = 1 \\ \hat\alpha_{\gamma,X} & \text{for } \gamma = \pm\infty, \end{cases}$$

where the equality is to be interpreted as equality of sets in $\mathbb{R}^n$.

Monotone mean–variance preferences correspond to $\gamma = -1$, and for them we readily recover

$$\hat\Phi_{-1,X} = \frac{\hat F_{-1,X}}{1 - 2\hat F_{-1,X}},$$
$$\hat\beta_{-1,X} = \frac{\hat\alpha_{-1,X}}{1 - 2\hat F_{-1,X}},$$

which parallel the classical relations (1.6) and (1.7). Černý (2003) shows that the quantity $\sqrt{2\hat F_{-1,X}(1 - 2\hat F_{-1,X})^{-1}}$ can be interpreted as the arbitrage-adjusted Sharpe ratio (denoted by $\overline{\text{SR}}$) of the

Table 1
Optimal investment in three correlated lognormally distributed assets for various preferences.

ρ_{12}	$\gamma=-1$				$\gamma=-2$		$\gamma=-3$	
	$\begin{bmatrix} f^{-1}(F) \\ \alpha \end{bmatrix}$	$\begin{bmatrix} \Phi \\ \beta \end{bmatrix}$	$\begin{bmatrix} f^{-1}(F) \\ \alpha \end{bmatrix}$	$\begin{bmatrix} \Phi \\ \beta \end{bmatrix}$	$\begin{bmatrix} f^{-1}(F) \\ \alpha \end{bmatrix}$	$\begin{bmatrix} \Phi \\ \beta \end{bmatrix}$	$\begin{bmatrix} f^{-1}(F) \\ \alpha \end{bmatrix}$	$\begin{bmatrix} \Phi \\ \beta \end{bmatrix}$
	Non-monotone		Monotone					
0.5	0.20	0.27	0.21	0.30	0.26	0.31	0.27	0.31
	1.11	1.71	1.44	2.32	1.80	2.21	1.93	2.19
	0.63	0.98	0.78	1.25	0.96	1.18	1.02	1.16
	4.37	6.75	5.22	8.39	6.37	7.83	6.76	7.68
0.75	0.18	0.24	0.19	0.26	0.23	0.26	0.24	0.27
	1.24	1.83	1.58	2.40	1.96	2.35	2.10	2.35
	0.20	0.30	0.22	0.33	0.26	0.31	0.27	0.30
	3.99	5.88	4.63	7.03	5.60	6.71	5.91	6.61
0.9	0.18	0.25	0.20	0.27	0.24	0.28	0.25	0.28
	1.92	2.86	2.51	3.88	3.11	3.76	3.33	3.75
	−0.69	−1.03	−0.93	−1.43	−1.15	−1.39	−1.23	−1.39
	3.68	5.50	4.32	6.69	5.22	6.31	5.52	6.21
0.99	0.35	0.69	0.45	1.15	0.61	0.97	0.68	0.93
	8.43	19.99	16.90	55.63	21.56	37.22	23.46	33.13
	−8.08	−19.15	−15.74	−51.82	−20.00	−34.52	−21.70	−30.64
	2.24	5.30	3.64	11.98	4.69	8.09	5.12	7.23

optimal portfolio $\hat{\alpha}_{-1,X}X$. This allows us to write the result for the monotone mean–variance portfolio more intuitively as follows:

$$\hat{\Phi}_{-1,X} = \frac{1}{2}\overline{\mathrm{SR}}^2,$$

$$\hat{\beta}_{-1,X} = \hat{\alpha}_{-1,X}(1+\overline{\mathrm{SR}}^2)^{-1}.$$

The $n+1$ equations which characterize the optimal value $\hat{\beta}_{-1,X}$ in Maccheroni et al. (2009) are now readily seen to be the first-order conditions of the optimization over η and β in (2.12) for the specific choice of $f(x) = f_{-1}(x) = -\frac{1}{2}(((1-x)^+)^2 - 1)$.

3. Numerical example

To illustrate our results, consider a model with lognormally distributed asset returns, where

$$X_i = e^{Z_i} - e^r, \quad i = 1, 2, 3,$$

$$r = 0.02,$$

and variables $\{Z_i\}_{i=1}^3$ are jointly normally distributed with mean μ_Z and covariance matrix Σ_Z:

$$\mu^Z = \begin{bmatrix} 0.12 \\ 0.07 \\ 0.04 \end{bmatrix}, \qquad \Sigma^Z = \begin{bmatrix} 0.09 & 0.09\rho_{12} & -0.15 \\ 0.09\rho_{12} & 0.09 & -0.15 \\ -0.15 & -0.15 & 0.01 \end{bmatrix}.$$

The example is designed in such a way that for $\rho_{12} = 1$ there is an arbitrage opportunity when one goes long the first risky asset and shorts the second risky asset. For $|\rho_{12}| < 1$, the market is arbitrage free. We report optimal portfolio allocation for $\rho_{12} \in \{0.5, 0.75, 0.9, 0.99\}$ in Table 1.

Strictly speaking, $X \notin L^\infty(P)^3$. Here, one has two options to justify the numerical results below. The less appealing option is to truncate Z_i at a given number of standard deviations (say 8) which makes the returns bounded. Alternatively, one can generalize our theory to $L^{1-\gamma}(P)^n$ for $\gamma \in (-\infty, 0)$. We discuss the latter option within the concluding remarks in Section 5.

The numerical analysis is performed for monotone and non-monotone quadratic preferences ($\gamma = -1$) and monotone third and fourth power HARA utilities ($\gamma = -2$ and $\gamma = -3$) whose non-monotone versions constitute preferences over the first three (four) moments of asset returns. The results illustrate several important points:

1. monotone and non-monotone quadratic preferences lead to different proportions invested in risky assets, and the same is true for the corresponding divergence preferences;
2. non-monotone quadratic preferences lead to underinvestment (lower leverage) compared to their monotone counterparts;
3. non-monotone preferences report the investment opportunities as less attractive (in terms of certainty equivalents);
4. in the monotone case, expected utility preferences do not exploit near-arbitrage opportunities as fully as their divergence counterpart.

To check the robustness of our analysis we have recomputed the numerical example with Z generated from multivariate uniform and multivariate truncated Student t-distributions. Results from these alternative specifications (available from the authors on request) confirm that the main conclusions reached above remain valid and are thus robust to distributional assumptions.

4. Existence of optimal portfolios in a general framework

In this section, we develop general results on the existence of optimal portfolios that go beyond the HARA class of expected utility and divergence preferences.

4.1. Results from convex analysis

Here we summarize key results from convex analysis (we refer the interested reader to Rockafellar, 1970 for details) that underpin the existence of optimal portfolios. Given a closed and convex subset C of $\mathbb{R}^n$, intuitively its recession cone consists of all directions along which C is unbounded. Formally, the *recession cone* R_C of C is defined by

$$R_C = \left\{ y \in \mathbb{R}^n : x + ty \in C \text{ for all } x \in C \text{ and all } t \geq 0 \right\},$$

with the convention $R_\emptyset = \mathbb{R}^n$. The vectors in R_C are called *directions of recession*. It is easy to see that R_C is a closed convex cone.

The lineality space L_C is given by $R_C \cap R_{-C}$. In particular, L_C is a vector subspace, with

$$L_C = \left\{ y \in \mathbb{R}^n : x + ty \in C \text{ for all } x \in C \text{ and all } t \in \mathbb{R} \right\}.$$

For a proper concave and upper semicontinuous function $f : \mathbb{R}^n \to [-\infty, \infty)$, the recession cone R_f and the lineality space L_f are given by $R_f = \bigcap_{\lambda\in\mathbb{R}} R_{(f\geq\lambda)}$ and $L_f = \bigcap_{\lambda\in\mathbb{R}} L_{(f\geq\lambda)}$. Clearly, $R_f \supseteq L_f$.

The following lemma gives a simple criterion to check whether a vector belongs to R_f or L_f (it actually holds in normed vector spaces; see Rockafellar, 1966, Theorem 2A).

Lemma 2. *For a proper concave and upper semicontinuous function $f : \mathbb{R}^n \to [-\infty, \infty)$, the following statements are equivalent:*

(i) $y \in R_f$ *(respectively,* $y \in L_f$*);*
(ii) $\lim_{\lambda\to\infty} f(x+\lambda y) > -\infty$ *(respectively,* $\lim_{\lambda\to\pm\infty} f(x+\lambda y) > -\infty$*) for all* $x \in \operatorname{dom} f$*;*
(iii) $\lim_{\lambda\to\infty} f(x+\lambda y) > -\infty$ *(respectively,* $\lim_{\lambda\to\pm\infty} f(x+\lambda y) > -\infty$*) for some* $x \in \operatorname{dom} f$*;*
(iv) $\lim_{\lambda\to\infty} f(x+\lambda y) > -\infty$ *(respectively,* $\lim_{\lambda\to\pm\infty} f(x+\lambda y) > -\infty$*) for some* $x \in \mathbb{R}^n$.

The following existence result summarizes several results in Rockafellar (1970); see Theorem 27.3 and subsequent discussion.

Theorem 3. *Let $f : \mathbb{R}^n \to [-\infty, \infty)$ be an upper semicontinuous proper concave function. Then* $\arg\max_{x\in\mathbb{R}^n} f(x)$ *is non-empty provided* $R_f = L_f$.

4.2. Preferences

For a concave function $f : \mathbb{R} \to [-\infty, \infty)$, we denote by $\operatorname{dom}_+ f$ the largest open interval on which f is strictly increasing. It can be checked that $\operatorname{dom}_+ f$ is a well-defined set that coincides with the (possibly empty) set $\left\{x \in \operatorname{int}\operatorname{dom} f : f'_+(x) > 0\right\}$. We set $\operatorname{sd}_+ f = \sup \operatorname{dom}_+ f$.

Assumption 1. Throughout the remainder of the paper, $f : \mathbb{R} \to [-\infty, \infty)$ is a proper, concave, increasing, and upper semicontinuous function, with $0 \in \operatorname{dom}_+ f$.

The preference functionals F and Φ are defined by dropping the subscript γ in Eqs. (2.8) and (2.9). The next result collects the basic properties of the preference functional F.

Lemma 4. *The preference functional $F : L^\infty \to [-\infty, \infty)$ is proper, concave, increasing, and upper semicontinuous.*

In order to study the preference functional Φ, we will restrict our attention to the following class of functions.

Definition 5. $\mathcal{H}$ denotes the set of functions f satisfying Assumption 1 and such that $f(0) = 0, f'_+(0) \le 1 \le f'_-(0)$, and there exist $x < 0 < y$ in $\operatorname{dom} f$ with $f'_+(x) > 1$ and $1 > f'_+(y) > 0$.

For example, f belongs to $\mathcal{H}$ if it satisfies Assumption 1 and is twice continuously differentiable in a neighborhood of 0, with $f(0) = 0, f'(0) = 1$, and $f''(0) < 0$. One easily verifies that the monotone HARA utility functions introduced in Section 2 satisfy $f_\gamma \in \mathcal{H}$ for all $\gamma \in \bar{\mathbb{R}} \setminus \{0\}$.

Lemma 6. *If $f \in \mathcal{H}$, then $1 \in \operatorname{int}\operatorname{dom} f^*$ and f^* attains its supremum at 1, with $f^*(1) = 0$.*

The next theorem, essentially due to Ben-Tal and Teboulle (2007), provides the main link between Φ and F and establishes the basic properties of the functional Φ.

Theorem 7. *If $f \in \mathcal{H}$, then, for all $Y \in L^\infty(P)$,*

$$\Phi(Y) = \sup_{\eta\in\mathbb{R}}\{\eta + F(Y-\eta)\} = \max_{\eta\in[\operatorname{ess\,inf} Y, \operatorname{ess\,sup} Y]}\{\eta + F(Y-\eta)\}.$$

Moreover, Φ is concave, increasing, normalized, translation invariant, finite, and Lipschitz.

We conclude this section by characterizing the recession directions of F and Φ, respectively. We need some notation: set $f'(\infty) = \lim_{x\to\infty} f'_+(x)$ and, with a slight abuse of notation,

$$f'(-\infty) = \begin{cases} \lim_{x\to-\infty} f'_-(x) & \text{if } \operatorname{dom} f = \mathbb{R}, \\ \infty & \text{otherwise.} \end{cases}$$

Proposition 8. *For all $Y \in L^\infty$,*

$$\lim_{\lambda\to\infty} \frac{F(\lambda Y)}{\lambda} = -f'(-\infty)\,E(Y^-) + f'(\infty)E(Y^+), \tag{4.1}$$

with the convention $0 \cdot \infty = 0$.

Corollary 9. *$R_F \subseteq L^\infty_+$ iff f satisfies*

$$\frac{f'(\infty)}{f'(-\infty)} = 0. \tag{4.2}$$

The next proposition reports a novel link between recession directions of Φ, asymptotic slopes of f, and the *average value at risk* (*AVaR*) described in Appendix A. This corrects, *inter alia*, a result of Ben-Tal and Teboulle (2007). Their Theorem 2.2a and Lemma 4.1 assert, in our notation,

$$\lim_{\lambda\to\infty} \Phi(\lambda Y)/\lambda = \operatorname{ess\,inf} Y, \quad \forall f \in \mathcal{H}, \forall Y \in L^\infty,$$

which in fact only holds when f satisfies the conditions $f'(\infty) = 0$ and $f'(-\infty) = \infty$ appearing in Eq. (4.3).

Proposition 10. *If $f \in \mathcal{H}$, then, for all $Y \in L^\infty$,*

$$\lim_{\lambda\to\infty} \frac{\Phi(\lambda Y)}{\lambda} = (1 - f'(\infty))\left(-\operatorname{AVaR}_Y\left(\frac{1-f'(\infty)}{f'(-\infty)-f'(\infty)}\right)\right) + f'(\infty)E(Y).$$

Corollary 11. *$R_\Phi \subseteq L^\infty_+$ iff f satisfies*

$$f'(\infty) = 0 \quad \text{and} \quad f'(-\infty) = \infty. \tag{4.3}$$

Corollaries 9 and 11 are a straightforward consequence of Lemma 2. The "if and only if" statement holds in the sense that if condition (4.2) (respectively, (4.3)) is violated then one can construct a two-state model in which $R_F \not\subseteq L^\infty_+$ (respectively, $R_\Phi \not\subseteq L^\infty_+$).

4.3. Portfolio selection problems

Given a random vector $X \in L^\infty(P)^n$ that represents the excess return of n securities, define the preference functionals $F_X, \Phi_X : \mathbb{R}^n \to [-\infty, \infty)$ over portfolios by setting

$$F_X(\alpha) = F(\alpha X) \quad \text{and} \quad \Phi_X(\beta) = \Phi(\beta X). \tag{4.4}$$

Provided the optimizers exist, we want to study the mutual relationship of the following portfolio selection problems: $\max_{\alpha\in\mathbb{R}^n} F_X(\alpha)$, and $\max_{\beta\in\mathbb{R}^n} \Phi_X(\beta)$. The former is a classical expected utility portfolio problem and, when f is the truncated quadratic (1.8), it is the problem considered by the monotonization of Černý (2003). The latter is instead a novel, non-expected utility, portfolio problem that uses divergence preferences. In particular, the monotone mean–variance preferences studied in Maccheroni et al. (2009) correspond to the preference functional Φ determined by the conjugate f^* of f.

We first confirm that F_X and Φ_X inherit their basic properties from F and Φ, respectively.

Lemma 12. *The function $F_X : \mathbb{R}^n \to [-\infty, \infty)$ is proper, concave, and upper semicontinuous. If $f \in \mathcal{H}$, then the function $\Phi_X : \mathbb{R}^n \to [-\infty, \infty)$ is real valued, concave, and Lipschitz.*

Next, we define arbitrage-free portfolios, which will play a key role in the solution of the optimization problems we consider.

Definition 13. $X \in L^\infty(P)^n$ is arbitrage free if, for all $\alpha \in \mathbb{R}^n$ such that $\alpha X \geq 0$, one has $\alpha X = 0$.

Complete characterization of the existence of optimal portfolios is now an easy consequence of Theorem 3 and Lemma 12, together with Corollaries 9 and 11 for expected utility F and divergence utility Φ, respectively.

Theorem 14. (i) *If $X \in L^\infty(P)^n$ is arbitrage free and condition (4.2) is satisfied, then* $\arg\max_{\alpha\in\mathbb{R}^n} F_X(\alpha) \neq \emptyset$ *and, moreover,* $f(0) \leq \hat{F}_X < f(\mathrm{sd}_+ f)$. *Conversely, if condition (4.2) is not satisfied, then there is a two-state arbitrage-free model in which* $\arg\max_{\alpha\in\mathbb{R}^n} F_X(\alpha) = \emptyset$.

(ii) *Suppose f belongs to $\mathcal{H}$. If $X \in L^\infty(P)^n$ is arbitrage free and condition (4.3) is satisfied, then* $\arg\max_{\beta\in\mathbb{R}^n} \Phi_X(\beta) \neq \emptyset$. *Conversely, if condition (4.3) is not satisfied, then there is a two-state arbitrage-free model in which* $\arg\max_{\beta\in\mathbb{R}^n} \Phi_X(\beta) = \emptyset$.

Part (i) generalizes Černý (2003, Theorem 2), to which we refer for further relevant references. Note that the HARA utility functions f_γ in (2.1)–(2.4) all satisfy the requirement (4.3), which is more stringent than condition (4.2).

Having just established their existence, we now proceed to examine optimal portfolios generated by divergence preferences. Along with Theorem 7, Theorem 14 implies the existence of $\hat{\beta}_X$ and $\hat{\eta}_X$ such that

$$\hat{\Phi}_X = \Phi(\hat{\beta}_X X) = \max_{\eta\in\mathbb{R},\beta\in\mathbb{R}^n} \{\eta + F(\beta X - \eta)\}$$
$$= \max_{\eta\in\mathbb{R}} \{\eta + F(\hat{\beta}_X X - \eta)\} = \hat{\eta}_X + F(\hat{\beta}_X X - \hat{\eta}_X).$$

The quantity $\hat{\eta}_X$ may in general depend on $\hat{\beta}_X$ if the latter is not unique. In the next lemma, we show that $-\hat{\eta}_X$ actually belongs to $\mathrm{dom}_+ f$, that is, to the strict monotonicity domain of f.

Lemma 15. *Suppose that $f \in \mathcal{H}$ and that $X \in L^\infty(P)^n$ is arbitrage free. If $\hat{\beta}_X$ and $\hat{\eta}_X$ satisfy*

$$\hat{\Phi}_X = \hat{\eta}_X + F(\hat{\beta}_X X - \hat{\eta}_X),$$

then $-\hat{\eta}_X \in \mathrm{dom}_+ f$.

The proof of the main Theorem 1 is now a simple matter of exploiting Lemma 15 together with the specific functional form of the HARA utility functions f_γ defined in (2.1)–(2.4).

5. Final remarks

We have studied, side by side, the optimal portfolio allocation for expected utility and for divergence preferences over $L^\infty(P)^n$. This framework contains multiperiod finite state models, as well as one-period models with general bounded asset returns. Assuming no arbitrage and using general upper semicontinuous utility f, normalized to have 0 inside the domain of strict monotonicity, we have provided necessary and sufficient conditions under which the optimal portfolios exist, and we have shown that if these conditions are not met one may construct a two-state arbitrage-free model in which the optimal portfolio allocation problem fails to have a maximizer (see Examples 17 and 18 in Appendix B). Our minimal conditions are as follows:

- $f'(\infty)/f'(-\infty) = 0$ for the expected utility criterion;
- $(f'(\infty) = 0) \wedge (f'(-\infty) = \infty)$ for divergence preferences, together with further normalization $f(0) = 0$, $1 \in \partial f(0)$.

Note that we do not impose strict monotonicity or strict concavity on f, and therefore we have to deal with non-uniqueness of optimal portfolios.

On the way to the second condition, we have generalized the result of Ben-Tal and Teboulle (2007, Theorem 2.2a) by providing a novel link between the recession directions of divergence preferences, asymptotic slopes of f and the *average value at risk*; see Proposition 10. Ours seems to be the first study to point out that the first condition (the one on expected utility) is weaker than the second; most papers on expected utility maximization use the stronger assumption from the outset.

Maximization of truncated quadratic utility has appeared in Jin et al. (2006) in the context of mean–risk optimization where the risk is represented by below-the-mean semivariance. Jin et al. (2006) allow for unbounded excess returns in $L^2(P)^n$. We expect that our main result (Theorem 1) can be generalized to semimartingale trading using the Orlicz space duality and the definition of admissible trading strategies proposed in Biagini and Černý (2011). Since for $\gamma = -1$ the relevant Orlicz space is $L^2(P)$, this would encompass the existence result for optimal strategies in Jin et al. (2006) as a special case.

And finally, Cherny and Kupper (2007) have studied optimal portfolio allocation for divergence preferences, making use of Orlicz space duality. They require $\mathrm{dom}\, f = \mathbb{R}$, strict monotonicity, strict concavity, differentiability, boundedness above and further growth conditions on f. However, among the HARA utilities which are the subject of our Theorem 1, the only case satisfying the conditions of Cherny and Kupper is the exponential utility, $\gamma = \infty$. In this case, the divergence preferences coincide with the expected utility ones and the optimal strategies are the same.

Appendix A. Average value at risk

Consider a random variable $X \in L^\infty$ representing a financial position and its cumulative distribution function (cdf) $F_X(x) = P(X \leq x)$. Denote by q_X^- and q_X^+ the left- and right-continuous inverse of the cdf, respectively. That is,

$$\left.\begin{aligned} q_X^-(\lambda) &= \inf\{x \in \mathbb{R} : F_X(x) \geq \lambda\} \\ q_X^+(\lambda) &= \inf\{x \in \mathbb{R} : F_X(x) > \lambda\} \end{aligned}\right\} \quad \text{for } \lambda \in (0, 1),$$
$$q_X^-(0) = q_X^+(0) = \operatorname{ess\,inf} X,$$
$$q_X^-(1) = q_X^+(1) = \operatorname{ess\,sup} X.$$

Following Föllmer and Schied (2004, A.14), the average value at risk (AVaR) at level $\lambda \in (0, 1]$ is defined by

$$\mathrm{AVaR}_X(\lambda) = -\frac{1}{\lambda}\int_0^\lambda q_X^+(z)dz.$$

For convenience, we extend this definition to $\lambda = 0$ by setting

$$\mathrm{AVaR}_X(0) = -\operatorname{ess\,inf} X = \lim_{\lambda\to 0} \mathrm{AVaR}_X(\lambda). \tag{A.1}$$

The following result provides an alternative variational formula for AVaR.

Theorem 16 (*Föllmer and Schied, 2004, Lemma 4.46*)**.** *Consider a random variable $X \in L^\infty$ and $\lambda \in (0, 1]$. For any $q \in [q_X^-(\lambda), q_X^+(\lambda)]$,*

$$\mathrm{AVaR}_X(\lambda) = \frac{1}{\lambda}E((q - X)^+) - q = \frac{1}{\lambda}\inf_{\eta\in\mathbb{R}}(E((\eta - X)^+) - \lambda\eta).$$

Appendix B. Proofs and related analysis

With a common abuse, for measurable functions we will write $\geq$ to denote both the pointwise and the almost sure inequality. When we want to be more precise, we denote the latter by $\geq_{\mathrm{a.s.}}$.

Proof of Lemma 4. The proof is standard, and hence is omitted. But notice that, under the assumptions of Lemma 4, we also have

$$\begin{aligned}&\{Y \in L^\infty : \operatorname{ess\,inf} Y \in \operatorname{dom} f\}\\ &\quad\subseteq \operatorname{dom} F\\ &\quad\subseteq \left\{Y \in L^\infty : \operatorname{ess\,inf} Y \in \operatorname{cl}\operatorname{dom} f\right\}.\end{aligned} \tag{B.1}$$

If $\operatorname{dom} f = (-\infty, \infty)$, the three sets in (B.1) clearly coincide. Thus we assume $\operatorname{cl}\operatorname{dom} f = [d, \infty)$ with $d \in \mathbb{R}$. If $Y \in L^\infty$ and $\operatorname{ess\,inf} Y \in \operatorname{dom} f$, then $[\operatorname{ess\,inf} Y, \operatorname{esssup} Y] \subseteq \operatorname{dom} f$, $\operatorname{ess\,inf} Y \leq_{\text{a.s.}} Y \leq_{\text{a.s.}} \operatorname{esssup} Y$ implies

$$-\infty < f(\operatorname{ess\,inf} Y) \leq_{\text{a.s.}} f(Y) \leq_{\text{a.s.}} f(\operatorname{esssup} Y) < \infty.$$

Thus $f(Y) \in L^\infty$ and $F(Y) \in \mathbb{R}$. Now, assume per contra $Y \in \operatorname{dom} F$ and $\operatorname{ess\,inf} Y \notin \operatorname{cl}\operatorname{dom} f$. Then

$$d > \operatorname{ess\,inf} Y = \max\{a \in \mathbb{R} : P(\{Y \geq a\}) = 1\}$$

and $P(\{Y \geq d\}) < 1$; thus $P(\{Y < d\}) > 0$. But $Y(\omega) < d$ implies $f(Y(\omega)) = -\infty$; thus $P(\{f(Y) = -\infty\}) > 0$ and $F(Y) = -\infty$, contradicting $Y \in \operatorname{dom} F$. □

The following example suggests why we cannot obtain a tighter result. Let $\Omega = (0, 1)$ endowed with the Lebesgue σ-algebra and the Lebesgue measure. Set $Y_t(\omega) = \omega^t$ for all $\omega \in (0, 1)$ and choose $f(x) = -x^{-1}$ if $x > 0$ and $-\infty$ otherwise. Then, considering $Y_{1/2}$ and Y_1, it turns out that

$$\begin{aligned}&\{Y \in L^\infty : \operatorname{ess\,inf} Y \in \operatorname{dom}\} \subset \operatorname{dom} F\\ &\quad\subset \left\{Y \in L^\infty : \operatorname{ess\,inf} Y \in \operatorname{cl}\operatorname{dom} f\right\},\end{aligned}$$

since $Y_{1/2}$ belongs to the second but not to the first set, while Y_1 belongs to the third but not to the second set.

Proof of Lemma 6. Since $f : \mathbb{R} \to [-\infty, \infty)$ is a proper, concave, and upper semicontinuous function, the Young–Fenchel theorem guarantees that, for $x, y \in \mathbb{R}$,

$$y \in \partial f(x) \iff x \in \partial f^*(y) \iff f^*(y) + f(x) = xy. \tag{B.2}$$

In particular,

$$\operatorname{dom} f^* \supseteq \bigcup_{x \in \operatorname{dom} f} \partial f(x) \supseteq \operatorname{int}\operatorname{dom} f^*. \tag{B.3}$$

Since $f'_+(0) \leq 1 \leq f'_-(0)$, then $1 \in \partial f(0)$, by (B.2), $0 \in \partial f^*(1)$; thus the maximum of f^* is attained at 1. By (B.2) again $f^*(1) = 0$. By (B.3), $\operatorname{int}\bigcup_{x \in \operatorname{dom} f} \partial f(x) = \operatorname{int}\operatorname{dom} f^*$ and $\bigcup_{x \in \operatorname{dom} f} \partial f(x)$ is an interval. Since Definition 5 implies that there exist $z, w \in \mathbb{R}$ such that both $\partial f(z) \cap (0, 1)$ and $\partial f(w) \cap (1, \infty)$ are non-empty, $\bigcup_{x \in \operatorname{dom} f} \partial f(x)$ contains an element strictly smaller than 1 and an element strictly greater than 1. Thus, $1 \in \operatorname{int}\bigcup_{x \in \operatorname{dom} f} \partial f(x)$ and $1 \in \operatorname{int}\operatorname{dom} f^*$. □

Proof of Theorem 7. The proof is a minor modification of Ben-Tal and Teboulle (2007, Proposition 2.1 and Theorem 4.2). □

Proof of Proposition 8. Without loss of generality, assume $f(0) = 0$. For every $x < 0 < y$ in $\operatorname{dom} f$ and arbitrary $\xi \in \partial f(x)$, $\theta \in \partial f(y)$, we have

$$\begin{aligned}f(z) \leq\ & (f(x) + \xi(z - x))1_{(-\infty,0)}(z)\\ &+ (f(y) + \theta(z - y))1_{(0,\infty)}(z),\end{aligned}$$

where 1_A is the indicator function of set A and $z \in \mathbb{R}$. This implies

$$\frac{E(f(\lambda Y))}{\lambda} \leq \frac{f(x) - \xi x}{\lambda} - \xi E(Y^-) + \frac{f(y) - \theta y}{\lambda} + \theta E(Y^+), \tag{B.4}$$

and consequently

$$\limsup_{\lambda \to \infty} F(\lambda Y)/\lambda \leq -\xi E(Y^-) + \theta E(Y^+). \tag{B.5}$$

Let $y_n = n$, $\theta_n = f'_+(y_n)$. Depending on the $\operatorname{dom} f$,

$$\begin{aligned}&x_n = (1/d - 1/n)^{-1} \quad \text{and} \quad \xi_n = f'_-(x_n) \quad \text{if } \operatorname{dom} f = (d, \infty),\\ &x_n = d \quad \text{and} \quad \xi_n = f'_+(d) + n \quad \text{if } \operatorname{dom} f = [d, \infty).\end{aligned}$$

Note that, by Assumption 1, $d < 0$. One easily verifies $x_n < 0 < y_n \in \operatorname{dom} f$, and $\xi_n \in \partial f(x_n)$, $\theta_n \in \partial f(y_n)$ with $\xi_n \nearrow f'(-\infty)$ and $\theta_n \searrow f'(\infty)$. From (B.5),

$$\begin{aligned}\limsup_{\lambda \to \infty} F(\lambda Y)/\lambda \ &\leq \inf_{n \in \mathbb{N}}\{-\xi_n E(Y^-) + \theta_n E(Y^+)\}\\ &= -f'(-\infty)\, E\left(Y^-\right) + f'(\infty)\, E\left(Y^+\right).\end{aligned} \tag{B.6}$$

We prove the converse inequalities case by case.

Case 1: $f'(-\infty) = \infty$ and $E\left(Y^-\right) > 0$. The right-hand side in (B.6) equals $-\infty$, which proves $\lim_{\lambda \to \infty} F(\lambda Y)/\lambda = -\infty$.

Case 2: $f'(-\infty) < \infty$. Recall that $f(0) = 0$. For all $z \in \mathbb{R}$, we have

$$\begin{aligned}&f(z) \geq -f'(-\infty)z^- + f'(\infty)z^+,\\ &\frac{F(\lambda Y)}{\lambda} \geq -f'(-\infty)E(Y^-) + f'(\infty)E(Y^+).\end{aligned}$$

Case 3: $E\left(Y^-\right) = 0$. From $f(z) \geq f'(\infty)z$ for $z \geq 0$ we conclude that $\frac{F(\lambda Y)}{\lambda} \geq f'(\infty)E(Y^+) = -f'(-\infty)E(Y^-) + f'(\infty)E(Y^+)$ due to the convention $0 \cdot \infty = 0$. □

Proof of Corollary 9. Consider $Y \notin L^\infty_+$. Then, $E(Y^-) > 0$. If $f'(-\infty) < \infty$, then (4.2) amounts to $f'(\infty) = 0$, and so Proposition 8 implies $\lim_{\lambda \to \infty} F(\lambda Y)/\lambda = -f'(-\infty)\, E\left(Y^-\right) < 0$. In turn, this implies $\lim_{\lambda \to \infty} F(\lambda Y) = -\infty$. On the other hand, if $f'(-\infty) = \infty$, then by Proposition 8 we again have $\lim_{\lambda \to \infty} F(\lambda Y)/\lambda = -\infty$. Hence, also in this case, $\lim_{\lambda \to \infty} F(\lambda Y) = -\infty$. By Lemma 2, $Y \notin R_F$, and therefore $R_F \subseteq L^\infty_+$. Condition (4.2) is also necessary for this inclusion, in the sense that, as the next example shows, when it is violated one can exhibit a finite probability space and $Y \notin L^\infty_+$ such that Y is a recession direction of F. □

Example 17. If $f'(\infty)/f'(-\infty) > 0$, then $\infty > f'(-\infty) \geq f'(\infty) > 0$. With $f(0) = 0$, we have

$$f(z) \geq f'(\infty)z^+ - f'(-\infty)z^-.$$

Consider a finite probability space $\Omega = \{\omega_1, \omega_2\}$. Set $P(\{\omega_1\}) = p$ and take a random variable Y such that $Y(\omega_1) = -1$ and $Y(\omega_2) = 1$. If $p = f'(\infty)/(f'(\infty) + f'(-\infty))$, then the above implies

$$E(f(\lambda Y)) \geq f'(\infty)\lambda(1 - p) - f'(-\infty)\lambda p = 0 > -\infty.$$

By Lemma 4, F is upper semicontinuous. By Lemma 2, $Y \in R_F$, and yet $Y \notin L^\infty_+$. □

Proof of Proposition 10. Fix $\lambda > 0$. By Theorem 7, we have

$$\begin{aligned}\Phi(\lambda Y)/\lambda &= \sup_{\eta \in \mathbb{R}}(\eta + F(\lambda Y - \eta))/\lambda\\ &= \sup_{\eta \in \mathbb{R}}\{\eta + F(\lambda(Y - \eta))/\lambda\}.\end{aligned} \tag{B.7}$$

Consider the sequences $\{x_n\}$, $\{y_n\}$, $\{\xi_n\}$, $\{\theta_n\}$ defined in the proof of Proposition 8. Set $g(x, y, \xi, \theta) = f(x) + f(y) - \xi x - \theta y$. Eqs. (B.4) and (B.7) yield, for each $n \in \mathbb{N}$,

$$\begin{aligned}\Phi(\lambda Y)/\lambda \leq\ & g(x_n, y_n, \xi_n, \theta_n)/\lambda + \sup_{\eta \in \mathbb{R}}\{\eta + \theta_n E((Y - \eta)^+)\\ &- \xi_n E((Y - \eta)^-)\},\end{aligned}$$

and hence, for each $n \in \mathbb{N}$,

$$\begin{aligned}
&\limsup_{\lambda\to\infty} \Phi(\lambda Y)/\lambda \\
&\le \sup_{\eta\in\mathbb{R}} \left\{\eta + \theta_n E(Y-\eta) - (\xi_n - \theta_n)E((Y-\eta)^-)\right\} \\
&= -(1-\theta_n) \inf_{\eta\in\mathbb{R}} \left\{\frac{\xi_n-\theta_n}{1-\theta_n} E((\eta - Y)^+) - \eta\right\} \\
&= -(1-\theta_n)\left(-\mathrm{AVaR}_Y\left(\frac{1-\theta_n}{\xi_n-\theta_n}\right)\right) + \theta_n E(Y), \qquad \text{(B.8)}
\end{aligned}$$

where the last equality follows from Theorem 16. On taking $\inf_{n\in\mathbb{N}}$ in (B.8), and in view of the continuity of AVaR_Y, we obtain

$$\begin{aligned}
\limsup_{\lambda\to\infty} \frac{\Phi(\lambda Y)}{\lambda} \le\ & (1-f'(\infty))\left(-\mathrm{AVaR}_Y\left(\frac{1-f'(\infty)}{f'(-\infty)-f'(\infty)}\right)\right) \\
& + f'(\infty)E(Y). \qquad \text{(B.9)}
\end{aligned}$$

We consider the converse inequalities case by case.

Case 1: $f'(-\infty) < \infty$. From (B.7), (B.6), and Theorem 16, we obtain

$$\begin{aligned}
&\Phi(\lambda Y)/\lambda \\
&\ge \sup_{\eta\in\mathbb{R}} \{\eta + f'(\infty)E((Y-\eta)^+) - f'(-\infty)E((Y-\eta)^-)\} \\
&= (1-f'(\infty))\left(-\mathrm{AVaR}_Y\left(\frac{1-f'(\infty)}{f'(-\infty)-f'(\infty)}\right)\right) \\
&\quad + f'(\infty)E(Y). \qquad \text{(B.10)}
\end{aligned}$$

Combination of (B.9) and (B.10) completes the proof for $f'(-\infty) < \infty$.

Case 2: $f'(-\infty) = \infty$. On choosing $\eta = \operatorname{ess\,inf} Y$, Eq. (B.7) yields, for all $\lambda > 0$,

$$\Phi(\lambda Y)/\lambda \ge \operatorname{ess\,inf} Y + F(\lambda(Y - \operatorname{ess\,inf} Y))/\lambda.$$

From the proof of Case 3 in Proposition 8, we obtain

$$\begin{aligned}
&\liminf_{\lambda\to\infty} \Phi(\lambda Y)/\lambda \ge \operatorname{ess\,inf} Y + f'(\infty)E((Y - \operatorname{ess\,inf} Y)^+) \\
&= (1-f'(\infty))\operatorname{ess\,inf} Y + f'(\infty)E(Y) \\
&= (1-f'(\infty))\,(-\mathrm{AVaR}_Y(0)) + f'(\infty)E(Y). \qquad \text{(B.11)}
\end{aligned}$$

Combination of (B.9) and (B.11) completes the proof for $f'(-\infty) = \infty$. □

Proof of Corollary 11. Let $Y \notin L^\infty_+$, so that $\operatorname{ess\,inf} Y < 0$. By Proposition 10 and Eq. (A.1),

$$\begin{aligned}
\lim_{\lambda\to\infty} \frac{\Phi(\lambda Y)}{\lambda} &= (1-f'(\infty))\left(-\mathrm{AVaR}_Y\left(\frac{1-f'(\infty)}{f'(-\infty)-f'(\infty)}\right)\right) \\
&\quad + f'(\infty)E(Y) \\
&= -\mathrm{AVaR}_Y(0) = \operatorname{ess\,inf} Y < 0.
\end{aligned}$$

This implies $\lim_{\lambda\to\infty} \Phi(\lambda Y) = -\infty$. By Lemma 2, $Y \notin R_\Phi$, and therefore $R_\Phi \subseteq L^\infty_+$. Condition (4.3) is also necessary for this inclusion to hold in the sense that when it is violated one can exhibit a finite probability space and $Y \notin L^\infty_+$ such that Y is a recession direction of Φ. This is accomplished by the following example. □

Example 18. Set $\lambda = \frac{1-f'(\infty)}{f'(-\infty)-f'(\infty)}$. Consider a finite probability space $\Omega = \{\omega_1, \omega_2\}$. Set $P(\{\omega_1\}) = p$ and a random variable Y such that $Y(\omega_1) = -1$ and $Y(\omega_2) = y > 0$. Clearly, $Y \notin L^\infty_+$. There are two cases to consider. (i) If $f'(-\infty) = \infty$, then $f'(\infty) > 0$ and $\lambda = 0$. Let $p = 1/2$ and $y = 2/f'(\infty)$. This yields

$$-(1-f'(\infty))\mathrm{AVaR}_Y(\lambda) + f'(\infty)E(Y) = f'(\infty)/2 > 0.$$

(ii) If $f'(-\infty) < \infty$, we have $0 < \lambda < 1$, and we let $p = \lambda/2$, $y = 1$. This yields

$$-(1-f'(\infty))\mathrm{AVaR}_Y(\lambda) + f'(\infty)E(Y) = (1-\lambda)f'(\infty) \ge 0.$$

In both cases, $\liminf_{\lambda\to\infty} \Phi(\lambda Y) \ge 0$. By Theorem 7, Φ is finite valued and Lipschitz. By Lemma 2, $Y \in R_\Phi$, and yet $Y \notin L^\infty_+$. □

Proof of Lemma 12. It is easy to see that F_X and Φ_X inherit the properties of F and Φ established in Lemma 4 and Theorem 7. We just check that Φ_X is Lipschitz. Let $\alpha, \beta \in \mathbb{R}^n$, and set $L = \max_{i=1,\dots,n} \|X_i\|_\infty$. Since Φ is Lipschitz, say without loss of generality with constant $M = 1$, we can write

$$\begin{aligned}
|\Phi_X(\alpha) - \Phi_X(\beta)| &= |\Phi(\alpha X) - \Phi(\beta X)| \le \|\alpha X - \beta X\| \\
&= \operatorname{ess\,sup} |\alpha X - \beta X| \\
&= \operatorname{ess\,sup} \left|\sum_{i=1}^n (\alpha_i X_i - \beta_i X_i)\right| \\
&\le \operatorname{ess\,sup} \sum_{i=1}^n |\alpha_i - \beta_i|\,|X_i| \le L\sum_{i=1}^n |\alpha_i - \beta_i|,
\end{aligned}$$

as desired. □

Lemma 19. *If X is arbitrage free and $f'(\infty)/f'(-\infty) = 0$, then*

$$R_{F_X} = L_{F_X} = \left\{\alpha \in \mathbb{R}^n : \alpha X = 0\right\}. \qquad \text{(B.12)}$$

Proof. By Lemma 2, $\alpha \in R_{F_X}$ only if $\lim_{\lambda\to\infty} F(\lambda\alpha X) > -\infty$. By Proposition 8, this implies $\alpha X \in L^\infty_+$. Since X is arbitrage free, we conclude that $\alpha X = 0$. Hence, $R_{F_X} \subseteq \{\alpha \in \mathbb{R}^n : \alpha X = 0\}$. It remains to show that $\{\alpha \in \mathbb{R}^n : \alpha X = 0\} \subseteq L_{F_X}$. Let $\alpha \in \mathbb{R}^n$ be such that $\alpha X = 0$. Then, $\lim_{\lambda\to\infty} F(\lambda\alpha X) = \lim_{\lambda\to-\infty} F(\lambda\alpha X) = f(0) > -\infty$. Hence, $\alpha \in L_{F_X}$. □

Lemma 20. *If X is arbitrage free, $f'(\infty) = 0$, and $f'(-\infty) = \infty$, then*

$$R_{\Phi_X} = L_{\Phi_X} = \left\{\alpha \in \mathbb{R}^n : \alpha X = 0\right\}. \qquad \text{(B.13)}$$

Proof. Analogous to Lemma 19. □

Proof of Theorem 14. (i) Since, by Lemma 12, F_X is proper, concave, and upper semicontinuous, Theorem 3 and Lemma 19 imply $\arg\max_{\alpha\in\mathbb{R}^n} F_X(\alpha) \neq \emptyset$.

To show that $f(0) \le \hat{F}_X < f(\mathrm{sd}_+ f)$, suppose $f(\mathrm{sd}_+ f) < \infty$; otherwise, the statement is trivial. We have $f'_+(t + \mathrm{sd}_+ f) = 0$ for all $t > 0$. Then, $f(x) \le f(t + \mathrm{sd}_+ f)$ for all $t > 0$ and all $x \in \mathbb{R}$, and so $f(\mathrm{sd}_+ f) = \max_{x\in\mathbb{R}} f(x)$. Since $0 \in \mathrm{dom}_+ f$, we have $f(0) \le \hat{F}_X$ and $f(0) < \max_{x\in\mathbb{R}} f(x) = f(\mathrm{sd}_+ f)$.

From the absence of arbitrage, $\alpha X \notin L^\infty_+$, i.e., $P(\alpha X \le 0) > 0$. Set $p = P(\alpha X \le 0)$. Since $f(0) < f(\mathrm{sd}_+ f) < \infty$, we have

$$E(f(\alpha X)) \le pf(0) + (1-p)f(\mathrm{sd}_+ f) < f(\mathrm{sd}_+ f), \quad \forall \alpha \in \mathbb{R}^n,$$

and we conclude that $f(0) \le \hat{F}_X < f(\mathrm{sd}_+ f)$.

The required counterexample is obtained by setting $n = 1$ and $X = Y$ in Example 17.

(ii) The proof proceeds completely analogously on replacing Lemma 19 with Lemma 20 and Example 17 with Example 18. □

Proof of Lemma 15. To ease notation, we omit the X subscript in $\hat{\beta}$ and $\hat{\eta}$. First, we show that $-\hat{\eta} < \mathrm{sd}_+ f$. Suppose $\mathrm{sd}_+ f < \infty$ (otherwise the inequality is trivially true). Suppose, per contra, that $-\hat{\eta} \ge \mathrm{sd}_+ f$. Since $f(\mathrm{sd}_+ f) = \max_{x\in\mathbb{R}} f(x)$, we have $f(\hat{\beta}X - \hat{\eta}) \le f(\mathrm{sd}_+ f) = f(0X - (-\mathrm{sd}_+ f))$. Moreover, it is easy to check that $f \in \mathcal{H}$ implies $f(\mathrm{sd}_+ f) < \mathrm{sd}_+ f$. Then

$$\begin{aligned}
\hat{\eta} + E\left(f(\hat{\beta}X - \hat{\eta})\right) &\le -\mathrm{sd}_+ f + f(\mathrm{sd}_+ f) < 0 \\
&= 0 + E(f(0X - 0)),
\end{aligned}$$

which contradicts the optimality of $\hat\beta, \hat\eta$. We conclude that $-\hat\eta < \mathrm{sd}_+ f$.

Since $f \in \mathcal{H}$, we have $\operatorname{int}\operatorname{dom} f = (d, \infty)$, with $d \in [-\infty, 0)$. Hence, $\operatorname{dom}_+ f = (d, \mathrm{sd}_+ f)$. It thus remains to show that $-\hat\eta > d$. Suppose $d > -\infty$ (otherwise the inequality is trivially true). By Theorem 7, $\operatorname{ess\,sup}(\hat\beta X) \geq \hat\eta \geq \operatorname{ess\,inf}(\hat\beta X)$. Moreover, $\operatorname{ess\,inf}(\hat\beta X) - \hat\eta \geq d$, since otherwise $E(f(\hat\beta X - \hat\eta)) = -\infty$.

Hence, we either have $\operatorname{ess\,inf}(\hat\beta X) < 0$, and so $-\hat\eta > d$, or $\operatorname{ess\,inf}(\hat\beta X) = 0$. In the latter case, the absence of arbitrage implies $\operatorname{ess\,sup}(\hat\beta X) = 0$, and therefore $-\hat\eta = 0 > d$. In all cases, $-\hat\eta > d$. □

Next, we collect some elementary properties of the utility functions f_γ.

Lemma 21. *Consider the family of functions $\{f_\gamma\}_{\gamma\in\bar{\mathbb{R}}\setminus\{0\}}$. Then,*

(i) *$f_\gamma \in \mathcal{H}$ for all γ, with $\operatorname{dom} f_\gamma = \mathbb{R}$ if $\gamma \in [-\infty, 0)$;*
(ii) *$\operatorname{dom}_+ f_\gamma = \{x \in \mathbb{R} : 1 + x/\gamma > 0\}$ for all γ, and*

$$f(\mathrm{sd}_+ f_\gamma) = \begin{cases} \dfrac{\gamma}{\gamma - 1} & \text{for } \gamma \in [-\infty, 0) \cup (1, \infty] \\ \infty & \text{for } \gamma \in (0, 1] \end{cases}$$

(iii) *$f_\gamma''(0) = -1, f_\gamma'(-\infty) = \infty$, and $f_\gamma'(\infty) = 0$ for all γ;*
(iv) *for $-\eta \in \operatorname{dom}_+ f_\gamma$, we have*

$$f_\gamma(x - \eta) = \begin{cases} \left(1 - \dfrac{\eta}{\gamma}\right)^{1-\gamma} f_\gamma\left(\dfrac{x}{1 - \eta/\gamma}\right) + f_\gamma(-\eta), \\ \quad \gamma \in \mathbb{R} \setminus \{0\} \\ f_\gamma(x)(1 - f_\gamma(-\eta)) + f_\gamma(-\eta), \\ \quad \gamma = \pm\infty. \end{cases}$$

Proof. Parts (i)–(iii) are straightforward. In part (iv), it suffices to observe that one can express (2.1) equivalently as $f_\gamma(x) = \frac{((1+x/\gamma)^+)^{1-\gamma} - 1}{1/\gamma - 1}$, $\gamma < 0$, which makes the remaining computations straightforward.

Proof of Theorem 1. We prove all statements for $\gamma \in \mathbb{R} \setminus \{0, 1\}$. The proof for $\gamma = 1$ and $\gamma = \pm\infty$ follows along the same lines. First, observe that the maximizers $\hat\alpha_{\gamma,X}$ and $\hat\beta_{\gamma,X}$ exist by Theorem 14.

(i) Fix a particular maximizer $\hat\beta$, and denote the corresponding optimal value of η in (2.12) by $\hat\eta$. Since $f_\gamma \in \mathcal{H}$, either $f_\gamma(\mathrm{sd}_+ f_\gamma) < \mathrm{sd}_+ f_\gamma$ or $\mathrm{sd}_+ f_\gamma = \infty$. By Lemma 15, $-\hat\eta \in \operatorname{dom}_+ f_\gamma$. Since $f_\gamma \in \mathcal{H}$, by Theorem 7,

$$\begin{aligned} \hat\Phi_{\gamma,X} &= \max_{\eta\in\mathbb{R}} \{\eta + \max_{\beta\in\mathbb{R}^n} F_\gamma(\beta X - \eta)\} \\ &= \max_{-\eta\in\operatorname{dom}_+ f_\gamma} \{\eta + \max_{\beta\in\mathbb{R}^n} E(f_\gamma(\beta X - \eta))\}. \end{aligned}$$

Using Lemma 21,

$$\begin{aligned} \hat\Phi_{\gamma,X} &= \max_{\eta\in\mathbb{R}: 1-\eta/\gamma>0} \{\eta + f_\gamma(-\eta) + (1 - \eta/\gamma)^{1-\gamma} \\ &\quad \times \max_{\beta\in\mathbb{R}^n} E(f_\gamma(\beta X(1 - \eta/\gamma)^{-1}))\} \\ &= \max_{\eta\in\mathbb{R}: 1-\eta/\gamma>0} \{\eta + f_\gamma(-\eta) + (1 - \eta/\gamma)^{1-\gamma} \\ &\quad \times \max_{\alpha\in\mathbb{R}^n} F_\gamma(\alpha X)\}. \end{aligned} \tag{B.14}$$

This proves that $\hat\alpha = \left(1 - \hat\eta/\gamma\right)^{-1} \hat\beta$ is a maximizer for $F_{\gamma,X}$ (otherwise the optimality of $\hat\beta, \hat\eta$ would be contradicted). We thus obtain

$$\hat\Phi_{\gamma,X} = \max_{\eta\in\mathbb{R}: 1-\eta/\gamma>0} \{\eta + f_\gamma(-\eta) + (1 - \eta/\gamma)^{1-\gamma} \hat F_{\gamma,X}\}. \tag{B.15}$$

By Theorem 14, $0 \leq \hat F_{\gamma,X} < f_\gamma(\mathrm{sd}_+ f_\gamma)$, which implies

$$1 + \hat F_{\gamma,X}(1/\gamma - 1) > 0. \tag{B.16}$$

Define

$$h_{\gamma,\delta}(x) = \begin{cases} x + f_\gamma(-x) + (1 - x/\gamma)^{1-\gamma}\delta & \text{for } 1 - x/\gamma > 0 \\ -\infty & \text{for } 1 - x/\gamma < 0, \end{cases}$$

and define $h_{\gamma,\delta}(\gamma)$ to make the function upper semicontinuous. On $\operatorname{int}\operatorname{dom} f$, we have

$$h'_{\gamma,\delta}(x) = 1 - (1 - x/\gamma)^{-\gamma}(1 + \delta(1/\gamma - 1)),$$

$$h''_{\gamma,\delta}(x) = -(1 - x/\gamma)^{-\gamma-1}(1 + \delta(1/\gamma - 1)).$$

Therefore, $h_{\gamma,\delta}$ is strictly concave as long as

$$1 + \delta(1/\gamma - 1) > 0. \tag{B.17}$$

Assuming (B.17) holds, one easily verifies that $h_{\gamma,\delta}$ has no directions of recession, and therefore it possesses a unique maximizer. The point

$$\hat x_{\gamma,\delta} := \gamma(1 - ((1/\gamma - 1)\delta + 1)^{1/\gamma}),$$

satisfies $h'_{\gamma,\delta}(\hat x_{\gamma,\delta}) = 0$ and

$$1 - \hat x_{\gamma,\delta}/\gamma = ((1/\gamma - 1)\delta + 1)^{1/\gamma} > 0,$$

and so $\hat x_{\gamma,\delta}$ is an interior optimum of $h_{\gamma,\delta}$.

On taking $\delta = \hat F_{\gamma,X}$, and in view of (B.16), Eq. (B.15) implies

$$\hat\eta = \gamma(1 - ((1/\gamma - 1)\hat F_{\gamma,X} + 1)^{1/\gamma}),$$

and therefore $\hat\eta$ does not depend on $\hat\beta$.

(ii) We have shown above that

$$\hat\eta_{\gamma,X} = \gamma\left(1 - \left((1/\gamma - 1)\hat F_{\gamma,X} + 1\right)^{1/\gamma}\right). \tag{B.18}$$

On substituting Eq. (B.18) into (B.15), we obtain the desired result.

(iii) Eq. (B.14) implies that $\hat\alpha$ is a maximizer for $\hat F_{\gamma,X}$ if and only if $\hat\beta = \hat\alpha\left(1 - \hat\eta_{\gamma,X}/\gamma\right)$ is a maximizer for $\hat\Phi_{\gamma,X}$. Substitution from (B.18) yields the desired result. □

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