

Existence of Subgame Perfect Equilibria

THEOREM: Every finite extensive form game has a subgame perfect equilibrium.

PROOF: Let $\Gamma = (X, P, C, A, p, u)$ be a finite EFG (as usual we index the players by $i \in I$ and their information sets by $j \in J^i$). Consider the finite sequence of games: $(\Gamma_k)_{k=0}^K = (X_k, P_k, C_k, A_k, p_k, u_k)_{k=0}^K$ ($K \in \mathbb{N}$ dependent on Γ will be specified later), defined recursively as follows:

$$\Gamma_0 = \Gamma,$$

and for $0 \leq k \leq K - 1$:

$$\begin{aligned} Z_{k+1} &= \{y \in X \setminus Z : \Gamma_{k,y} \text{ is a minimal subgame of } \Gamma_k\}, \\ X_{k+1} &= X_k \setminus \bigcup_{y \in Z_{k+1}} S(y), \\ \succeq_{k+1} &= \succeq_k |_{X_{k+1}}, \\ P_{k+1}^i &= P_k^i \cap X_{k+1}, \\ P_{k+1,j}^i &= P_{k,j}^i \cap X_{k+1}, \\ a_{k+1}^i &= a_k^i |_{C_k^i}, \\ p_{k+1} &= p_k |_{C^0}, \\ u_{k+1}(y) &= \sum_{z \in Z_k} P_{b_y}(y, z) u(z), \text{ where } b_y \text{ is some equilibrium of } \Gamma_y. \end{aligned}$$

The K is defined as the lowest index k , such that the above recursive formula gives the trivial outcome (i.e. the whole game is minimal).

The above algorithm produces also a behavioral strategy profile for Γ , defined as:

$$b = \prod_{k=1}^K \times_{y \in Z_k} b_y.$$

(observe, that the above cartesian product make sense, i.e. defines a strategy profile for Γ).

We claim that such a b is a subgame perfect equilibrium for Γ . The proof will be done by induction with respect to K . For $K = 1$ the statement is trivially true, because the only subgame is the whole game, so any Nash equilibrium is subgame perfect. Now suppose that the statement is true for any game, for which $K = m$. We will show that it also holds for $K = m + 1$. Indeed, consider the player i 's expected payoff, coming from playing the behavioral strategy profile $d^i \in B^i$ against b^{-i} of other players, which was obtained via the above algorithm:

$$\begin{aligned} E_{(d^i, b^{-i})}(u^i) &= \sum_{z \in Z} P_{(d^i, b^{-i})}(z) u^i(z) \\ &= \sum_{y \in Z_m} P_{(d^i, b^{-i})}(\alpha, y) \sum_{z \in Z \cap S(y)} P_{(d^i, b^{-i})}(y, z) u^i(z) \\ &= \sum_{y \in Z_m} P_{(d^i |_{\Gamma_m}, b^{-i} |_{\Gamma_m})}^{\Gamma_m}(y) \sum_{z \in Z \cap S(y)} P_{(d^i |_{\Gamma_y}, b^{-i} |_{\Gamma_y})}^{\Gamma_y}(z) u^i(z) \\ \text{(by our induction assumption } b^i |_{\Gamma_y} \text{ is a NE of } \Gamma_y) &\geq \sum_{y \in Z_m} P_{(d^i |_{\Gamma_m}, b^{-i} |_{\Gamma_m})}^{\Gamma_m}(y) \sum_{z \in Z \cap S(y)} P_{(b^i |_{\Gamma_y}, b^{-i} |_{\Gamma_y})}^{\Gamma_y}(z) u^i(z) \\ &= \sum_{y \in Z_m} P_{(d^i |_{\Gamma_m}, b^{-i} |_{\Gamma_m})}^{\Gamma_m}(y) u_m(y) \\ \text{(because } b^i |_{\Gamma_m} \text{ is a NE of } \Gamma_m) &\geq \sum_{y \in Z_m} P_{(b^i |_{\Gamma_m}, b^{-i} |_{\Gamma_m})}^{\Gamma_m}(y) u_m(y) = E_{(b^i, b^{-i})}(u^i). \end{aligned}$$

So we conclude that $b^i \in B^i(b^{-i})$ for all i (i was arbitrary), so b is an equilibrium of Γ and also, since it coincides with equilibria for all subgames (by construction), it is a subgame perfect equilibrium. \square