MACROECONOMIC THEORY **T.J. KEHOE**
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ANSWERS TO MIDTERM EXAMINATION

1. (a) With sequential market markets structure, there are markets for goods, labor services, capital services, and bonds open every period. Consumers sell labor services and rent capital to the firm. They buy goods from the firm, some of which they consume and some of which they save as capital. They trade bonds among themselves.

A **sequential markets equilibrium** is sequences of rental rates $\hat{r}_0^k, \hat{r}_1^k, \ldots$, interest rates $\hat{r}_0^b, \hat{r}_1^b, \ldots$, wages $\hat{w}_0, \hat{w}_1, \ldots$, consumption levels $\hat{c}_0, \hat{c}_1, \ldots$, capital stocks $\hat{k}_0, \hat{k}_1, \ldots$, and bond holdings $\hat{b}_0, \hat{b}_1, \ldots$, such that

- Given $\hat{r}_0^k, \hat{r}_1^k, \ldots$, $\hat{r}_0^b, \hat{r}_1^b, \ldots$, and $\hat{w}_0, \hat{w}_1, \ldots$, the consumer chooses $\hat{c}_0, \hat{c}_1, \ldots$, $\hat{k}_0, \hat{k}_1, \ldots$, and $\hat{b}_0, \hat{b}_1, \ldots$ to solve

$$\begin{aligned} &\max \sum_{t=0}^{\infty} \beta^t u(c_t) \\ \text{s.t. } & c_t + k_{t+1} + b_{t+1} \le \hat{w}_t + \left(1 + \hat{r}_t^k - \delta\right) k_t + \left(1 + \hat{r}_t^b\right) b_t,\ t = 0,\ 1,\ \ldots \\ & k_0 = \bar{k}_0 \\ & b_t \ge -B \\ & c_t, k_t \ge 0. \end{aligned}$$

Here $b_t \ge -B$, where $B > 0$, rules out Ponzi schemes but does not otherwise bind in equilibrium. [It is also possible to define a labor choice $\hat{\ell}_t$ if we impose the constraint $0 \le \ell_t \le 1$.]

- $\hat{r}_t^k = F_K(\hat{k}_t, 1)$, $t = 0,\ 1,\ \ldots$
  $\hat{w}_t = F_L(\hat{k}_t, 1)$, $t = 0,\ 1,\ \ldots$.

[A good answer would explain that these are the profit maximization conditions for constant returns.]

- $\hat{c}_t + \hat{k}_{t+1} - (1-\delta)\hat{k}_t = F(\hat{k}_t, 1),\ t = 0,\ 1,\ \ldots$

- $\hat{b}_t = 0,\ t = 0,\ 1,\ \ldots$

(b) A Pareto efficient allocation is sequences $\hat{c}_0, \hat{c}_1, \ldots, \hat{k}_0, \hat{k}_1, \ldots$ that are feasible,

$$\hat{c}_t + \hat{k}_{t+1} - (1-\delta)\hat{k}_t \le F\left(\hat{k}_t, 1\right),\ t = 0,\ 1,\ \ldots$$

$$\hat{k}_0 \le \bar{k}_0,$$

and such that there exists no alternative allocation $\tilde{c}_t, \tilde{k}_t$ that is also feasible and such that

$$\sum_{t=0}^{\infty} \beta^t u(\tilde{c}_t) > \sum_{t=0}^{\infty} \beta^t u(\hat{c}_t).$$

In other words, the allocation $\hat{c}_t, \hat{k}_t$ solves

$$\begin{aligned} &\max \sum_{t=0}^{\infty} \beta^t u(c_t) \\ \text{s.t. } & c_t + k_{t+1} - (1-\delta)k_t \le F(k_t, 1),\ t = 0,\ 1,\ \dots \\ & k_0 \le \bar{k}_0 \\ & c_t, k_t \ge 0. \end{aligned}$$

[It is also possible to include ℓ_t in $F(k_t, \ell_t)$ and impose the constraints $0 \le \ell_t \le 1$.]

(c) Suppose that $\hat{r}_t^k, \hat{r}_t^b, \hat{w}_t, \hat{c}_t, \hat{k}_t, \hat{b}_t$ is an equilibrium. Then

$$\hat{c}_t + \hat{k}_{t+1} - (1-\delta)\hat{k}_t = F(\hat{k}_t, 1),\ t = 0,\ 1,\ \dots$$
$$\hat{k}_0 = \bar{k}_0.$$

and, since $\hat{c}_0, \hat{c}_1, \dots,$ $\hat{k}_0, \hat{k}_1, \dots,$ and $\hat{b}_0, \hat{b}_1, \dots$ solve the consumer's problem, there exist Lagrange multipliers $p_t \ge 0,\ t = 0,\ 1,\ \dots$, such that

$$\beta^t u'(\hat{c}_t) - p_t = 0,\ t = 0,\ 1,\ \dots$$
$$-p_t + p_{t+1}(1 + \hat{r}_{t+1}^k - \delta) = 0,\ t = 0,\ 1,\ \dots$$
$$\lim_{t\to\infty} p_t \hat{k}_{t+1} = 0.$$

From the profit maximization conditions, we know that $\hat{r}_{t+1}^k = F_K(\hat{k}_{t+1}, 1)$. [We also know other things, like $\hat{b}_t = 0$, $-p_t + p_{t+1}(1 + \hat{r}_{t+1}^b) = 0$, and $\lim_{t\to\infty} p_t \hat{b}_{t+1} = 0$, but we do not need to use these conditions. Our assumptions on u and F do not rule out corner solutions, so, to be strictly correct, we should write the first order conditions as inequalities with complementary slackness conditions.]

The necessary and sufficient conditions for $\tilde{c}_t, \tilde{k}_t$ to be a Pareto efficient allocation are

$$\tilde{c}_t + \tilde{k}_{t+1} - (1-\delta)\tilde{k}_t = F(\tilde{k}_t, 1),\ t = 0,\ 1,\ \dots$$
$$\tilde{k}_0 = \bar{k}_0,$$

and that there exist some Lagrange multipliers $\pi_t \geq 0,\ t = 0,\ 1,\dots$, such that

$$\beta^t u'(\tilde{c}_t) - \pi_t = 0\,,\ t = 0,\ 1,\ \dots$$
$$-\pi_t + \pi_{t+1}\left(1 + F_K(\tilde{k}_{t+1}, 1) - \delta\right) = 0\,,\ t = 0,\ 1,\ \dots$$
$$\lim_{t\to\infty} \pi_t \tilde{k}_{t+1} = 0\,.$$

Given that $\hat{r}_t^k, \hat{r}_t^b, \hat{w}_t, \hat{c}_t, \hat{k}_t, \hat{b}_t$ is an equilibrium, we can set $\tilde{c}_t = \hat{c}_t$, $\tilde{k}_t = \hat{k}_t$, and $\pi_t = p_t$ and thus construct an allocation that satisfies the necessary and sufficient conditions for Pareto efficiency.

(d) A **sequential markets equilibrium** is sequences of rental rates $\hat{r}_0^k, \hat{r}_1^k, \dots$, interest rates $\hat{r}_0^b, \hat{r}_1^b, \dots$, wages $\hat{w}_0, \hat{w}_1, \dots$, consumption levels $\hat{c}_0^i, \hat{c}_0^i, \dots$, capital stocks $\hat{k}_0^i, \hat{k}_1^i, \dots$, and bond holdings $\hat{b}_0^i, \hat{b}_1^i, \dots$, $i = 1,\ 2$, such that

- Given $\hat{r}_0^k, \hat{r}_1^k, \dots$, $\hat{r}_0^b, \hat{r}_1^b, \dots$, and $\hat{w}_0, \hat{w}_1, \dots$, consumer i, $i = 1,\ 2$, chooses $\hat{c}_0^i, \hat{c}_0^i, \dots$, $\hat{k}_0^i, \hat{k}_1^i, \dots$, and $\hat{b}_0^i, \hat{b}_1^i, \dots$ to solve

$$\max \sum_{t=0}^{\infty} \beta_i^t u_i(c_t^i)$$
$$\text{s.t. } c_t^i + k_{t+1}^i + b_{t+1}^i \leq \hat{w}_t + \left(1 + \hat{r}_t^k - \delta\right)k_t^i + \left(1 + \hat{r}_t^b\right)b_t^i,\ t = 0,\ 1,\ \dots$$
$$k_0^i = \overline{k}_0^i$$
$$b_t^i \geq -B$$
$$c_t^i, k_t^i \geq 0.$$

- $\hat{r}_t^k = F_K\left(\hat{k}_t^1 + \hat{k}_t^2, \overline{\ell}^1 + \overline{\ell}^2\right),\ t = 0,\ 1,\ \dots$
  $\hat{w}_t = F_L\left(\hat{k}_t^1 + \hat{k}_t^2, \overline{\ell}^1 + \overline{\ell}^2\right),\ t = 0,\ 1,\ \dots.$

- $(\hat{c}_t^1 + \hat{c}_t^1) + (\hat{k}_{t+1}^1 + \hat{k}_{t+1}^2) - (1-\delta)(\hat{k}_t^1 + \hat{k}_t^2) = F(\hat{k}_t^1 + \hat{k}_t^2, \overline{\ell}^1 + \overline{\ell}^2),\ t = 0,\ 1,\ \dots$

- $\hat{b}_t^1 + \hat{b}_t^1 = 0,\ t = 0,\ 1,\ \dots$

(e) A Pareto efficient allocation plan is sequences $\hat{c}_0^i, \hat{c}_0^i, \dots$, $\hat{k}_0^i, \hat{k}_1^i, \dots$, $i = 1,\ 2$, that are feasible,

$$(\hat{c}_t^1 + \hat{c}_t^1) + (\hat{k}_{t+1}^1 + \hat{k}_{t+1}^2) - (1-\delta)(\hat{k}_t^1 + \hat{k}_t^2) = F(\hat{k}_t^1 + \hat{k}_t^2, \overline{\ell}^1 + \overline{\ell}^2),\ t = 0,\ 1,\ \dots$$
$$\hat{k}_0 \leq \overline{k}_0,$$

and such that there exists no alternative allocation $\tilde{c}_t^i, \tilde{k}_t^i$ that is also feasible and such that

$$\sum_{t=0}^{\infty} \beta_i^t u_i(\tilde{c}_t^i) \geq \sum_{t=0}^{\infty} \beta_i^t u_i(\hat{c}_t^i),\ i = 1,\ 2,$$

with strict inequality for some i.

2. (a) With an Arrow-Debreu markets structure futures markets for goods are open in period 1. Consumers trade futures contracts among themselves.

An **Arrow-Debreu equilibrium** is a sequence of prices $\hat{p}_1, \hat{p}_2, \ldots$ and an allocation $\hat{c}_1^0, (\hat{c}_1^1, \hat{c}_2^1), (\hat{c}_2^2, \hat{c}_3^2), \ldots$ such that

- Given $\hat{p}_1$, consumer 0 chooses $\hat{c}_1^0$ to solve

$$\begin{aligned} &\max\ c_1^0 \\ &\text{s.t}\ \ \hat{p}_1 c_1^0 \leq \hat{p}_1 w_2 + m \\ &c_1^0 \geq 0. \end{aligned}$$

- Given $\hat{p}_t, \hat{p}_{t+1}$, consumer t, $t = 1,\ 2,\ \ldots$, chooses $(\hat{c}_t^t, \hat{c}_{t+1}^t)$ to solve

$$\begin{aligned} &\max\ \ \log c_t^t + c_{t+1}^t \\ &\text{s.t.}\ \ \hat{p}_t c_t^t + \hat{p}_{t+1} c_{t+1}^t \leq \hat{p}_t w_1 + \hat{p}_{t+1} w_2 \\ &c_t^t, c_{t+1}^t \geq 0. \end{aligned}$$

- $\hat{c}_t^{t-1} + \hat{c}_t^t = w_2 + w_1,\ t = 1,\ 2,\ \ldots.$

(b) With sequential market markets structure, there are markets for goods and assets open every period. The consumers in generations $t-1$ and t trade goods and assets among themselves.

A **sequential markets equilibrium** is a sequence of interest rates $\hat{r}_1, \hat{r}_2, \ldots$, an allocation $\hat{c}_1^0, (\hat{c}_1^1, \hat{c}_2^1), (\hat{c}_2^2, \hat{c}_3^2), \ldots$, and asset holdings $\hat{s}_1^1, \hat{s}_2^2, \ldots$ such that

- Consumer 0 chooses $\hat{c}_1^0$ to solve

$$\begin{aligned} &\max\ c_1^0 \\ &\text{s.t}\ \ c_1^0 \leq w_2 + m \\ &c_1^0 \geq 0. \end{aligned}$$

- Given $\hat{r}_t$, consumer t, $t = 1,\ 2,\ \ldots$, chooses $(\hat{c}_t^t, \hat{c}_{t+1}^t)$ and $\hat{s}_t^t$ to solve

$$\begin{aligned} &\max \ \ \log c_t^t + c_{t+1}^t \\ &\text{s.t. } \ c_t^t + s_t^t \le w_1 \\ &c_{t+1}^t \le w_2 + (1+\hat{r}_t) s_t^t \\ &c_t^t, c_{t+1}^t \ge 0. \end{aligned}$$

- $\hat{c}_t^{t-1} + \hat{c}_t^t = w_2 + w_1$, $t = 1, 2, \ldots$.

- $\hat{s}_1^1 = m$
  $\hat{s}_t^t = \left[\prod_{\tau=1}^{t-1}(1+\hat{r}_\tau)\right] m$, $t = 2, 3, \ldots$.

(c) **Proposition 1:** Suppose that $\hat{p}_1, \hat{p}_2, \ldots, \hat{c}_1^0, (\hat{c}_1^1, \hat{c}_2^1), (\hat{c}_2^2, \hat{c}_3^2), \ldots$ is an Arrow-Debreu equilibrium. Then $\hat{r}_1, \hat{r}_2, \ldots, \hat{c}_1^0, (\hat{c}_1^1, \hat{c}_2^1), (\hat{c}_2^2, \hat{c}_3^2), \ldots, \hat{s}_1^1, \hat{s}_2^2, \ldots$ is a sequential markets equilibrium where

$$\hat{r}_t = \frac{\hat{p}_t}{\hat{p}_{t+1}} - 1 \text{ and } \hat{s}_t^t = w_1 - \hat{c}_t^t, \ t = 1, 2, \ldots.$$

Proposition 2: Suppose that $\hat{r}_1, \hat{r}_2, \ldots, \hat{c}_1^0, (\hat{c}_1^1, \hat{c}_2^1), (\hat{c}_2^2, \hat{c}_3^2), \ldots, \hat{s}_1^1, \hat{s}_2^2, \ldots$ is a sequential markets equilibrium. Then $\hat{p}_1, \hat{p}_2, \ldots, \hat{c}_1^0, (\hat{c}_1^1, \hat{c}_2^1), (\hat{c}_2^2, \hat{c}_3^2), \ldots$ is an Arrow-Debreu equilibrium where $\hat{p}_1 = 1$ and

$$\hat{p}_t = \prod_{\tau=1}^{t-1} \frac{1}{(1+\hat{r}_\tau)}, \ t = 2, 3, \ldots.$$

(d) Solving the problem of consumer t in the Arrow-Debreu world, we obtain

$$c_t^t = \frac{p_{t+1}}{p_t}, \ c_{t+1}^t = \frac{p_t w_1 + p_{t+1}(w_2 - 1)}{p_{t+1}}.$$

[Corner solutions are possible in general, but that is not relevant here.]

Since there is no fiat money, the unique equilibrium is autarky:

$$\hat{c}_t^t = w_1, \ \hat{c}_{t+1}^t = w_2.$$

Consequently,

$$\hat{p}_t = w_1^{t-1}.$$

Using proposition 1 from part c, we obtain

$$\hat{r}_t = \frac{1}{w_1} - 1,\ \hat{s}_t^t = 0,\ t = 1,\ 2,\ \ldots$$

in the sequential markets world.

(e) An **Arrow-Debreu equilibrium** is a sequence of prices $\hat{p}_1, \hat{p}_2, \ldots$ and an allocation $\hat{c}_1^{-1}, (\hat{c}_1^0, \hat{c}_2^0), (\hat{c}_1^1, \hat{c}_2^1, \hat{c}_3^1), (\hat{c}_2^2, \hat{c}_3^2, \hat{c}_4^2), \ldots$ such that

- Given $\hat{p}_1$, consumer -1 chooses $\hat{c}_1^{-1}$ to solve

$$\begin{aligned} &\max\ c_1^{-1} \\ &\text{s.t}\ \ \hat{p}_1 c_1^{-1} \leq \hat{p}_1 w_3 + m^{-1} \\ &c_1^{-1} \geq 0. \end{aligned}$$

- Given $\hat{p}_1, \hat{p}_2$, consumer 0 chooses $(\hat{c}_1^0, \hat{c}_2^0)$ to solve

$$\begin{aligned} &\max\ \ \gamma \log c_1^0 + c_2^0 \\ &\text{s.t.}\ \ \hat{p}_1 c_1^0 + \hat{p}_2 c_2^0 \leq \hat{p}_1 w_2 + \hat{p}_2 w_3 \\ &c_1^0, c_2^0 \geq 0. \end{aligned}$$

- Given $\hat{p}_t, \hat{p}_{t+1}, \hat{p}_{t+2}$, consumer t, $t = 1,\ 2,\ \ldots$, chooses $(\hat{c}_t^t, \hat{c}_{t+1}^t, \hat{c}_{t+2}^t)$ to solve

$$\begin{aligned} &\max\ \ \log c_t^t + \gamma \log c_{t+1}^t + c_{t+2}^t \\ &\text{s.t.}\ \ \hat{p}_t c_t^t + \hat{p}_{t+1} c_{t+1}^t + \hat{p}_{t+2} c_{t+2}^t \leq \hat{p}_t w_1 + \hat{p}_{t+1} w_2 + \hat{p}_{t+2} w_3 \\ &c_t^t, c_{t+1}^t, c_{t+2}^t \geq 0. \end{aligned}$$

- $\hat{c}_t^{t-2} + \hat{c}_t^{t-1} + \hat{c}_t^t = w_3 + w_2 + w_1,\ t = 1,\ 2,\ \ldots.$

A **sequential markets equilibrium** is a sequence of interest rates $\hat{r}_1, \hat{r}_2, \ldots$, an allocation $\hat{c}_1^{-1}, (\hat{c}_1^0, \hat{c}_2^0), (\hat{c}_1^1, \hat{c}_2^1, \hat{c}_3^1), (\hat{c}_2^2, \hat{c}_3^2, \hat{c}_4^2), \ldots$, and asset holdings $\hat{s}_1^0, (\hat{s}_1^1, \hat{s}_2^1), (\hat{s}_2^2, \hat{s}_3^2), \ldots$ such that

- Consumer -1 chooses $\hat{c}_1^{-1}$ to solve

$$\max \; c_1^{-1}$$
$$\text{s.t } \; c_1^{-1} \le w_3 + m^{-1}$$
$$c_1^{-1} \ge 0.$$

- Consumer 0 chooses $\left(\hat{c}_1^0, \hat{c}_2^0\right)$ and $\hat{s}_1^0$ to solve

$$\max \; \gamma \log c_1^0 + c_2^0$$
$$\text{s.t. } \; c_1^0 + s_1^0 \le w_2 + m^0$$
$$c_2^0 \le w_3 + (1+\hat{r}_1)s_1^0$$
$$c_1^0, c_2^0 \ge 0.$$

- Given $\hat{r}_t, \hat{r}_{t+1}$, consumer t, $t = 1,\ 2,\ \ldots$, chooses $\left(\hat{c}_t^t, \hat{c}_{t+1}^t, \hat{c}_{t+2}^t\right)$ and $\left(\hat{s}_t^t, \hat{s}_{t+1}^t\right)$ to solve

$$\max \;\; \log c_t^t + \gamma \log c_{t+1}^t + c_{t+2}^t$$
$$\text{s.t. } \; c_t^t + s_t^t \le w_1$$
$$c_{t+1}^t + s_{t+1}^t \le w_2 + (1+\hat{r}_t)s_t^t$$
$$c_{t+2}^t \le w_3 + (1+\hat{r}_{t+1})s_{t+1}^t$$
$$c_t^t, c_{t+1}^t, c_{t+2}^t \ge 0.$$

- $\hat{c}_t^{t-2} + \hat{c}_t^{t-1} + \hat{c}_t^t = w_3 + w_2 + w_1$, $t = 1,\ 2,\ \ldots.$

- $\hat{s}_1^0 + \hat{s}_1^1 = m^{-1} + m^0$
  $\hat{s}_t^{t-1} + \hat{s}_t^t = \left[\prod_{\tau=1}^{t-1}(1+\hat{r}_\tau)\right]\left(m^{-1} + m^0\right)$, $t = 2,\ 3,\ \ldots.$

3. (a) With sequential market markets structure, there are markets for goods, labor services, capital services, and bonds open every period. Consumers sell labor services and rent capital to the firm. They buy goods from the firm, some of which they consume and some of which they save as capital. They trade bonds among themselves.

A **sequential markets equilibrium** is sequences of rental rates $\hat{r}_0^k, \hat{r}_1^k, \ldots$, interest rates $\hat{r}_0^b, \hat{r}_1^b, \ldots$, wages $\hat{w}_0, \hat{w}_1, \ldots$, consumption levels $\hat{c}_0, \hat{c}_1, \ldots$, capital stocks $\hat{k}_0, \hat{k}_1, \ldots$, and bond holdings $\hat{b}_0, \hat{b}_1, \ldots$, such that

- Given $\hat{r}_0^k, \hat{r}_1^k, \ldots,$ $\hat{r}_0^b, \hat{r}_1^b, \ldots,$ and $\hat{w}_0, \hat{w}_1, \ldots$, the consumer chooses $\hat{c}_0, \hat{c}_1, \ldots,$ $\hat{k}_0, \hat{k}_1, \ldots,$ and $\hat{b}_0, \hat{b}_1, \ldots$ to solve

$$\max \sum_{t=0}^{\infty} \beta^t \log c_t$$

$$\text{s.t. } c_t + k_{t+1} + b_{t+1} \leq \hat{w}_t + \hat{r}_t^k k_t + \left(1+\hat{r}_t^b\right) b_t,\ t = 0,\ 1,\ \ldots$$

$$k_0 = \bar{k}_0$$

$$b_t \geq -B$$

$$c_t, k_t \geq 0.$$

- $\hat{r}_t^k = \alpha\theta\hat{k}_t^{\alpha-1}$, $t = 0,\ 1,\ \ldots$
  $\hat{w}_t = (1-\alpha)\theta\hat{k}_t^{\alpha}$, $t = 0,\ 1,\ \ldots.$

- $\hat{c}_t + \hat{k}_{t+1} = \theta\hat{k}_t^{\alpha},\ t = 0,\ 1,\ \ldots$

- $\hat{b}_t = 0,\ t = 0,\ 1,\ \ldots$

(b) A Pareto efficient allocation is sequences $\hat{c}_0, \hat{c}_1, \ldots, \hat{k}_0, \hat{k}_1, \ldots$ that are feasible,

$$\hat{c}_t + \hat{k}_{t+1} = \theta\hat{k}_t^{\alpha},\ t = 0,\ 1,\ \ldots$$
$$\hat{k}_0 \leq \bar{k}_0,$$

and such that there exists no alternative allocation $\tilde{c}_t, \tilde{k}_t$ that is also feasible and such that

$$\sum_{t=0}^{\infty} \beta^t \log \tilde{c}_t > \sum_{t=0}^{\infty} \beta^t \log \hat{c}_t\ .$$

In other words, the allocation $\hat{c}_t, \hat{k}_t$ solves

$$\max \sum_{t=0}^{\infty} \beta^t \log c_t$$

$$\text{s.t. } c_t + k_{t+1} \leq \theta\hat{k}_t^{\alpha},\ t = 0,\ 1,\ \ldots$$

$$k_0 \leq \bar{k}_0$$

$$c_t, k_t \geq 0.$$

(c) Suppose that $\hat{r}_t^k, \hat{r}_t^b, \hat{w}_t, \hat{c}_t, \hat{k}_t, \hat{b}_t$ is an equilibrium. Then

$$\hat{c}_t + \hat{k}_{t+1} = \theta\hat{k}_t^{\alpha},\ t = 0,\ 1,\ \ldots$$
$$\hat{k}_0 \leq \bar{k}_0,$$

and there exist Lagrange multipliers $p_t \geq 0,\ t = 0,\ 1,\ \ldots$, such that

$$\frac{\beta^t}{\hat{c}_t} - p_t = 0\,,\ t = 0,\ 1,\ \ldots$$
$$-p_t + p_{t+1}\alpha\theta\hat{k}_{t+1}^{\alpha-1} = 0\,,\ t = 0,\ 1,\ \ldots$$
$$\lim_{t\to\infty} p_t\hat{k}_{t+1} = 0\,.$$

The necessary and sufficient conditions for $\tilde{c}_t, \tilde{k}_t$ to be a Pareto efficient allocation are

$$\tilde{c}_t + \tilde{k}_{t+1} = \theta\tilde{k}_t^{\alpha}\,,\ t = 0,\ 1,\ \ldots$$
$$\tilde{k}_0 = \bar{k}_0,$$

and that there exist some Lagrange multipliers $\pi_t \geq 0,\ t = 0,\ 1, \ldots$, such that

$$\frac{\beta^t}{\tilde{c}_t} - \pi_t = 0\,,\ t = 0,\ 1,\ \ldots$$
$$-\pi_t + \pi_{t+1}\alpha\theta\tilde{k}_{t+1}^{\alpha-1} = 0\,,\ t = 0,\ 1,\ \ldots$$
$$\lim_{t\to\infty} \pi_t\tilde{k}_{t+1} = 0\,.$$

Given that $\hat{r}_t^k, \hat{r}_t^b, \hat{w}_t, \hat{c}_t, \hat{k}_t, \hat{b}_t$ is an equilibrium, we can set $\tilde{c}_t = \hat{c}_t,\ \tilde{k}_t = \hat{k}_t$, and $\pi_t = p_t$ and thus construct an allocation that satisfies the necessary and sufficient conditions for Pareto efficiency.

(d) Bellman's equation is

$$\begin{aligned} V(k) &= \max\ \ \log c + \beta V(k') \\ \text{s.t. }\ & c + k' \leq \theta k^{\alpha} \\ & c, k' \geq 0. \end{aligned}$$

Guessing that $V(k)$ has the form $a_0 + a_1 \log k$, we can solve for c and k':

$$c = \frac{1}{1+\beta a_1}\theta k^{\alpha}\,,\ k' = \frac{\beta a_1}{1+\beta a_1}\theta k^{\alpha}\,.$$

We can plug these solutions back into Bellman's equation to obtain

$$a_0 + a_1 \log k = \log\left(\frac{1}{1+\beta a_1}\theta k^{\alpha}\right) + \beta\left[a_0 + a_1 \log\left(\frac{\beta a_1}{1+\beta a_1}\theta k^{\alpha}\right)\right].$$

Collecting all the terms on the right-hand side that involve $\log k$, we can solve for a_1:

$$a_1 = \alpha + \alpha\beta a_1$$
$$a_1 = \frac{\alpha}{1-\alpha\beta},$$

which implies that

$$c = (1-\alpha\beta)\theta k^{\alpha},\ k' = \alpha\beta\theta k^{\alpha}.$$

[We can also solve for a_0:

$$a_0 = \frac{1}{1-\beta}\left[\log\left(\frac{\theta}{1+\beta a_1}\right) + \beta a_1 \log\left(\frac{\beta a_1 \theta}{1+\beta a_1}\right)\right]$$
$$a_0 = \frac{1}{1-\beta}\left[\log\left((1-\alpha\beta)\theta\right) + \frac{\alpha\beta}{1-\alpha\beta}\log\left(\alpha\beta\theta\right)\right],$$

but this is tedious, and, besides, the question does not ask us to do it.]

(e) To calculate the sequential markets equilibrium, we just run the first order difference equation

$$k_{t+1} = \alpha\beta\theta k_t^{\alpha}$$

forward, starting at $k_0 = \bar{k}_0$. We set

$$c_t = (1-\alpha\beta)\theta k_t^{\alpha}$$
$$b_t = 0$$
$$r_t^k = \alpha\theta k_t^{\alpha-1}$$
$$r_t^b = \alpha\theta k_t^{\alpha-1} - 1$$
$$w_t = (1-\alpha)\theta k_t^{\alpha}.$$

Notice that this problem actually has an analytical solution:

$$k_t = \alpha\beta\theta k_{t-1}^{\alpha} = \alpha\beta\theta\left(\alpha\beta\theta k_{t-2}^{\alpha}\right)^{\alpha} = \left(\alpha\beta\theta\right)^{\sum_{\tau=0}^{t-1}\alpha^{\tau}}\ \bar{k}_0^{\alpha^t} = \left(\alpha\beta\theta\right)^{\frac{1-\alpha^t}{1-\alpha}}\ \bar{k}_0^{\alpha^t}$$