MACROECONOMIC THEORY **T. J. KEHOE**
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ANSWERS TO MIDTERM EXAMINATION

1. (a) With an Arrow-Debreu markets structure futures markets for goods are open in period 0. Consumers trade futures contracts among themselves.

An **Arrow-Debreu equilibrium** is sequence of prices $\hat{p}_0, \hat{p}_1, \hat{p}_2, \ldots$ and consumption levels $\hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots; \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots$ such that

- Given $\hat{p}_0, \hat{p}_1, \hat{p}_2, \ldots$ $\hat{p}_1, \hat{p}_2, \ldots$, consumer i, $i = 1, 2$, chooses $\hat{c}_0^i, \hat{c}_1^i, \hat{c}_2^i, \ldots$ to solve

$$\max \ \sum_{t=0}^{\infty} \beta^t u_i(c_t^i)$$
$$\text{s.t.} \ \sum_{t=0}^{\infty} \hat{p}_t c_t^i \leq \sum_{t=0}^{\infty} \hat{p}_t w_t^i$$
$$c_t^i \geq 0 .$$

- $\hat{c}_t^1 + \hat{c}_t^2 = w_t^1 + w_t^2$, $t = 0,1,\ldots$.

(b) With sequential market markets structure, there are markets for goods and bonds open every period. Consumers trade goods and bonds among themselves.

A **sequential markets equilibrium** is sequences of interest rates $\hat{r}_0, \hat{r}_1, \hat{r}_2, \ldots$, consumption levels $\hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots; \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots$, and bond holdings $\hat{b}_0^1, \hat{b}_1^1, \hat{b}_2^1, \ldots$; $\hat{b}_0^2, \hat{b}_1^2, \hat{b}_2^2, \ldots$ such that

- Given $\hat{r}_0, \hat{r}_1, \hat{r}_2, \ldots$, the consumer i, $i = 1, 2$, chooses $\hat{c}_0^i, \hat{c}_1^i, \hat{c}_2^i, \ldots; \hat{b}_0^i, \hat{b}_1^i, \hat{b}_2^i, \ldots$ to solve

$$\max \sum_{t=0}^{\infty} \beta^t u_i(c_t)$$
$$\text{s.t.} \ c_0^i + b_0^i \leq w_0^i$$
$$c_t^i + b_t^i \leq w_t^i + \left(1 + \hat{r}_{t-1}\right) b_{t-1}^i, \ t = 1, 2, \ldots$$
$$b_t^i \geq -B$$
$$c_t^i \geq 0 .$$

Here $b_t \geq -B$, where $B > 0$ is chosen large enough, rules out Ponzi schemes but does not otherwise bind in equilibrium.

- $\hat{c}_t^1 + \hat{c}_t^2 = w_t^1 + w_t^2$, $t = 0,1,\ldots$

- $\hat{b}_t^1 + \hat{b}_t^2 = 0$, $t = 0,1,\ldots$

(c) **Proposition 1:** Suppose that $\hat{p}_0, \hat{p}_1, \hat{p}_2, \ldots; \hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots;\ \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots$ is an Arrow-Debreu equilibrium. Then $\hat{r}_0, \hat{r}_1, \hat{r}_2, \ldots,\ \hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots; \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots; \hat{b}_0^1, \hat{b}_1^1, \hat{b}_2^1, \ldots;\ \hat{b}_0^2, \hat{b}_1^2, \hat{b}_2^2, \ldots$ is a sequential markets equilibrium where

$$\hat{r}_t = \frac{\hat{p}_t}{\hat{p}_{t+1}} - 1$$
$$\hat{b}_0^i = w_0^i - \hat{c}_0^i$$
$$\hat{b}_t^i = w_t^i + (1+\hat{r}_{t-1})\hat{b}_{t-1}^i - \hat{c}_t^i,\ t = 1, 2, \ldots.$$

Proposition 2: Suppose that $\hat{r}_0, \hat{r}_1, \hat{r}_2, \ldots,\ \hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots; \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots; \hat{b}_0^1, \hat{b}_1^1, \hat{b}_2^1, \ldots;$ $\hat{b}_0^2, \hat{b}_1^2, \hat{b}_2^2, \ldots$ is a sequential markets equilibrium. Then $\hat{p}_0, \hat{p}_1, \hat{p}_2, \ldots; \hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots;$ $\hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots$ is an Arrow-Debreu equilibrium where

$$\hat{p}_0 = 1$$
$$\hat{p}_t = \prod_{s=0}^{t-1} \frac{1}{(1+\hat{r}_s)},\ t = 1, 2, \ldots.$$

(d) (There are a number of ways to do this.) The continuous differentiability and concavity of u_i implies that the necessary and sufficient conditions for a solution $\hat{c}_0^i, \hat{c}_1^i, \hat{c}_2^i, \ldots$ to the utility maximization problem of consumer i, $i = 1, 2$, in the Arrow-Debreu equilibrium is that there exists a Lagrange multiplier $\hat{\lambda}_i$ such that

$$\beta^t u_i'(\hat{c}_t^i) = \hat{\lambda}_i \hat{p}_t,\ t = 0, 1, \ldots$$
$$\sum_{t=0}^{\infty} \hat{p}_t c_t^i = \sum_{t=0}^{\infty} \hat{p}_t w_t^i .$$

Combining the two first order conditions in period t, $t = 0, 1, \ldots$, with the feasibility condition, we obtain

$$\frac{u_1'(\hat{c}_t^1)}{u_2'(\hat{c}_t^2)} = \frac{u_1'(\hat{c}_t^1)}{u_2'(w_t^1 + w_t^2 - \hat{c}_t^1)} = \frac{u_1'(\hat{c}_t^1)}{u_2'(4 - \hat{c}_t^1)} = \frac{\hat{\lambda}_1}{\hat{\lambda}_2}.$$

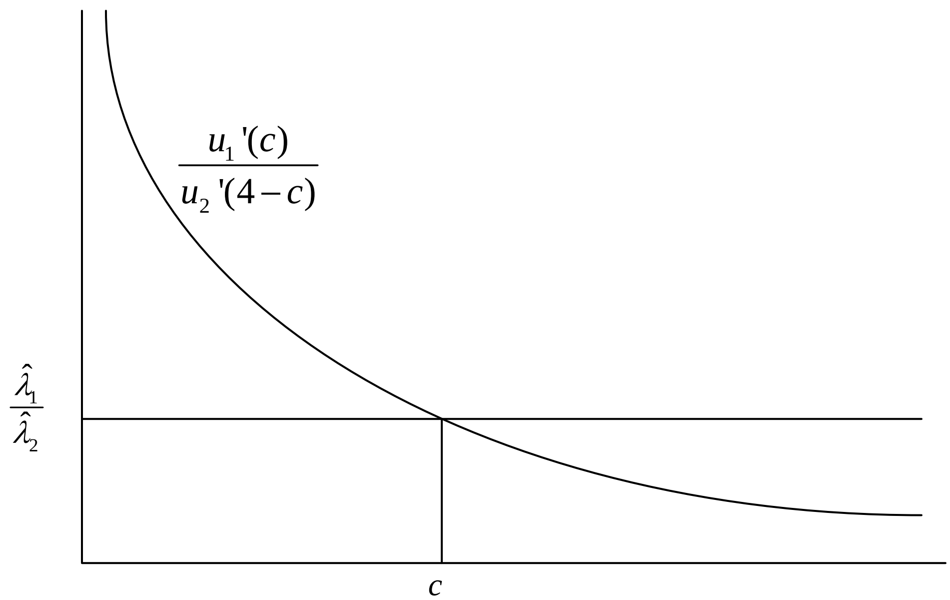
The assumptions that $u_i'(c) > 0$ and $u_i''(c) < 0$ for all $c > 0$ guarantee that $u_1'(c)/u_2'(4-c)$ is decreasing in c (in fact, the derivative of this function is

$$\frac{u_2'(4-c)u_1''(c) + u_1'(c)u_2''(4-c)}{u_2''(4-c)^2} < 0),$$

and $\lim_{c\to 0} u_i\,'(c) = \infty$ implies that

$$\lim_{c\to 0} \frac{u_1\,'(c)}{u_2\,'(4-c)} = \infty$$
$$\lim_{c\to \infty} \frac{u_1\,'(c)}{u_2\,'(4-c)} = 0\,.$$

Consequently, for any $\hat{\lambda}_1 / \hat{\lambda}_2 > 0$ these is a unique value of $c = \hat{c}_t^1$ that solves this equation. $\hat{c}_t^2 = 4 - c$.



Since $\beta^t u_i\,'(\hat{c}^i) = \hat{\lambda}_i \hat{p}_t$, if we normalize $\hat{p}_0 = 1$, $\hat{p}_t = \beta^t$. We use the budget constraint for consumer 1 to calculate $\hat{c}^1$:

$$\sum_{t=0}^{\infty} \hat{p}_t \hat{c}_t^1 = \sum_{t=0}^{\infty} \beta^t \hat{c}^1 = \sum_{t=0}^{\infty} \hat{p}_t w_t^1 = \sum_{t=0}^{\infty} \beta^t w_t^i = \sum_{t=0}^{\infty} \beta^{2t} 3 + \sum_{t=0}^{\infty} \beta^{2t+1}$$
$$\frac{1}{1-\beta}\hat{c}^1 = \frac{3}{1-\beta^2} + \frac{\beta}{1-\beta^2}$$
$$\hat{c}^1 = \frac{3+\beta}{1+\beta}$$

Consequently, $\hat{c}^2 = (1+3\beta)/(1+\beta)$. (Incidentally, $\hat{\lambda}_i = u_i\,'(\hat{c}^i)$, $i = 1, 2$.)

(e) A **sequential markets equilibrium** is sequences of interest rates $\hat{r}_0, \hat{r}_1, \hat{r}_2, \dots$, consumption levels $\hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \dots; \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \dots$, bond holdings $\hat{b}_0^1, \hat{b}_1^1, \hat{b}_2^1, \dots;\ \hat{b}_0^2, \hat{b}_1^2, \hat{b}_2^2, \dots$, and storage levels $\hat{x}_0^1, \hat{x}_1^1, \hat{x}_2^1, \dots;\ \hat{x}_0^2, \hat{x}_1^2, \hat{x}_2^2, \dots$ such that

- Given $\hat{r}_0, \hat{r}_1, \hat{r}_2, \dots$, the consumer i, $i = 1,2$, chooses $\hat{c}_0^i, \hat{c}_1^i, \hat{c}_2^i, \dots; \hat{b}_0^i, \hat{b}_1^i, \hat{b}_2^i, \dots;$ $\hat{x}_0^i, \hat{x}_1^i, \hat{x}_2^i, \dots$ to solve

$$\max \sum_{t=0}^{\infty} \beta^t u_i(c_t)$$
$$\text{s.t. } c_0^i + b_0^i + x_0^i \le w_0^i$$
$$c_t^i + b_t^i + x_t^i \le w_t^i + \left(1 + \hat{r}_{t-1}\right) b_{t-1}^i + \theta x_{t-1}^i,\ t = 1, 2, \dots$$
$$b_t^i \ge -B$$
$$c_t^i \ge 0.$$

- $\hat{c}_t^1 + \hat{c}_t^2 + \hat{x}_t^1 + \hat{x}_t^2 = w_t^1 + w_t^2 + \theta \hat{x}_{t-1}^1 + \theta \hat{x}_{t-1}^2$, $t = 0, 1, \dots$

- $\hat{b}_t^1 + \hat{b}_t^2 = 0$, $t = 0, 1, \dots$

(Alternatively, we could change to the notation that dates assets by the period in which they are paid off, which is more appropriate for economies with production, making the feasibility, for example,

$$\hat{c}_t^1 + \hat{c}_t^2 + \hat{x}_{t+1}^1 + \hat{x}_{t+1}^2 = w_t^1 + w_t^2 + \theta \hat{x}_t^1 + \theta \hat{x}_t^2 .)$$

2. (a) With an Arrow-Debreu markets structure futures markets for goods are open in period 1. Consumers trade futures contracts among themselves.

An **Arrow-Debreu equilibrium** is a sequence of prices $\hat{p}_1, \hat{p}_2, \dots$ and an allocation $\hat{c}_1^0, \left(\hat{c}_1^1, \hat{c}_2^1\right), \left(\hat{c}_2^2, \hat{c}_3^2\right), \dots$ such that

- Given $\hat{p}_1$, consumer 0 chooses $\hat{c}_1^0$ to solve

$$\max\ \log c_1^0$$
$$\text{s.t } \hat{p}_1 c_1^0 \le \hat{p}_1 w_2 + m$$
$$c_1^0 \ge 0.$$

- Given $\hat{p}_t, \hat{p}_{t+1}$, consumer t, $t = 1, 2, \dots$, chooses $\left(\hat{c}_t^t, \hat{c}_{t+1}^t\right)$ to solve

$$\max \ \log c_t^t + \log c_{t+1}^t$$
$$\text{s.t. } \hat{p}_t c_t^t + \hat{p}_{t+1} c_{t+1}^t \le \hat{p}_t w_1 + \hat{p}_{t+1} w_2$$
$$c_t^t, c_{t+1}^t \ge 0 .$$

- $\hat{c}_t^{t-1} + \hat{c}_t^t = w_2 + w_1$, $t = 1, 2, \ldots$.

(b) With sequential market markets structure, there are markets for goods and assets open every period. The consumers in generations $t-1$ and t trade goods and assets among themselves.

A **sequential markets equilibrium** is a sequence of interest rates $\hat{r}_1, \hat{r}_2, \ldots$, an allocation $\hat{c}_1^0, (\hat{c}_1^1, \hat{c}_2^1), (\hat{c}_2^2, \hat{c}_3^2), \ldots$, and asset holdings $\hat{s}_1^1, \hat{s}_2^2, \ldots$ such that

- Consumer 0 chooses $\hat{c}_1^0$ to solve

$$\max \ \log c_1^0$$
$$\text{s.t } c_1^0 \le w_2 + m$$
$$c_1^0 \ge 0 .$$

- Given $\hat{r}_t$, consumer t, $t = 1, 2, \ldots$, chooses $(\hat{c}_t^t, \hat{c}_{t+1}^t)$ and $\hat{s}_t^t$ to solve

$$\max \ \log c_t^t + \log c_{t+1}^t$$
$$\text{s.t. } c_t^t + s_t^t \le w_1$$
$$c_{t+1}^t \le w_2 + (1 + \hat{r}_t) s_t^t$$
$$c_t^t, c_{t+1}^t \ge 0 .$$

- $\hat{c}_t^{t-1} + \hat{c}_t^t = w_2 + w_1$, $t = 1, 2, \ldots$.

- $\hat{s}_1^1 = m$

  $\hat{s}_t^t = \left[\prod_{\tau=1}^{t-1} (1 + \hat{r}_\tau) \right] m$, $t = 2, 3, \ldots$.

(c) Since there is no fiat money, there is only one good per period, there is only one consumer type in each generation, and consumers live for only two periods, the equilibrium allocation is autarky:

$$\hat{c}_1^0 = w_2$$
$$(\hat{c}_t^t, \hat{c}_{t+1}^t) = (w_1, w_2)$$

The first order conditions from the consumers' problems in the Arrow-Debreu equilibrium imply that

$$\frac{\hat{p}_{t+1}}{\hat{p}_t} = \frac{\hat{c}_t^t}{\hat{c}_{t+1}^t} = \frac{w_1}{w_2}.$$

Normalizing $\hat{p}_1 = 1$, we obtain $\hat{p}_t = (w_1 / w_2)^{t-1}$. Similarly, the first order conditions from the consumers' problems in the sequential markets equilibrium, imply that

$$\frac{1}{1+\hat{r}_t} = \frac{\hat{c}_t^t}{\hat{c}_{t+1}^t} = \frac{w_1}{w_2}$$

or $\hat{r}_t = (w_2 / w_1) - 1$. Since the equilibrium allocation is autarky, $\hat{s}_t^t = 0$.

(d) An allocation $\hat{c}_1^0, (\hat{c}_1^1, \hat{c}_2^1), (\hat{c}_2^2, \hat{c}_3^2), \ldots$ is **feasible** if

$$\hat{c}_t^{t-1} + \hat{c}_t^t \leq w_2 + w_1,\ t = 1,\ 2,\ \ldots.$$

An allocation is **Pareto efficient** if it is feasible and thee exists no other allocation $\overline{c}_1^0, (\overline{c}_1^1, \overline{c}_2^1), (\overline{c}_2^2, \overline{c}_3^2), \ldots$ that is also feasible and satisfies

$$\log \overline{c}_1^0 \geq \log \hat{c}_1^0$$
$$\log \overline{c}_t^t + \log \overline{c}_{t+1}^t \geq \log \hat{c}_t^t + \log \hat{c}_{t+1}^t$$

with at least one inequality strict.

If $w_1 > w_2$, the equilibrium allocation, which is autarky, is not Pareto efficient because the alternative allocation

$$\overline{c}_1^0 = \frac{w_1 + w_2}{2}$$
$$(\overline{c}_t^t, \overline{c}_{t+1}^t) = \left(\frac{w_1 + w_2}{2}, \frac{w_1 + w_2}{2} \right)$$

is feasible and strictly dominates it. Let us verify:

$$\log \overline{c}_1^0 = \log \frac{w_1 + w_2}{2} > \log \hat{c}_1^0 = \log w_2$$
$$\frac{w_1 + w_2}{2} > w_2$$
$$w_1 > w_2$$

for generation 0, and

$$\log \bar{c}_t^t + \log \bar{c}_{t+1}^t = \log \frac{w_1 + w_2}{2} + \log \frac{w_1 + w_2}{2} > \log \hat{c}_t^t + \log \hat{c}_{t+1}^t = \log w_1 + \log w_2$$
$$\left(\frac{w_1 + w_2}{2}\right)^2 > w_1 w_2$$
$$w_1^2 + w_2^2 + 2w_1 w_2 > 4w_1 w_2$$
$$\left(w_1 - w_2\right)^2 > 0$$

for generation t, $t = 1, 2, \ldots$. (This also follows from the strict concavity of the log function.)

(e) A **sequential markets equilibrium** is a sequence of interest rates $\hat{r}_1, \hat{r}_2, \ldots$, an allocation $\hat{c}_1^{10}, \hat{c}_1^{20}, \left(\hat{c}_1^{11}, \hat{c}_2^{11}\right), \left(\hat{c}_1^{21}, \hat{c}_2^{21}\right), \left(\hat{c}_2^{12}, \hat{c}_3^{12}\right), \left(\hat{c}_2^{22}, \hat{c}_3^{22}\right) \ldots$, and asset holdings $\hat{s}_1^{11}, \hat{s}_1^{21}, \hat{s}_2^{12}, \hat{s}_2^{22} \ldots$ such

- Consumer $i0$ chooses $\hat{c}_1^{i0}$, $i = 1, 2$, to solve

$$\begin{aligned} &\max \ (1-\gamma_i) \log c_1^{i0} \\ &\text{s.t} \ \ c_1^{i0} \le w_2^i + m^i \\ &\qquad c_1^{i0} \ge 0. \end{aligned}$$

- Given $\hat{r}_t$, consumer it, $i = 1, 2$, $t = 1, 2, \ldots$, chooses $\left(\hat{c}_t^{it}, \hat{c}_{t+1}^{it}\right)$ and $\hat{s}_t^{it}$ to solve

$$\begin{aligned} &\max \ \gamma \log c_t^{it} + (1-\gamma) \log c_{t+1}^{it} \\ &\text{s.t.} \ \ c_t^{it} + s_t^{it} \le w_1^i \\ &\qquad c_{t+1}^{it} \le w_2^i + (1+\hat{r}_t) s_t^{it} \\ &\qquad c_t^{it}, c_{t+1}^{it} \ge 0. \end{aligned}$$

- $\hat{c}_t^{1t-1} + \hat{c}_t^{2t-1} + \hat{c}_t^{1t} + \hat{c}_t^{2t} = w_2^1 + w_2^2 + w_1^1 + w_1^2$, $t = 1, 2, \ldots$.

- $\hat{s}_1^1 + \hat{s}_1^2 = m^1 + m^2$
  $\hat{s}_t^{1t} + \hat{s}_t^{2t} = \left[\prod_{\tau=1}^{t-1} (1+\hat{r}_\tau)\right] (m^1 + m^2)$, $t = 2, 3, \ldots$.

3. (a) With Arrow-Debreu markets, there are futures markets of goods, capital, labor services, capital services open in period 0. Consumers sell labor services to firms. They buy goods from firms. Who makes the capital accumulation decision can be modeled different ways. We could have consumers buy and sell future claims to capital and sell claims to capital services to firms, or we could have consumers sell their initial capital to

firms and have firms buy and sell future claims to capital and sell claims to capital services to other firms.

An **Arrow-Debreu equilibrium** is sequences of prices of goods $\hat{p}_0, \hat{p}_1, \hat{p}_2, \ldots$, wages $\hat{w}_0, \hat{w}_1, \ldots$, rental rates $\hat{r}_0, \hat{r}_1, \hat{r}_2, \ldots$, consumption levels $\hat{c}_0, \hat{c}_1, \ldots$, and capital stocks $\hat{k}_0, \hat{k}_1, \ldots$ such that

- Given $\hat{p}_0, \hat{p}_1, \hat{p}_2, \ldots$, $\hat{w}_0, \hat{w}_1, \ldots$, and $\hat{r}_0$, the consumer chooses $\hat{c}_0, \hat{c}_1, \ldots$ to solve

$$\max \sum_{t=0}^{\infty} \beta^t \log c_t$$
$$\text{s.t. } \sum_{t=0}^{\infty} \hat{p}_t c_t \leq \sum_{t=0}^{\infty} \hat{w}_t + \hat{r}_0 \bar{k}_0$$
$$c_t \geq 0 .$$

(Here we have the consumers sell their initial capital to firms and have firms make capital accumulation decisions. If we have consumers make capital accumulation decisions, then consumers choose $\hat{k}_0, \hat{k}_1, \ldots$ and the budget constraint is $\sum_{t=0}^{\infty} \hat{p}_t \left(c_t + k_{t+1}\right) \leq \sum_{t=0}^{\infty} \left(\hat{w}_t + \hat{r}_t k_t\right)$.)

- $\hat{r}_t = \hat{p}_t \alpha \theta \hat{k}_t^{\alpha-1}$, $t = 0,1,\ldots$
  $\hat{w}_t = \hat{p}_t (1-\alpha) \theta \hat{k}_t^{\alpha}$, $t = 0,1,\ldots$
  $\hat{r}_{t+1} - \hat{p}_t \leq 0$, $= 0$ if $\hat{k}_{t+1} > 0$, $t = 0,1,\ldots$

(A good answer would explain that these are the profit maximization conditions for constant returns. Notice that, if we have consumers make capital accumulation decisions, then the zero profit condition on accumulating capital is a first order condition for utility maximization and does not need to be included as a separate equilibrium condition.)

- $\hat{c}_t + \hat{k}_{t+1} = \theta \hat{k}_t^{\alpha}$, $t = 0,1,\ldots$

(b) With sequential market markets structure, there are markets for goods, labor services, capital services, and bonds open every period. Consumers sell labor services and rent capital to the firm. They buy goods from the firm, some of which they consume and some of which they save as capital. They trade bonds among themselves.

A **sequential markets equilibrium** is sequences of rental rates $\hat{r}_0^k, \hat{r}_1^k, \ldots$, interest rates $\hat{r}_0^b, \hat{r}_1^b, \ldots$, wages $\hat{w}_0, \hat{w}_1, \ldots$, consumption levels $\hat{c}_0, \hat{c}_1, \ldots$, capital stocks $\hat{k}_0, \hat{k}_1, \ldots$, and bond holdings $\hat{b}_0, \hat{b}_1, \ldots$, such that

- Given $\hat{r}_0^k, \hat{r}_1^k, \ldots$, $\hat{r}_0^b, \hat{r}_1^b, \ldots$, and $\hat{w}_0, \hat{w}_1, \ldots$, the consumer chooses $\hat{c}_0, \hat{c}_1, \ldots$, $\hat{k}_0, \hat{k}_1, \ldots$, and $\hat{b}_0, \hat{b}_1, \ldots$ to solve

$$\max \sum_{t=0}^{\infty} \beta^t \log c_t$$
$$\text{s.t. } c_t + k_{t+1} + b_{t+1} \le \hat{w}_t + \hat{r}_t^k k_t + \left(1+\hat{r}_t^b\right) b_t ,\ t = 0,1,\ldots$$
$$k_0 = \bar{k}_0 ,\ b_0 = 0$$
$$b_t \ge -B ,\ \ c_t , k_t \ge 0 .$$

- $\hat{r}_t^k = \alpha\theta\hat{k}_t^{\alpha-1} ,\ t = 0,1,\ldots$
  $\hat{w}_t = (1-\alpha)\theta\hat{k}_t^{\alpha} ,\ t = 0,1,\ldots .$

- $\hat{c}_t + \hat{k}_{t+1} = \theta\hat{k}_t^{\alpha} ,\ t = 0,1,\ldots$

- $\hat{b}_t = 0 ,\ t = 0,1,\ldots$

(c) **Proposition 1:** Suppose that $\hat{p}_0, \hat{p}_1, \hat{p}_2, \ldots; \hat{w}_0, \hat{w}_1, \ldots; \hat{r}_0, \hat{r}_1, \hat{r}_2, \ldots; \hat{c}_0, \hat{c}_1, \ldots; \hat{k}_0, \hat{k}_1, \ldots$ is an Arrow-Debreu equilibrium. Then $\hat{r}_0^k, \hat{r}_1^k, \ldots; \hat{r}_0^b, \hat{r}_1^b, \ldots; \tilde{w}_0, \tilde{w}_1, \ldots; \hat{c}_0, \hat{c}_1, \ldots; \hat{k}_0, \hat{k}_1, \ldots;$ $\hat{b}_0, \hat{b}_1, \ldots$ is a sequential markets equilibrium where

$$\hat{r}_t^k = \frac{\hat{r}_t}{\hat{p}_t}$$
$$\hat{r}_t^b = \hat{r}_t^k - 1$$
$$\tilde{w}_t = \frac{\hat{w}_t}{\hat{p}_t}$$
$$\hat{b}_t = 0 ,\ t = 0,1,\ldots .$$

Proposition 2: Suppose that $\hat{r}_0^k, \hat{r}_1^k, \ldots; \hat{r}_0^b, \hat{r}_1^b, \ldots; \hat{w}_0, \hat{w}_1, \ldots; \hat{c}_0, \hat{c}_1, \ldots; \hat{k}_0, \hat{k}_1, \ldots;$ $\hat{b}_0, \hat{b}_1, \ldots$ is a sequential markets equilibrium. Then $\hat{p}_0, \hat{p}_1, \hat{p}_2, \ldots; \tilde{w}_0, \tilde{w}_1, \ldots; \hat{r}_0, \hat{r}_1, \hat{r}_2, \ldots;$ $\hat{c}_0, \hat{c}_1, \ldots; \hat{k}_0, \hat{k}_1, \ldots$ is an Arrow-Debreu equilibrium where

$$\hat{p}_0 = 1$$
$$\hat{p}_t = \prod_{s=1}^{t} \frac{1}{\hat{r}_s^k} ,\ t = 1, 2, \ldots$$
$$\hat{r} = \hat{p}_t \hat{r}_t^k$$
$$\tilde{w}_t = \hat{p}_t \hat{w}_t .$$

(Notice that, according to the way in which we have done things, we need to use separate symbols for the Arrow-Debreu wage, the price of labor services in period t in terms of period 0 goods, and the sequential markets wage, of labor services in period t in terms of period t goods.)

(d) A Pareto efficient allocation/production plan is sequences $\hat{c}_0, \hat{c}_1, \ldots, \hat{k}_0, \hat{k}_1, \ldots$ that are feasible,

$$\hat{c}_t + \hat{k}_{t+1} = \theta \hat{k}_t^{\alpha}, \; t = 0, \; 1, \; \ldots$$
$$\hat{k}_0 \leq \bar{k}_0,$$

and such that there exists no alternative allocation $\tilde{c}_t, \tilde{k}_t$ that is also feasible and such that

$$\sum_{t=0}^{\infty} \beta^t \log \tilde{c}_t > \sum_{t=0}^{\infty} \beta^t \log \hat{c}_t \, .$$

In other words, the allocation $\hat{c}_t, \hat{k}_t$ solves

$$\max \sum_{t=0}^{\infty} \beta^t \log c_t$$
$$\text{s.t. } \; c_t + k_{t+1} \leq \theta \hat{k}_t^{\alpha}, \; t = 0,1,\ldots$$
$$k_0 \leq \bar{k}_0$$
$$c_t, k_t \geq 0 \, .$$

Proposition: The allocation/production plan in a sequential markets equilibrium is Pareto efficient.

Proof: Suppose that $\hat{r}_t^k, \hat{r}_t^b, \hat{w}_t, \hat{c}_t, \hat{k}_t, \hat{b}_t$ is an equilibrium. Then

$$\hat{c}_t + \hat{k}_{t+1} = \theta \hat{k}_t^{\alpha}, t = 0,1,\ldots$$
$$\hat{k}_0 \leq \bar{k}_0 \, ,$$

and there exist Lagrange multipliers $p_t \geq 0$, $t = 0,1,\ldots$, such that

$$\frac{\beta^t}{\hat{c}_t} - p_t = 0 \, , \; t = 0,1,\ldots$$
$$-p_t + p_{t+1} \alpha \theta \hat{k}_{t+1}^{\alpha-1} = 0 \, , \; t = 0,1,\ldots$$
$$\lim_{t \to \infty} p_t \hat{k}_{t+1} = 0 \, .$$

The necessary and sufficient conditions for $\tilde{c}_t, \tilde{k}_t$ to be a Pareto efficient allocation/production plan are

$$\tilde{c}_t + \tilde{k}_{t+1} = \theta \tilde{k}_t^{\alpha}, \; t = 0,1,\ldots$$
$$\tilde{k}_0 = \bar{k}_0,$$

and that there exist some Lagrange multipliers $\pi_t \geq 0,\ t = 0,\ 1, \ldots$, such that

$$\frac{\beta^t}{\tilde{c}_t} - \pi_t = 0\,,\ t = 0,1,\ldots$$
$$-\pi_t + \pi_{t+1}\alpha\theta\tilde{k}_{t+1}^{\alpha-1} = 0\,,\ t = 0,1,\ldots$$
$$\lim_{t\to\infty} \pi_t \tilde{k}_{t+1} = 0\,.$$

Given that $\hat{r}_t^k, \hat{r}_t^b, \hat{w}_t, \hat{c}_t, \hat{k}_t, \hat{b}_t$ is an equilibrium, we can set $\tilde{c}_t = \hat{c}_t$, $\tilde{k}_t = \hat{k}_t$, and $\pi_t = p_t$ and thus construct an allocation that satisfies the necessary and sufficient conditions for Pareto efficiency.

(e) Bellman's equation is

$$V(k) = \max\ \log c + \beta V(k')$$
$$\text{s.t. } c + k' \leq \theta k^{\alpha}$$
$$c, k' \geq 0\,.$$

Guessing that $V(k)$ has the form $a_0 + a_1 \log k$, we can solve for c and k':

$$c = \frac{1}{1+\beta a_1}\theta k^{\alpha}\,,\ k' = \frac{\beta a_1}{1+\beta a_1}\theta k^{\alpha}\,.$$

We can plug these solutions back into Bellman's equation to obtain

$$a_0 + a_1 \log k = \log\left(\frac{1}{1+\beta a_1}\theta k^{\alpha}\right) + \beta\left[a_0 + a_1 \log\left(\frac{\beta a_1}{1+\beta a_1}\theta k^{\alpha}\right)\right].$$

Collecting all the terms on the right-hand side that involve $\log k$, we can solve for a_1:

$$a_1 = \alpha + \alpha\beta a_1$$
$$a_1 = \frac{\alpha}{1-\alpha\beta}\,,$$

which implies that

$$c = (1-\alpha\beta)\theta k^{\alpha}\,,\ k' = \alpha\beta\theta k^{\alpha}\,.$$

[We can also solve for a_0:

$$a_0 = \frac{1}{1-\beta}\left[\log\left(\frac{\theta}{1+\beta a_1}\right) + \beta a_1 \log\left(\frac{\beta a_1 \theta}{1+\beta a_1}\right)\right]$$

$$a_0 = \frac{1}{1-\beta}\left[\log\left((1-\alpha\beta)\theta\right) + \frac{\alpha\beta}{1-\alpha\beta}\log\left(\alpha\beta\theta\right)\right],$$

but this is tedious, and, besides, the question does not ask us to do it.]

(f) To calculate the sequential markets equilibrium, we just run the first order difference equation

$$k_{t+1} = \alpha\beta\theta k_t^{\alpha}$$

forward, starting at $k_0 = \bar{k}_0$. We set

$$\begin{aligned} c_t &= (1-\alpha\beta)\theta k_t^{\alpha} \\ b_t &= 0 \\ r_t^k &= \alpha\theta k_t^{\alpha-1} \\ r_t^b &= \alpha\theta k_t^{\alpha-1} - 1 \\ w_t &= (1-\alpha)\theta k_t^{\alpha}\ . \end{aligned}$$

Notice that this problem actually has an analytical solution:

$$k_t = \alpha\beta\theta k_{t-1}^{\alpha} = \alpha\beta\theta\left(\alpha\beta\theta k_{t-2}^{\alpha}\right)^{\alpha} = \left(\alpha\beta\theta\right)^{\sum_{\tau=0}^{t-1}\alpha^{\tau}}\ \bar{k}_0^{\alpha^t} = \left(\alpha\beta\theta\right)^{\frac{1-\alpha^t}{1-\alpha}}\ \bar{k}_0^{\alpha^t}$$