MACROECONOMIC THEORY **T. J. KEHOE**
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ANSWERS TO MIDTERM EXAMINATION

1. (a) With an Arrow-Debreu markets structure futures markets for goods are open in period 0. Consumers trade futures contracts among themselves.

An **Arrow-Debreu equilibrium** is sequence of prices $\hat{p}_0, \hat{p}_1, \hat{p}_2, \ldots$ and consumption levels $\hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots$; $\hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots$ such that

- Given $\hat{p}_0, \hat{p}_1, \hat{p}_2, \ldots$, consumer i, $i = 1,2$, chooses $\hat{c}_0^i, \hat{c}_1^i, \hat{c}_2^i, \ldots$ to solve

$$\max \ \sum_{t=0}^{\infty} \beta^t \log c_t^i$$
$$\text{s.t. } \sum_{t=0}^{\infty} \hat{p}_t c_t^i \leq \sum_{t=0}^{\infty} \hat{p}_t w_t^i$$
$$c_t^i \geq 0.$$

- $\hat{c}_t^1 + \hat{c}_t^2 = w_t^1 + w_t^2$, $t = 0,1,\ldots$.

(b) With sequential market markets structure, there are markets for goods and bonds open every period. Consumers trade goods and bonds among themselves.

A **sequential markets equilibrium** is sequences of interest rates $\hat{r}_1, \hat{r}_2, \hat{r}_3, \ldots$, consumption levels $\hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots$; $\hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots$, and asset holdings $\hat{b}_1^1, \hat{b}_2^1, \hat{b}_3^1, \ldots$; $\hat{b}_1^2, \hat{b}_2^2, \hat{b}_3^2, \ldots$ such that

- Given $\hat{r}_1, \hat{r}_2, \hat{r}_3, \ldots$, the consumer i, $i = 1,2$, chooses $\hat{c}_0^i, \hat{c}_1^i, \hat{c}_2^i, \ldots; \hat{b}_1^i, \hat{b}_2^i, \hat{b}_3^i, \ldots$ to solve

$$\max \sum_{t=0}^{\infty} \beta^t \log c_t^i$$
$$\text{s.t. } c_0^i + b_1^i \leq w_0^i$$
$$c_t^i + b_{t+1}^i \leq w_t^i + (1 + \hat{r}_t) b_t^i, \ t = 1,2,\ldots$$
$$b_t^i \geq -B, \ c_t^i \geq 0$$
$$b_0^i = 0.$$

Here $b_t^i \geq -B$, where $B > 0$ is chosen large enough, rules out Ponzi schemes but does not otherwise bind in equilibrium.

- $\hat{c}_t^1 + \hat{c}_t^2 = w_t^1 + w_t^2$, $t = 0,1,\ldots$

- $\hat{b}_t^1 + \hat{b}_t^2 = 0$, $t = 0,1,\ldots$

(c) **Proposition 1:** Suppose that $\hat{p}_0, \hat{p}_1, \hat{p}_2, \ldots;\ \hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots;\ \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots$ is an Arrow-Debreu equilibrium. Then $\hat{r}_1, \hat{r}_2, \hat{r}_3, \ldots;\ \hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots;\ \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots;\ \hat{b}_1^1, \hat{b}_2^1, \hat{b}_3^1, \ldots;$ $\hat{b}_1^2, \hat{b}_2^2, \hat{b}_3^2, \ldots$ is a sequential markets equilibrium where

$$\hat{r}_t = \frac{\hat{p}_{t-1}}{\hat{p}_t} - 1$$
$$\hat{b}_1^i = w_0^i - \hat{c}_0^i$$
$$\hat{b}_{t+1}^i = w_t^i + (1+\hat{r}_t)\hat{b}_t^i - \hat{c}_t^i,\ t = 1,2,\ldots.$$

Proposition 2: Suppose that $\hat{r}_1, \hat{r}_2, \hat{r}_3, \ldots;\ \hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots;\ \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots;\ \hat{b}_1^1, \hat{b}_2^1, \hat{b}_3^1, \ldots;$ $\hat{b}_1^2, \hat{b}_2^2, \hat{b}_3^2, \ldots$ is a sequential markets equilibrium. Then $\hat{p}_0, \hat{p}_1, \hat{p}_2, \ldots; \hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots;$ $\hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots$ is an Arrow-Debreu equilibrium where

$$\hat{p}_0 = 1$$
$$\hat{p}_t = \prod_{s=1}^{t} \frac{1}{(1+\hat{r}_s)},\ t = 1,2,\ldots.$$

(d) Using the two consumers' first order conditions

$$\frac{\beta^t}{c_t^i} = \lambda^i p_t,$$

we can write

$$\frac{c_t^1}{c_t^2} = \frac{\lambda^2}{\lambda^1}.$$

In even periods,

$$c_t^1 + c_t^2 = 4$$
$$c_t^1 + \frac{\lambda^1}{\lambda^2} c_t^1 = 4$$
$$c_t^1 = \frac{\lambda^2}{\lambda^1 + \lambda^2} 4.$$

Similarly, in odd periods,

$$c_t^1 = \frac{\lambda^2}{\lambda^1 + \lambda^2} 2 .$$

Normalizing $p_0 = 1$, we can use the first order condition to write

$$p_t = \begin{cases} \beta^t & \text{if } t \text{ is even} \\ 2\beta^t & \text{if } t \text{ is odd} \end{cases},$$

which implies that

$$p_t c_t^1 = \beta^t \frac{4\lambda^2}{\lambda^1 + \lambda^2} .$$

Consequently,

$$\begin{aligned}
\sum\nolimits_{t=0}^{\infty} p_t c_t^1 = \frac{4\lambda^2}{\lambda^1 + \lambda^2} \sum\nolimits_{t=0}^{\infty} \beta^t &= \frac{1}{1-\beta} \frac{4\lambda^2}{\lambda^1 + \lambda^2} = \sum\nolimits_{t=0}^{\infty} p_t w_t^1 \\
\frac{1}{1-\beta} \frac{4\lambda^2}{\lambda^1 + \lambda^2} &= 3\sum\nolimits_{t=0}^{\infty} p_{2t} + \sum\nolimits_{t=0}^{\infty} p_{2t+1} \\
\frac{1}{1-\beta} \frac{4\lambda^2}{\lambda^1 + \lambda^2} &= 3\sum\nolimits_{t=0}^{\infty} \beta^{2t} + 2\beta \sum\nolimits_{t=0}^{\infty} \beta^{2t} \\
\frac{1}{1-\beta} \frac{4\lambda^2}{\lambda^1 + \lambda^2} &= \frac{3+2\beta}{1-\beta^2} \\
\frac{\lambda^2}{\lambda^1 + \lambda^2} &= \frac{3+2\beta}{4(1+\beta)},
\end{aligned}$$

which implies that

$$\frac{\lambda^1}{\lambda^1 + \lambda^2} = \frac{1+2\beta}{4(1+\beta)} .$$

$$c_t^1 = \begin{cases} \dfrac{3+2\beta}{1+\beta} & \text{if } t \text{ is even} \\ \dfrac{3+2\beta}{2(1+\beta)} & \text{if } t \text{ is odd} \end{cases}$$

$$c_t^2 = \begin{cases} \dfrac{1+2\beta}{1+\beta} & \text{if } t \text{ is even} \\ \dfrac{1+2\beta}{2(1+\beta)} & \text{if } t \text{ is odd} \end{cases} .$$

(We can even work out λ^1 and λ^2, although the question does not require this and it would be a waste of precious time to do so during the exam.

$$\lambda^1 = \frac{1}{c_0^1} = \frac{1+\beta}{3+2\beta}$$
$$\lambda^2 = \frac{1}{c_0^2} = \frac{1+\beta}{1+2\beta}.$$

Check:

$$\frac{\lambda^1}{\lambda^1+\lambda^2} = \frac{\frac{1+\beta}{3+2\beta}}{\frac{1+\beta}{3+2\beta}+\frac{1+\beta}{1+2\beta}} = \frac{\frac{1}{3+2\beta}}{\frac{1}{3+2\beta}+\frac{1}{1+2\beta}} = \frac{1+2\beta}{1+2\beta+3+2\beta} = \frac{1+2\beta}{4(1+\beta)}.)$$

To calculate the sequential markets equilibrium, we just use the formulas from proposition 1 in part c. For example,

$$r_t = \frac{\hat{p}_{t-1}}{\hat{p}_t} - 1 = \begin{cases} \frac{1}{2\beta} - 1 & \text{if } t \text{ is odd} \\ \frac{2}{\beta} - 1 & \text{if } t \text{ is even} \end{cases}.$$

Notice that, in $t = 0$,

$$\hat{b}_1^1 = 3 - \frac{3+2\beta}{1+\beta} = \frac{\beta}{1+\beta}.$$

That is, in even periods, consumer 1 lends $\frac{\beta}{1+\beta}$ to consumer 2, who pays back $\frac{1}{2(1+\beta)}$ in odd periods.

(e) A **Pareto efficient allocation** is an allocation $\hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots; \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots$ that is **feasible**,

$$\hat{c}_t^1 + \hat{c}_t^2 \le w_t^1 + w_t^2,\ t = 0,1,\ldots,$$

and is such that there is no other feasible allocation $\overline{c}_0^1, \overline{c}_1^1, \overline{c}_2^1, \ldots; \overline{c}_0^2, \overline{c}_1^2, \overline{c}_2^2, \ldots$ that is also feasible,

$$\overline{c}_t^1 + \overline{c}_t^2 \le w_t^1 + w_t^2,\ t = 0,1,\ldots,$$

and satisfies

$$\sum_{t=0}^{\infty} \beta^t \log \overline{c}_t^i \geq \sum_{t=0}^{\infty} \beta^t \log \hat{c}_t^i \, , \; i = 1,2 \, ,$$

with at least one of the two inequalities being strict.

Proposition 3. Suppose that $\hat{p}_0, \hat{p}_1, \hat{p}_2, \ldots; \; \hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots; \; \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots$ is an Arrow-Debreu equilibrium. Then $\hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots; \; \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots$ is a Pareto efficient allocation.

Proof. Suppose not, that there is an allocation $\overline{c}_0^1, \overline{c}_1^1, \overline{c}_2^1, \ldots; \; \overline{c}_0^2, \overline{c}_1^2, \overline{c}_2^2, \ldots$ that is feasible and Pareto superior. $\sum_{t=0}^{\infty} \beta^t \log \overline{c}_t^i > \sum_{t=0}^{\infty} \beta^t \log \hat{c}_t^i$ implies that

$$\sum_{t=0}^{\infty} \hat{p}_t \overline{c}_t^i > \sum_{t=0}^{\infty} \hat{p}_t w_t^i$$

because, otherwise, $\hat{c}_0^i, \hat{c}_1^i, \hat{c}_2^i, \ldots$ would not be utility maximizing.
$\sum_{t=0}^{\infty} \beta^t \log \overline{c}_t^i \geq \sum_{t=0}^{\infty} \beta^t \log \hat{c}_t^i$ implies that

$$\sum_{t=0}^{\infty} \hat{p}_t \overline{c}_t^i \geq \sum_{t=0}^{\infty} \hat{p}_t w_t^i \, .$$

Otherwise, we could set $\tilde{c}_0^i = \overline{c}_0^i + \left(\sum_{t=0}^{\infty} \hat{p}_t w_t^i - \sum_{t=0}^{\infty} \hat{p}_t \overline{c}_t^i \right) / \, \hat{p}_0$ and $\tilde{c}_t^i = \overline{c}_t^i$, $t = 1,2,...$ and obtain a consumption plan $\tilde{c}_0^i, \tilde{c}_1^i, \tilde{c}_2^i, \ldots$ that satisfies the budget constraint and yields strictly higher utility than $\hat{c}_0^i, \hat{c}_1^i, \hat{c}_2^i, \ldots$. Adding the inequalities for the two consumers together yields

$$\sum_{t=0}^{\infty} \hat{p}_t (\overline{c}_t^1 + \overline{c}_t^2) > \sum_{t=0}^{\infty} \hat{p}_t (w_t^1 + w_t^2) \, .$$

Notice that $\sum_{t=0}^{\infty} \hat{p}_t w_t^i < \infty$, $i = 1,2$, for utility maximization to make sense, so that this last inequality makes sense. (That is, we are not saying $\infty > \infty$, which is nonsense.) Since utility is strictly increasing, prices $\hat{p}_t$ are strictly positive. Multiply the condition that $\overline{c}_0^1, \overline{c}_1^1, \overline{c}_2^1, \ldots; \; \overline{c}_0^2, \overline{c}_1^2, \overline{c}_2^2, \ldots$ be feasible in period t by $\hat{p}_t$ and adding up $t = 0,1,...$, we obtain

$$\sum_{t=0}^{\infty} \hat{p}_t (\overline{c}_t^1 + \overline{c}_t^2) \leq \sum_{t=0}^{\infty} \hat{p}_t (w_t^1 + w_t^2) \, ,$$

which is a contradiction.■

Proposition 4. Suppose that $\hat{r}_1, \hat{r}_2, \hat{r}_3, \ldots;\ \hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots;\ \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots;\ \hat{b}_1^1, \hat{b}_2^1, \hat{b}_3^1, \ldots;$ $\hat{b}_1^2, \hat{b}_2^2, \hat{b}_3^2, \ldots$ is a sequential markets equilibrium. Then $\hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots;\ \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots$ is a Pareto efficient allocation.

Proof: Proposition 2 implies that $\hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots;\ \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots$ is the equilibrium allocation of an Arrow-Debreu equilibrium. Proposition 3 implies that it is Pareto efficient.■

We could also answer this question using first order conditions from the consumers' problems and first order conditions from the Pareto problem.

2. (a) With an Arrow-Debreu markets structure futures markets for goods are open in period 1. Consumers trade futures contracts among themselves.

An **Arrow-Debreu equilibrium** is a sequence of prices $\hat{p}_1, \hat{p}_2, \ldots$ and an allocation $\hat{c}_1^0, (\hat{c}_1^1, \hat{c}_2^1), (\hat{c}_2^2, \hat{c}_3^2), \ldots$ such that

- Given $\hat{p}_1$, consumer 0 chooses $\hat{c}_1^0$ to solve

$$\begin{aligned} &\max\ 3\log c_1^0 \\ &\text{s.t}\ \ \hat{p}_1 c_1^0 \le \hat{p}_1 w_2 + m \\ &c_1^0 \ge 0. \end{aligned}$$

- Given $\hat{p}_t, \hat{p}_{t+1}$, consumer *t*, $t = 1, 2, \ldots$, chooses $(\hat{c}_t^t, \hat{c}_{t+1}^t)$ to solve

$$\begin{aligned} &\max\ \ \log c_t^t + 3\log c_{t+1}^t \\ &\text{s.t.}\ \ \hat{p}_t c_t^t + \hat{p}_{t+1} c_{t+1}^t \le \hat{p}_t w_1 + \hat{p}_{t+1} w_2 \\ &c_t^t, c_{t+1}^t \ge 0. \end{aligned}$$

- $\hat{c}_t^{t-1} + \hat{c}_t^t = w_2 + w_1$, $t = 1,\ 2,\ \ldots$.

(b) With sequential market markets structure, there are markets for goods and assets open every period. The consumers in generations $t-1$ and t trade goods and assets among themselves.

A **sequential markets equilibrium** is a sequence of interest rates $\hat{r}_2, \hat{r}_3, \ldots$, an allocation $\hat{c}_1^0, (\hat{c}_1^1, \hat{c}_2^1), (\hat{c}_2^2, \hat{c}_3^2), \ldots$, and asset holdings $\hat{b}_2^1, \hat{b}_3^2, \ldots$ such that

- Consumer 0 chooses $\hat{c}_1^0$ to solve

$$\begin{aligned} &\max \ 3\log c_1^0 \\ &\text{s.t } c_1^0 \le w_2 + m \\ &c_1^0 \ge 0 . \end{aligned}$$

- Given $\hat{r}_{t+1}$, consumer t, $t = 1, 2, \ldots$, chooses $(\hat{c}_t^t, \hat{c}_{t+1}^t)$ and $\hat{b}_{t+1}^t$ to solve

$$\begin{aligned} &\max \ \log c_t^t + 3\log c_{t+1}^t \\ &\text{s.t. } c_t^t + b_{t+1}^t \le w_1 \\ &c_{t+1}^t \le w_2 + (1+\hat{r}_{t+1}) b_{t+1}^t \\ &c_t^t, c_{t+1}^t \ge 0 . \end{aligned}$$

- $\hat{c}_t^{t-1} + \hat{c}_t^t = w_2 + w_1$, $t = 1, 2, \ldots$.
- $\hat{b}_2^1 = m$, $\hat{b}_{t+1}^t = \left[\prod_{\tau=2}^{t} (1+\hat{r}_\tau)\right] m$, $t = 2, 3, \ldots$.

(c) Since there is no fiat money, there is only one good per period, there is only one consumer type in each generation, and consumers live for only two periods, the equilibrium allocation is autarky:

$$\begin{aligned} \hat{c}_1^0 &= w_2 \\ (\hat{c}_t^t, \hat{c}_{t+1}^t) &= (w_1, w_2) \end{aligned}$$

The first order conditions from the consumers' problems in the Arrow-Debreu equilibrium imply that

$$\frac{\hat{c}_{t+1}^t}{3\hat{c}_t^t} = \frac{\hat{p}_t}{\hat{p}_{t+1}} .$$

Normalizing $\hat{p}_1 = 1$, we obtain $\hat{p}_t = (3w_1 / w_2)^{t-1}$. Similarly, the first order conditions from the consumers' problems in the sequential markets equilibrium, imply that

$$1 + \hat{r}_{t+1} = \frac{\hat{c}_{t+1}^t}{3\hat{c}_t^t} = \frac{w_2}{3w_1}$$

or $\hat{r}_t = w_2 / (3w_1) - 1$. Since the equilibrium allocation is autarky, $\hat{b}_{t+1}^t = 0$.

(d) An allocation $\hat{c}_1^0, (\hat{c}_1^1, \hat{c}_2^1), (\hat{c}_2^2, \hat{c}_3^2), \ldots$ is **feasible** if

$$\hat{c}_t^{t-1} + \hat{c}_t^t \le w_2 + w_1 , \ t = 1,\ 2,\ \ldots .$$

An allocation is **Pareto efficient** if it is feasible and there exists no other allocation $\bar{c}_1^0, (\bar{c}_1^1, \bar{c}_2^1), (\bar{c}_2^2, \bar{c}_3^2), \ldots$ that is also feasible and satisfies

$$\log \bar{c}_1^0 \ge \log \hat{c}_1^0$$

$$\log \overline{c}_t^t + \log \overline{c}_{t+1}^t \geq \log \hat{c}_t^t + \log \hat{c}_{t+1}^t, \; t = 1, 2, \ldots,$$

with at least one inequality strict.

If $(w_1, w_2) = (2, 2)$, the equilibrium allocation is not Pareto efficient. To see this, we consider the alternative allocation

$$\overline{c}_1^0 = 3$$
$$(\overline{c}_t^t, \overline{c}_{t+1}^t) = (1, 3).$$

Notice that, since

$$\hat{c}_t^{t-1} + \hat{c}_t^t = w_2 + w_1, \; t = 1, \; 2, \; \ldots,$$

this alternative allocation is feasible. Since

$$\overline{c}_1^0 = 3 > 1 = \hat{c}_1^0,$$
$$\log \overline{c}_1^0 > \log \hat{c}_1^0.$$

Notice that

$$\log \overline{c}_t^t + 3 \log \overline{c}_{t+1}^t = \log 1 + 3 \log 3 = \log 27$$

while

$$\log \hat{c}_t^t + 3 \log \hat{c}_{t+1}^t = \log 2 + 3 \log 2 = \log 16.$$

Consequently,

$$\log \overline{c}_t^t + 3 \log \overline{c}_{t+1}^t > \log \hat{c}_t^t + 3 \log \hat{c}_{t+1}^t.$$

(How do we get $(\overline{c}_t^t, \overline{c}_{t+1}^t) = (1, 3)$? Solve the consumer's problem in the Arrow-Debreu equilibrium where $p_{t-1} = p_t = 1$.)

(e) A **sequential markets equilibrium** is a sequence of interest rates $\hat{r}_2, \hat{r}_3, \ldots$, an allocation $\hat{c}_1^0, (\hat{c}_1^1, \hat{c}_2^1), (\hat{c}_2^2, \hat{c}_3^2), \ldots$, asset holdings $\hat{b}_2^1, \hat{b}_3^2, \ldots$, and storage holdings $\hat{x}_2^1, \hat{x}_3^2, \ldots$ such that

- Consumer 0 chooses $\hat{c}_1^0$ to solve

$$\max \ 3\log c_1^0$$
$$\text{s.t } \ c_1^0 \le w_2 + m + \theta \overline{x}_1^0$$
$$c_1^0 \ge 0 \,.$$

Here $\overline{x}_1^0$ are the initial storage holdings brought into period 1 of generation 0.

- Given $\hat{r}_t$, consumer t, $t = 1,2,\dots$, chooses $(\hat{c}_t^t, \hat{c}_{t+1}^t)$ and $\hat{b}_{t+1}^t$ to solve

$$\max \ \log c_t^t + 3\log c_{t+1}^t$$
$$\text{s.t. } \ c_t^t + b_{t+1}^t + x_{t+1}^t \le w_1$$
$$c_{t+1}^t \le w_2 + (1+\hat{r}_{t+1})b_{t+1}^t + \theta x_{t+1}^t$$
$$c_t^t, c_{t+1}^t, x_{t+1}^t \ge 0 \,.$$

- $\hat{c}_t^{t-1} + \hat{c}_t^t + \hat{x}_{t+1}^t = w_2 + w_1 + \theta \hat{x}_t^{t-1}$, $t = 1,2,\dots.$
- $\hat{b}_2^1 = m$, $\hat{b}_{t+1}^t = \left[\prod_{\tau=2}^t (1+\hat{r}_\tau)\right] m$, $t = 2,3,\dots.$

3. (a) Suppose that there is a representative consumer with the utility function and the endowment $\overline{k}_0$ of capital in period 0 and the endowment 1 of labor in every period. The production function is

$$y_t = \theta k_t^\alpha \ell_t^{1-\alpha} \,.$$

Capital depreciates completely every period.

With sequential market markets structure, there are markets for goods, capital services, labor services, and bonds open every period. Consumers sell labor services and rent capital to the firm. They buy goods from the firm, some of which they consume and some of which they save as capital. They trade bonds among themselves.

A **sequential markets equilibrium** is sequences of wages $\hat{w}_0, \hat{w}_1, \dots$, rental rates $\hat{r}_0^k, \hat{r}_1^k, \dots$, interest rates $\hat{r}_0^b, \hat{r}_1^b, \dots$, consumption levels $\hat{c}_0, \hat{c}_1, \dots$, labor levels $\hat{\ell}_0, \hat{\ell}_1, \dots$, capital stocks $\hat{k}_0, \hat{k}_1, \dots$, and bond holdings $\hat{b}_0, \hat{b}_1, \dots$, such that

- Given $\hat{w}_0, \hat{w}_1, \dots$, $\hat{r}_0^k, \hat{r}_1^k, \dots$, $\hat{r}_0^b, \hat{r}_1^b, \dots$, the consumer chooses $\hat{c}_0, \hat{c}_1, \dots$, $\hat{k}_0, \hat{k}_1, \dots$, and $\hat{b}_0, \hat{b}_1, \dots$ to solve

$$\max \sum_{t=0}^\infty \beta^t \log c_t$$
$$\text{s.t. } \ c_t + k_{t+1} + b_{t+1} \le \hat{w}_t \ell_t + (1+\hat{r}_t^b) b_t \,, \ t = 0,1,\dots$$

$$c_t, k_t \geq 0\,,\; 1 \geq \ell_t \geq 0\,,\; b_t \geq -B$$
$$k_0 = \bar{k}_0\,,\; b_0 = 0\,.$$

- $\hat{w}_t = (1-\alpha)\theta \hat{k}_t^{\alpha} \hat{\ell}_t^{-\alpha}$, $t = 0,1,\ldots$
- $\hat{r}_t^k = \alpha\theta \hat{k}_t^{\alpha-1} \hat{\ell}_t^{1-\alpha}$, $t = 0,1,\ldots$
- $\hat{c}_t + \hat{k}_{t+1} = \theta \hat{k}_t^{\alpha}$, $t = 0,1,\ldots$
- $\hat{\ell}_t = 1$, $t = 0,1,\ldots$
- $\hat{b}_t = 0$, $t = 0,1,\ldots$

(b) A Pareto efficient allocation/production plan is consumption levels $\hat{c}_0, \hat{c}_1, \ldots$, labor levels $\hat{\ell}_0, \hat{\ell}_1, \ldots$, and capital stocks $\hat{k}_0, \hat{k}_1, \ldots$ that solve

$$\max \sum\nolimits_{t=0}^{\infty} \beta^t \log c_t$$
$$\text{s.t. } c_t + k_{t+1} \leq \theta k_t^{\alpha} \ell_t^{1-\alpha}\,,\; t = 0,1,\ldots$$
$$c_t, k_t \geq 0\,,\; 1 \geq \ell_t \geq 0$$
$$k_0 \leq \bar{k}_0\,.$$

Proposition: The allocation/production plan in a sequential markets equilibrium is Pareto efficient.

Proof: Suppose that $\hat{r}_t^k, \hat{r}_t^b, \hat{w}_t, \hat{c}_t, \hat{\ell}_t, \hat{k}_t, \hat{b}_t$ is an equilibrium. Then

$$\hat{c}_t + \hat{k}_{t+1} = \theta \hat{k}_t^{\alpha}\,, t = 0,1,\ldots$$
$$\hat{k}_0 = \bar{k}_0$$
$$\hat{\ell}_t = 1\,,$$

and there exist Lagrange multipliers $p_t \geq 0$, $t = 0,1,\ldots$, such that

$$\frac{\beta^t}{\hat{c}_t} - p_t = 0\,,\; t = 0,1,\ldots$$
$$-p_t + p_{t+1}\alpha\theta \hat{k}_{t+1}^{\alpha-1} = 0\,,\; t = 0,1,\ldots$$
$$\lim\nolimits_{t\to\infty} p_t \hat{k}_{t+1} = 0\,.$$

The necessary and sufficient conditions for $\tilde{c}_t, \tilde{\ell}_t, \tilde{k}_t$ to be a Pareto efficient allocation/production plan are

$$\tilde{c}_t + \tilde{k}_{t+1} = \theta \tilde{k}_t^{\alpha},\ t = 0,1,\ldots$$
$$\tilde{k}_0 = \bar{k}_0$$
$$\tilde{\ell}_t = 1$$

and that there exist some Lagrange multipliers $\pi_t \geq 0,\ t = 0,\ 1,\ldots$, such that

$$\frac{\beta^t}{\tilde{c}_t} - \pi_t = 0,\ t = 0,1,\ldots$$
$$-\pi_t + \pi_{t+1}\alpha\theta\tilde{k}_{t+1}^{\alpha-1} = 0,\ t = 0,1,\ldots$$
$$\lim_{t\to\infty} \pi_t \tilde{k}_{t+1} = 0.$$

Given that $\hat{r}_t^k, \hat{r}_t^b, \hat{w}_t, \hat{c}_t, \hat{k}_t, \hat{b}_t$ is an equilibrium, we can set $\tilde{c}_t = \hat{c}_t$, $\tilde{\ell}_t = \hat{\ell}_t = 1$, $\tilde{k}_t = \hat{k}_t$, and $\pi_t = p_t$ and thus construct an allocation that satisfies the necessary and sufficient conditions for Pareto efficiency.

(c) Bellman's equation is

$$V(k) = \max\ \ \log c + \beta V(k')$$
$$\text{s.t. } c + k' \leq \theta k^{\alpha}$$
$$c, k' \geq 0.$$

The k' that solves this problem is the policy function $k' = g(k)$.

Guessing that $V(k)$ has the form $a_0 + a_1 \log k$, we can solve for c and k':

$$c = \frac{1}{1+\beta a_1}\theta k^{\alpha},\ k' = \frac{\beta a_1}{1+\beta a_1}\theta k^{\alpha}.$$

We can plug these solutions back into Bellman's equation to obtain

$$a_0 + a_1 \log k = \log\left(\frac{1}{1+\beta a_1}\theta k^{\alpha}\right) + \beta\left[a_0 + a_1 \log\left(\frac{\beta a_1}{1+\beta a_1}\theta k^{\alpha}\right)\right].$$

Collecting all the terms on the right-hand side that involve $\log k$, we can solve for a_1:

$$a_1 = \alpha + \alpha\beta a_1$$

$$a_1 = \frac{\alpha}{1-\alpha\beta},$$

which implies that

$$c = (1-\alpha\beta)\theta k^{\alpha},\ k' = \alpha\beta\theta k^{\alpha}.$$

[We could also solve for a_0:

$$a_0 = \frac{1}{1-\beta}\left[\log\left(\frac{\theta}{1+\beta a_1}\right) + \beta a_1 \log\left(\frac{\beta a_1 \theta}{1+\beta a_1}\right)\right]$$
$$a_0 = \frac{1}{1-\beta}\left[\log\big((1-\alpha\beta)\theta\big) + \frac{\alpha\beta}{1-\alpha\beta}\log\big(\alpha\beta\theta\big)\right],$$

but this is tedious, and, besides, the question does not ask us to do it.]

(d) To calculate the sequential markets equilibrium, we just run the first order difference equation

$$k_{t+1} = \alpha\beta\theta k_t^{\alpha}$$

forward, starting at $k_0 = \bar{k}_0$. We set

$$\begin{aligned} c_t &= (1-\alpha\beta)\theta k_t^{\alpha} \\ b_t &= 0 \\ r_t^k &= \alpha\theta k_t^{\alpha-1} \\ r_t^b &= \alpha\theta k_t^{\alpha-1} - 1 \\ w_t &= (1-\alpha)\theta k_t^{\alpha}. \end{aligned}$$

Notice that this problem actually has an analytical solution:

$$k_t = \alpha\beta\theta k_{t-1}^{\alpha} = \alpha\beta\theta\left(\alpha\beta\theta k_{t-2}^{\alpha}\right)^{\alpha} = \alpha\beta\theta^{1+\alpha} k_{t-2}^{\alpha^2} = \left(\alpha\beta\theta\right)^{\sum_{\tau=0}^{t-1}\alpha^{\tau}} \bar{k}_0^{\alpha^t} = \left(\alpha\beta\theta\right)^{\frac{1-\alpha^t}{1-\alpha}} \bar{k}_0^{\alpha^t}.$$

(e) A **sequential markets equilibrium** is sequences of rental rates $\hat{r}_0^k, \hat{r}_1^k, \ldots$, interest rates $\hat{r}_0^b, \hat{r}_1^b, \ldots$, wages $\hat{w}_0, \hat{w}_1, \ldots$, consumption levels $\hat{c}_0, \hat{c}_1, \ldots$, labor levels $\hat{\ell}_0, \hat{\ell}_1, \ldots$, capital stocks $\hat{k}_0, \hat{k}_1, \ldots$, and bond holdings $\hat{b}_0, \hat{b}_1, \ldots$, such that

- Given $\hat{r}_0^k, \hat{r}_1^k, \ldots$, $\hat{r}_0^b, \hat{r}_1^b, \ldots$, and $\hat{w}_0, \hat{w}_1, \ldots$, the consumer chooses $\hat{c}_0, \hat{c}_1, \ldots$, $\hat{\ell}_0, \hat{\ell}_1, \ldots$, $\hat{k}_0, \hat{k}_1, \ldots$, and $\hat{b}_0, \hat{b}_1, \ldots$ to solve

$$\begin{gathered}\sum\nolimits_{t=0}^{\infty} \beta^t \left(\gamma \log c_t + (1-\gamma)\log(1-\ell_t)\right) \\ \text{s.t. } c_t + k_{t+1} + b_{t+1} \le \hat{w}_t \ell_t + \hat{r}_t^k k_t + (1+\hat{r}_t^b) b_t,\ t = 0,1,\ldots \\ c_t, k_t \ge 0,\ 1 \ge \ell_t \ge 0,\ b_t \ge -B \\ k_0 = \bar{k}_0,\ b_0 = 0.\end{gathered}$$

- $\hat{r}_t^k = \alpha\theta \hat{k}_t^{\alpha-1} \hat{\ell}_t^{1-\alpha}$, $t = 0,1,\ldots$
  $\hat{w}_t = (1-\alpha)\theta \hat{k}_t^{\alpha} \hat{\ell}_t^{-\alpha}$, $t = 0,1,\ldots$.

4. $\hat{c}_t + \hat{k}_{t+1} = \theta \hat{k}_t^{\alpha} \hat{\ell}_t^{1-\alpha}$, $t = 0,1,\ldots$
5. $\hat{b}_t = 0$, $t = 0,1,\ldots$