Profit maximization with constant returns-to-scale

Production function $f(k,\ell)$:

$$f(\theta k,\theta \ell)=\theta f(k,\ell)$$

$$f(k,\ell) \text{ is concave.}$$

Profit maximization

$$\hat{p}f(\hat{k},\hat{\ell})-\hat{r}\hat{k}-\hat{w}\hat{\ell}=0$$

$$\hat{p}f(k,\ell)-\hat{r}k-\hat{w}\ell \leq 0 \text{ for all } k,\ell$$

These conditions are as much restrictions on the prices $\hat{p},\hat{r},\hat{w}$ as they are on $\hat{k},\hat{\ell}$.

Suppose that

$$\hat{r}=\hat{p}f_k(\hat{k},\hat{\ell})$$

$$\hat{w} = \hat{p} f_\ell(\hat{k}, \hat{\ell})$$

Then we can show that $\hat{k}, \hat{\ell}$ solves

$$\min \ \hat{r}k + \hat{w}\ell$$

$$\text{s.t. } f(k, \ell) \geq f(\hat{k}, \hat{\ell})$$

and

$$\hat{p} f(\hat{k}, \hat{\ell}) = \hat{r}\hat{k} + \hat{w}\hat{\ell}.$$

Debreu's proof of the Pareto efficiency of equilibrium allocations

Suppose that $\hat{p}_0, \hat{p}_1, \hat{p}_2, \ldots$, $\hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots,$ $\hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots$ are an equilibrium:

- Given $\hat{p}_0, \hat{p}_1, \hat{p}_2, \ldots,$ consumer i, $i = 1, 2,$ chooses $\hat{c}_0^i, \hat{c}_1^i, \hat{c}_2^i, \ldots$ to solve

$$\max \sum_{t=0}^{\infty} \beta^t \log c_t^i$$
$$\text{s.t. } \sum_{t=0}^{\infty} \hat{p}_t c_t^i \leq \sum_{t=0}^{\infty} \hat{p}_t w_t^i$$
$$c_t^i \geq 0.$$

- $\hat{c}_t^1 + \hat{c}_t^2 \leq w_t^1 + w_t^2$, $t = 0, 1, \ldots.$

Then $\hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots,$ $\hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots$ is a Pareto efficient allocation:

It is feasible

- $\hat{c}_t^1 + \hat{c}_t^2 \leq w_t^1 + w_t^2$, $t = 0, 1, \ldots.$

and there exists no other feasible allocation $\overline{c}_0^1, \overline{c}_1^1, \overline{c}_2^1, \ldots, \overline{c}_0^2, \overline{c}_1^2, \overline{c}_2^2, \ldots$ such that

$$\sum_{t=0}^{\infty} \beta^t \log \overline{c}_t^i \geq \sum_{t=0}^{\infty} \beta^t \log \hat{c}_t^i, \; i = 1, 2,$$

with one inequality strict.

Steps in proof:

Proof by contradiction. Suppose that there is a feasible allocation that is Pareto superior.

If $\sum_{t=0}^{\infty} \beta^t \log \overline{c}_t^i > \sum_{t=0}^{\infty} \beta^t \log \hat{c}_t^i$, then $\sum_{t=0}^{\infty} \hat{p}_t \overline{c}_t^i > \sum_{t=0}^{\infty} \hat{p}_t w_t^i$.

If $\sum_{t=0}^{\infty}\beta^t \log \overline{c}_t^i \geq \sum_{t=0}^{\infty}\beta^t \log \hat{c}_t^i$, then $\sum_{t=0}^{\infty}\hat{p}_t\overline{c}_t^i \geq \sum_{t=0}^{\infty}\hat{p}_t w_t^i$.

Consequently,

$$\sum_{i=1}^{2}\sum_{t=0}^{\infty}\hat{p}_t\overline{c}_t^i > \sum_{i=1}^{2}\sum_{t=0}^{\infty}\hat{p}_t w_t^i$$

Multiplying each feasibility constraint by the respective price and adding up produces

$$\sum_{i=1}^{2}\sum_{t=0}^{\infty}\hat{p}_t\overline{c}_t^i \leq \sum_{i=1}^{2}\sum_{t=0}^{\infty}\hat{p}_t w_t^i .$$

which is a contradiction.