MACROECONOMIC THEORY **T.J. KEHOE**
ECON 8105 **FALL 2009**

PROBLEM SET #2

1. Consider an economy with a representative, infinitely lived consumer who has the utility function

$$\sum_{t=0}^{\infty} \beta^t \log c_t$$

where $0 < \beta < 1$. The consumer owns one unit of labor in each period and $\bar{k}_0$ units of capital in period 0. The depreciation rate on capital is δ.

a) Suppose that the consumer borrows b_{t+1} bonds in period *t* to be paid off in period $t+1$. The consumer's initial endowment of bonds is $\bar{b}_0 = 0$, the wage rate in period *t* is w_t, the rental rate on capital is r_t^k, and the interest rate on bonds is r_t^b. Write down the consumer's utility maximization problem in a sequential markets economy. Explain why you need to include a constraint to rule out Ponzi schemes. Write down the Euler conditions and the transversality conditions for this problem.

Suppose that feasible consumption/investment plans satisfy

$$c_t + k_{t+1} - (1-\delta)k_t \le F(k_t, \ell_t)$$

where $F(k_t, \ell_t) = \theta k_t^{\alpha} \ell_t^{1-\alpha}$.

b) Define a sequential markets equilibrium with borrowing and lending for this economy. Prove that in equilibrium $r_t^k - \delta = r_t^b$ if $k_t > 0$.

c) Suppose that the consumer sells his endowment of capital to the firm in period 0. Thereafter, firms buy and sell capital from each other. Describe the production set for the Arrow-Debreu economy, the set of feasible $k_0, k_1, \ldots, \ell_0, \ell_1, \ldots, c_0, c_1, \ldots$.

d) Define the Arrow-Debreu equilibrium for this economy.

e) Suppose that the consumer can buy new capital in each period and rent capital services to the firm. Define the Arrow-Debreu equilibrium for this economy.

f) Carefully state theorems that relate the equilibrium allocations in parts (b), (d), and (e).

2. Consider the social planning problem

$$\begin{aligned} &\max \ \sum_{t=0}^{\infty} \beta^t \log \ c_t \\ &\text{s.t.} \quad c_t + k_{t+1} \le \theta k_t^{\alpha} \\ &\qquad c_t, k_t \ge 0 \\ &\qquad k_0 \le \bar{k}_0 . \end{aligned}$$

a) Write down the Euler conditions and the transversality condition for this problem.

b) Let $v(k_t, k_{t+1})$ be the solution to

$$\begin{aligned} &\max_{c_t} \ \log \ c_t \\ &\text{s.t.} \quad c_t + k_{t+1} \le \theta k_t^{\alpha} \\ &\qquad c_t \ge 0 \end{aligned}$$

for fixed k_t, k_{t+1}. What is $v(k_t, k_{t+1})$? What conditions do k_t and k_{t+1} need to satisfy to ensure $c_t, k_{t+1}, \ge 0$? If we write these conditions as $k_{t+1} \in \Gamma(k_t)$, what is $\Gamma(k_t)$?

c) Write down the Euler conditions and the transversality condition for the problem

$$\begin{aligned} &\max \sum_{t=0}^{\infty} \beta^t v(k_t, k_{t+1}) \\ &\text{s.t.} \quad k_{t+1} \in \Gamma(k_t) \\ &\qquad k_0 \le \bar{k}_0 . \end{aligned}$$

d) Prove that a sequence $\hat{k}_0, \hat{k}_1, \ldots$ solves the conditions in part (c) if an only if there exist sequences of Lagrange multipliers $\hat{p}_0, \hat{p}_1, \ldots$ and of consumption $\hat{c}_0, \hat{c}_1, \ldots$ such that $(\hat{c}_0, \hat{c}_1, \ldots, \hat{k}_0, \hat{k}_1, \ldots, \hat{p}_{0,} \hat{p}_1, \ldots)$ satisfy the conditions in part (a).

e) Prove that, if a sequence $\hat{k}_0, \hat{k}_1, \ldots$ satisfies the Euler conditions and transversality condition in part (c), then it solves the related planning problem. (Hint: you can adapt the general proof on pp. 98-99 of Stokey, Lucas, and Prescott to these specific functions.)

3. Let $v(k_t, k_{t+1})$ and $\Gamma(k_t)$ be defined as in part (b) of question 2.

a) Consider the dynamic programming problem with functional equation

$$\begin{aligned} &V(k) = \max v(k, k') + \beta V(k') \\ &\text{s.t.} \quad k' \in \Gamma(k). \end{aligned}$$

Guess that $V(k)$ has the form $a_1 + a_2 \log k$ and solve for a_1 and a_2.

b) What is the policy function $g(k)$ such that $k' = g(k)$? Verify that $k_{t+1} = g(k_t)$ satisfies the Euler equations and the transversality condition in part (c) of question 2.

c) Try to approximate $V(k)$: Guess that $V_0(k) = 0$ for all k and use the iterative updating rule

$$\begin{aligned} V_{n+1}(k) = \max\ & v(k,k') + \beta V_n(k') \\ \text{s.t.}\quad & k' \in \Gamma(k). \end{aligned}$$

Calculate the functions V_1, V_2, V_3 and V_4.

4. Consider an economy specified as in question 1 except that the representative consumer has the utility function

$$\sum_{t=0}^{\infty} \beta^t \left(\log c_t + \gamma \log x_t\right)$$

where x_t is the consumption of leisure.

a) Define a sequential markets equilibrium for this economy.

b) Define an Arrow-Debreu equilibrium for this economy.

c) Let $v(k_t, k_{t+1})$ be the solution to

$$\begin{aligned} \max_{c_t, x_t, \ell_t}\ & \log c_t + \gamma \log x_t \\ \text{s.t.}\quad & c_t + k_{t+1} \le \theta k_t^{\alpha} \ell_t^{1-\alpha} \\ & x_t + \ell_t \le 1 \\ & c_t, x_t, \ell_t \ge 0 \end{aligned}$$

for fixed k_t, k_{t+1}. Explain carefully how you would calculate $v(k_t, k_{t+1})$. What conditions do k_t, k_{t+1} need to satisfy to ensure $c_t, x_t, \ell_t, k_{t+1} \ge 0$? If we write these conditions as $k_{t+1} \in \Gamma(k_t)$, what is $\Gamma(k_t)$?