MACROECONOMIC THEORY **T. J. KEHOE**
ECON 8107 **SPRING 2005**

PROBLEM SET #1

1. Consider an overlapping generations economy in which there is one good in each period and each generation, except the initial one, lives for two periods. The representative consumer in generation t, $t = 1,2,\ldots,$ has the utility function

$$\log\ c_t^t + \log\ c_{t+1}^t$$

and the endowment $(w_t^t,\ w_{t+1}^t) = (2,\ 1)$. The representative consumer in generation 0 lives only in period 1, prefers more consumption to less, and has the endowment $w_1^0 = 1$. There is no fiat money.

a) Define an Arrow-Debreu equilibrium for this economy. Calculate the unique Arrow-Debreu equilibrium.

b) Define a sequential markets equilibrium for this economy. Calculate the unique sequential markets equilibrium.

c) Define a Pareto efficient allocation. Letting the endowment process $w_1,\ w_2$ vary, where $w_1^0 = w_2$, $(w_t^t,\ w_{t+1}^t) = (w_1, w_2)$, produce examples of economies for which you can prove the competitive equilibrium allocation is Pareto efficient and ones for which it is not Pareto efficient.

d) Suppose now that there are two goods in each period and that the representative consumer in generation t, $t = 1,2,\ldots,$ has the utility function

$$\log\ c_{1t}^t + \log\ c_{2t}^t + \log\ c_{1t+1}^t + \log\ c_{2t+1}^t$$

and the endowment $(w_{1t}^t,\ w_{2t}^t,\ w_{1t+1}^t,\ w_{2t+1}^t) = (2,\ 4,\ 2,\ 1)$. The representative consumer in generation 0 lives only in period 1, has the utility function $\log\ c_{11}^0 + \log\ c_{21}^0$, and has the endowment $(w_{11}^0,\ w_{21}^0) = (2,\ 1)$. There is no fiat money. Define an Arrow-Debreu equilibrium for this economy.

d) In the equilibrium allocation is $c_{1t}^t = 2$? Explain carefully why or why not.

2. Consider a simple overlapping generations economy in which the consumer born in period t, $t = 1,\ 2,\ldots,$ has the utility function

$$u(c_t^t, c_{t+1}^t) = c_t^t + \left[\left(c_{t+1}^t\right)^b - 1\right]/b,$$

where $b < 1$. Suppose that his endowment is $(w_t^t, w_{t+1}^t) = (w_1, w_2)$.

a) What is the utility function in the case where $b = 0$? [Hint: use l'Hôpital's rule.]

b) Write down the utility maximization problem in an environment with Arrow-Debreu markets. Derive the excess demand functions $y(p_t, p_{t+1})$ and $y(p_t, p_{t+1})$. Demonstrate that they are homogeneous of degree zero and that they satisfy Walras's law.

c) Suppose that the first generation has an excess demand function of the form

$$z_0(p_1, m) = \frac{m}{p_1}.$$

Explain the role of m. Define an Arrow-Debreu equilibrium of this model. Write down the equilibrium conditions using the excess demand functions.

d) Find an expression for the offer curve for this model. (Hint: you have to solve for y as a function of z.)

e) Suppose that $w_1 = 1$ and $w_2 = 0.25$. Draw the offer curve for the three cases $b = 0.5$, $b = 0$, and $b = -1$.

3. Consider an overlapping generations economy in which the representative consumer in generation t, $t = 1, 2, \ldots,$ has preferences over the consumption of the single good in each of the two periods of her life given by the utility function

$$u(c_t^t, c_{t+1}^t) = \log\ c_t^t + \log\ c_{t+1}^t.$$

This consumer is endowed with quantities of labor $(\ell_t^t, \ell_{t+1}^t) = (\bar{\ell}_1, \bar{\ell}_2)$. In addition, there is a generation 0 who representative consumer lives only in period 1 and has the utility function

$$u^0(c_1^0) = \log\ c_1^0,$$

and the endowment of $\ell_1^0 = \bar{\ell}_2$ units of labor and $\bar{k}_1^0$ units of capital in period 1. This consumer also has an endowment of fiat money m, which can be positive, negative or zero.

The production function is

$$f(k_t, \ell_t) = \theta k_t^{\alpha} \ell_t^{1-\alpha},$$

and capital depreciates at the rate δ per period, $0 \le \delta \le 1$.

a) Define a sequential market equilibrium for this economy.

b) Define an Arrow-Debreu equilibrium for this economy. State and prove two theorems that establish the equivalence between a sequential market equilibrium and an Arrow-Debreu equilibrium.

c) Reduce the equilibrium conditions to a second-order difference equation in k_t, that is, an equation in k_{t+1}, k_t, k_{t-1} that includes no other endogenous variables.

d) Suppose that $m = 0$. Reduce the equilibrium conditions to a first-order difference equation in k_t. (Hint: in this case you know that the savings of generation t in period t in the sequential market equilibrium must equal k_{t+1}.)

4. Suppose that $\theta = 100$, $\alpha = 0.4$, $\delta = 0.8$, $\bar{\ell}_1 = 1$, and $\bar{\ell}_2 = 0$ in question 2.

a) Define a steady state for this economy. Calculate the two steady states.

b) Suppose that $\bar{k}_1^0 = 10$ and $m = 0$. Use your answer to question 2d to calculate the equilibrium in the first 10 periods both by hand and on the computer.

c) Suppose now that $\bar{\ell}_2 = 0.5$. Repeat parts a and b, doing your calculations on the computer.

d) Suppose now that $\bar{\ell}_2 = 2$. Repeat parts a and b, doing your calculations on the computer.

5. Consider now an economy like that in questions 3 and 4 except that consumers live for 4 periods rather 2. The utility function of the consumer born in period t, $t = 1, 2, \ldots$, is

$$u(c_t^t, c_{t+1}^t, c_{t+2}^t, c_{t+3}^t) = \log c_t^t + \beta \log c_{t+1}^t + \beta^2 \log c_{t+2}^t + \beta^3 \log c_{t+3}^t .$$

This consumer is endowed with quantities of labor $(\ell_t^t, \ell_{t+1}^t, \ell_{t+2}^t, \ell_{t+3}^t) = (\bar{\ell}_1, \bar{\ell}_2, \bar{\ell}_3, \bar{\ell}_4)$.

In addition, there are three other consumers, -2, -1, and 0. Consumer -2 has the utility function

$$u^{-2}(c_1^{-2}) = \log\ c_1^{-2}$$

and the endowments $\ell_1^{-2} = \overline{\ell}_4$, $\overline{k}_1^{-2}$. Consumer -1 has the utility function

$$u^{-1}(c_1^{-1}, c_2^{-1}) = \log\ c_1^{-1} + \beta \log\ c_2^{-1}$$

and the endowments $(\ell_1^{-1}, \ell_2^{-1}) = (\overline{\ell}_3, \overline{\ell}_4)$, $\overline{k}_1^{-1}$. Consumer 0 has the utility function

$$u^0(c_1^0, c_2^0, c_3^0) = \log\ c_1^0 + \beta \log\ c_2^0 + \beta^2 \log\ c_3^0$$

and endowments $(\ell_1^0, \ell_2^0, \ell_3^0) = (\overline{\ell}_2, \overline{\ell}_3, \overline{\ell}_4)$, $\overline{k}_1^0$. Consumers -2, -1, and 0 are also endowed with m^{-2}, m^{-1}, and m^0 units of fiat money respectively.

The production function remains

$$f(k_t, \ell_t) = \theta k_t^{\alpha} \ell_t^{1-\alpha},$$

and capital depreciates at the rate δ per period, $0 \le \delta \le 1$.

a) Define a sequential market equilibrium for this economy.

b) Define a steady state for this economy.

c) Reduce the equilibrium conditions in part a to a difference equation in k_t. Show that, if $m^{-2} = m^{-1} = m^0 = 0$, you can reduce the equilibrium conditions to a difference equation of lower order.

d) Suppose that, after $T = 9$ periods, the capital stock becomes constant at $\hat{k}$. That is,

$$k_{10} = k_{11} = ... = \hat{k}.$$

Considering the general case — not the case where $m^{-2} = m^{-1} = m^0 = 0$ — write out the equilibrium conditions in part c as a system of 8 equations in the 8 unknowns k_2, k_3, k_3, k_4, k_5, k_6, k_7, k_8, and k_9 by considering the market clearing conditions in periods 1, 2, 3, 4, 5, 6, 7, and 8.

e) Suppose that T — the date immediately after which the capital stock becomes constant — increases. What happens to the system of equations and unknowns in part d?

f) Redo parts d and e for the case where $m^{-2} = m^{-1} = m^0 = 0$. (Hint: you can reduce the number of equations and unknowns.)