MACROECONOMIC THEORY **T. J. KEHOE**
ECON 8107 **SPRING 2005**

PROBLEM SET #2

1. Consider an overlapping generations economy like that in question 5 on problem set 1 in which consumers live for 4 periods. The utility function of the consumer born in period t, $t = 1, 2, \ldots,$ is

$$u(c_t^t, c_{t+1}^t, c_{t+2}^t, c_{t+3}^t) = \log\ c_t^t + \beta \log\ c_{t+1}^t + \beta^2 \log\ c_{t+2}^t + \beta^3 \log\ c_{t+3}^t .$$

This consumer is endowed with quantities of labor

$$(\ell_t^t, \ell_{t+1}^t, \ell_{t+2}^t, \ell_{t+3}^t) = (\eta^{t-1}\bar{\ell}_1, \eta^{t-1}\bar{\ell}_2, \eta^{t-1}\bar{\ell}_3, \eta^{t-1}\bar{\ell}_4) .$$

In addition, there are three other consumers, -2, -1, and 0. Consumer -2 has the utility function

$$u^{-2}(c_1^{-2}) = \log\ c_1^{-2}$$

and the endowments $\ell_1^{-2} = \eta^{-3}\bar{\ell}_4$, $\bar{k}_1^{-2}$. Consumer -1 has the utility function

$$u^{-1}(c_1^{-1}, c_2^{-1}) = \log\ c_1^{-1} + \beta \log\ c_2^{-1}$$

and the endowments $(\ell_1^{-1}, \ell_2^{-1}) = (\eta^{-2}\bar{\ell}_3, \eta^{-2}\bar{\ell}_4)$, $\bar{k}_1^{-1}$. Consumer 0 has the utility function

$$u^0(c_1^0, c_2^0, c_3^0) = \log\ c_1^0 + \beta \log\ c_2^0 + \beta^2 \log\ c_3^0$$

and endowments $(\ell_1^0, \ell_2^0, \ell_3^0) = (\eta^{-1}\bar{\ell}_2, \eta^{-1}\bar{\ell}_3, \eta^{-1}\bar{\ell}_4)$, $\bar{k}_1^0$. The production function is

$$f(k_t, \ell_t) = A_t k_t^{\alpha} \ell_t^{1-\alpha} ,$$

where $A_{t+1} = \gamma A_t$, and capital depreciates at the rate δ per period, $0 \le \delta \le 1$. There is no fiat money.

a) Define a sequential markets equilibrium for this economy.

b) Define a balanced growth path for this economy. Find the value of $\lambda > 0$ such that $k_{t+1} = \lambda k_t$ along the balanced growth path.

c) Suppose that, after $T = 9$ periods, the capital stock grows at the rate $\lambda - 1$. That is, $k_{t+1} = \lambda k_t$, $t = 9, 10, \ldots$ Reduce the equilibrium conditions in part a to a system of 8 equations in the 8 unknowns k_2, k_3, k_3, k_4, k_5, k_6, k_7, k_8, and k_9 by considering the market clearing conditions in periods 1, 2, 3, 4, 5, 6, 7, and 8.

d) Suppose that $\beta = 1$, $\eta = 1.1$, $(\bar{\ell}_1, \bar{\ell}_2, \bar{\ell}_3, \bar{\ell}_4) = (1.0,\ 1.3,\ 1.5,\ 0.0)$, $A_1 = 100$, $\gamma = 1.2$, $\alpha = 0.4$, $\delta = 0.5$. Calculate the balanced growth path in which the savings is equal to capital stock that must be accumulated for next period, $\lambda \hat{k}$. (There is another balanced growth path in which $1 + \hat{r} - \delta = \lambda$, but to be in this balanced growth path requires a nonzero amount of fiat money.)

e) Write a computer program to solve the system of equations in part c. Test the program by verifying that, for the right choice of $\bar{k}_1^{-2}$, $\bar{k}_1^{-1}$, $\bar{k}_1^{0}$, the balanced growth path is an equilibrium. Let $\bar{k}_1^{-2}$, $\bar{k}_1^{-1}$, $\bar{k}_1^{0}$ be equal to 80 percent of their balanced growth path values. Calculate the solution to the system of equations. Check whether the equilibrium conditions in periods 9, 10, 11, and 12 are satisfied.

f) Now truncate the model at $T = 40$. Repeat part e and compare the results.

2. Using data for the United States or some other country, calibrate the parameters β, η, $(\bar{\ell}_1, \bar{\ell}_2, \bar{\ell}_3, \bar{\ell}_4)$, A_0, γ, α, and δ.

a) Repeat parts d, e and f of question 1.

b) Suppose now that the economy is on a balanced growth path in which the population growth rate, $\eta - 1$, is at its calibrated level. In period 1 consumers realize that the population growth rate will fall to $\tilde{\eta} - 1$. That is, $(\ell_t^1, \ell_{t+1}^1, \ell_{t+2}^1, \ell_{t+3}^1) = (\bar{\ell}_1, \bar{\ell}_2, \bar{\ell}_3, \bar{\ell}_4)$ and $(\ell_t^t, \ell_{t+1}^t, \ell_{t+2}^t, \ell_{t+3}^t) = (\eta^{t-1}\bar{\ell}_1, \eta^{t-1}\bar{\ell}_2, \eta^{t-1}\bar{\ell}_3, \eta^{t-1}\bar{\ell}_4)$, $t = 2, 3, \ldots$ Choose η and $\tilde{\eta}$, $\eta > \tilde{\eta}$, and calculate the equilibrium, truncating at $T = 40$. What are the predictions of the model that you can compare the data for an economy that has experienced a slowdown in population growth?

3. Consider an economy in which there are two dates, $t = 0, 1$, no uncertainty at $t = 0$, and uncertainty over three possible states at $t = 1$. There are two consumers, $i = 1, 2$, with utility functions

$$\pi_1 \log c_1^i + \pi_2 \log c_2^i + \pi_3 \log c_3^i$$

and endowments (w_1^i, w_2^i, w_3^i). Here $\pi_j > 0$ is the probability of event j, c_j^i is the consumption of the single good at event j by consumer i, and w_j^i is the endowment of this good. The consumers have no endowment of goods nor utility for consumption at $t = 0$.

a) Define an Arrow-Debreu equilibrium for this economy.

b) Define a Pareto efficient allocation for this economy.

c) Prove that any equilibrium allocation is Pareto efficient.

d) Suppose that $\pi_1 = \pi_2 = \pi_3 = 1/3$, $(w_1^1, w_2^1, w_3^1) = (2,\ 1,\ 1)$, and $(w_1^1, w_2^1, w_3^1) = (1,\ 2,\ 1)$. Calculate the Arrow-Debreu equilibrium of this economy.

4. Consider an economy in which consumers live forever. In every period, $t = 0, 1, \ldots$, one of two random events occurs, $\eta_t = 1$ or $\eta_t = 2$. At $t = 0$, the initial state is η_0 and a stationary Markov process given by a 2×2 matrix with elements $\pi_{ij} = \text{prob}(\eta_{t+1} = j \mid \eta_t = i)$ governs the probability of future states. Let $\pi(s)$ be the induced probability distribution over states. Suppose that there are two consumers, $i = 1, 2$, each of whom has the utility function

$$\sum\nolimits_{s \in S} \beta^{t(s)} \pi(s) u_i(c_s^i) .$$

Here S is the set of all states, $0 < \beta < 1$, $t(s)$ is the date that state s occurs in, and c_s^i is the consumption of the single good in that state by consumer i. Suppose that endowments depend only on the latest event, $w_s^i = w^i(\eta_s)$. There is no production.

a) Define an Arrow-Debreu equilibrium for this economy.

b) Define a Pareto efficient allocation for this economy. Assuming that u_i is monotonically increasing, prove that an equilibrium allocation is Pareto efficient.

c) Define a sequential markets equilibrium for this economy. Carefully state and prove two propositions that relate Arrow-Debreu equilibria to sequential market equilibria. Make explicit any assumptions that you make on utility functions, endowments, and so on.

d) Translate your definitions of Arrow-Debreu equilibria and sequential market equilibria to the sort of "double sum" notation used by Stokey, Lucas, and Prescott.

e) Suppose that $u_i(c) = \log c$, $w^1(1) = w^2(2) = 9$, and $w^1(2) = w^2(1) = 1$. Suppose too that $\pi_{12} = \pi_{21} = \pi$, $0 < \pi < 1$. Calculate the Arrow-Debreu equilibrium and the sequential markets equilibrium.

5. Consider the optimal growth problem

$$\max \sum_{t=0}^{\infty} (0.6)^t \log c_t$$
$$\text{s.t.} \quad c_t + k_{t+1} \le 20k_t^{0.3}$$
$$c_t, k_t \ge 0$$
$$k_0 = \bar{k}_0 .$$

a) Write down the Euler conditions and the transversality condition for this problem. Calculate the steady state values of c and k.

b) Write down the functional equation that defines the value function for this problem. Guess that the value function has the form $a_0 + a_1 \log k$. Calculate the value function and the policy function. Verify that the policy function generates a path for capital that satisfies the Euler conditions and transversality condition in part a.

c) Let capital take values for the discrete grid $(2, 4, 6, 8, 10)$. Make the initial guess $V_0(k) = 0$ for all k, and perform the first three steps of the value function iteration

$$V_{i+1}(k) = \max \ \log(20k^{0.3} - k') + (0.6)V_i(k') .$$

d) Perform the value function iterations until

$$\max_k \left| V_{i+1}(k) - V_i(k) \right| < 10^{-5} .$$

Report the value function and the policy function that you obtain. Compare these results with what you obtained in part b. (Hint: you probably want to use a computer.)

e) Repeat part d for the grid of capital stocks $(0.05, 0.10, \ldots, 9.95, 10.00)$. Compare your answer with those of parts b and d. (Hint: you need to use a computer).

f) Repeat part e for the problem

$$\max \ \sum_{t=0}^{\infty} (0.6)^t \log c_t$$
$$\text{s.t.} \quad c_t + k_{t+1} - 0.5k_t \le 20k_t^{0.3}$$
$$c_t, k_t \ge 0$$
$$k_0 = \bar{k}_0 .$$

(Here there is now no comparison with an analytical answer to be made.)

6. Consider the dynamic programming problem whose value function satisfies the functional equation.

$$V(k,\theta) = \max \ \log c + (0.6)E\left(V(k',\theta')\right)$$
$$\text{s.t.} \quad c + k' \leq \theta k^{0.3}$$
$$c, k' \geq 0 .$$

Here θ is a random variable that takes on the values $\theta_1 = 24$ and $\theta_2 = 16$ as governed by the first order, stationary Markov process given by the matrix

$$\begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{bmatrix} = \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix}.$$

a) Solve this dynamic programming problem analytically.

b) Describe an economic environment for which the solution in part a is an equilibrium allocation. Define the equilibrium. Calculate the equilibrium.

c) Let capital take values for the discrete grid $(2, 4, 6, 8, 10)$. Make the original guess $V_0(k,\theta) = 0$ for all k and θ, and perform the first three steps of the value function iteration

$$V_{i+1}(k,\theta) = \max \ \log(\theta k^{0.3} - k') + (0.6)E\left(V_i(k',\theta')\right) .$$

d) Perform the value function iterations until

$$\max_{k,\theta} \left|V_{i+1}(k,\theta) - V_i(k,\theta)\right| < 10^{-5} .$$

Report the value function and the policy function that you obtain. Compare these results with what you obtained in part b. (Hint: you probably want to use a computer.)

e) Repeat part d for the grid of capital stocks $(0.05, 0.10, \ldots, 9.95, 10.00)$. Compare your answer with those of parts a and d. (Hint: you need to use a computer).

f) Repeat part e for the problem

$$V(k,\theta) = \max \ \log c + (0.6)E\left(V(k',\theta')\right)$$
$$\text{s.t.} \quad c + k' - 0.5k \leq \theta k^{0.3}$$
$$c, k' \geq 0 .$$