

PROBLEM SET #3

1. Consider the dynamic programming problem whose value function satisfies the functional equation.

$$\begin{aligned} V(k, \theta) = \max & \log c + \lambda \log(1 - \ell) + (0.6)E(V(k', \theta')) \\ \text{s.t. } & c + k' \leq \theta k^{0.3} \ell^{0.7} \\ & c, k' \geq 0, 1 \geq \ell \geq 0. \end{aligned}$$

Here ℓ is the fraction of available time spent working, $1 - \ell$ is the fraction of available time spent consuming leisure, and $\lambda = 3$. θ is a random variable that takes on three possible values $\theta_1 = 32$, $\theta_2 = 28$, $\theta_3 = 24$, as governed by the first order, stationary Markov process given by the matrix

$$\begin{bmatrix} \pi_{11} & \pi_{12} & \pi_{13} \\ \pi_{21} & \pi_{22} & \pi_{23} \\ \pi_{31} & \pi_{32} & \pi_{33} \end{bmatrix} = \begin{bmatrix} 1 - 2\pi & \pi & \pi \\ \pi & 1 - 2\pi & \pi \\ \pi & \pi & 1 - 2\pi \end{bmatrix}$$

where $\pi = 0.1$.

- Solve this dynamic programming problem analytically.
- Describe an economic environment for which the solution in part a is an equilibrium allocation. Define the equilibrium. Calculate the equilibrium.
- Let capital take values for the discrete grid (2, 4, 6, 8, 10). Make the original guess $V_0(k, \theta) = 0$ for all k , θ and perform the value function iteration

$$V_{i+1}(k, \theta) = \max \log(\theta k^{0.3} \ell^{0.7} - k') + \gamma \log(1 - \ell) + (0.6)E(V_i(k', \theta')) .$$

until

$$\max_{k, \theta} |V_{i+1}(k, \theta) - V_i(k, \theta)| < 10^{-5} .$$

Report the value function and the policy function that you obtain. Compare these results with what you obtained in part b.

- Repeat part c for the grid of capital stocks (0.05, 0.10, ..., 9.95, 10). Compare your answer with those of parts a and c.

e) Repeat part d for the problem

$$V(k, \theta) = \max \log c + \lambda \log(1 - \ell) + (0.6)E(V(k', \theta'))$$

$$\text{s.t. } c + k' - 0.5k \leq \theta k^{0.3} \ell^{0.7}$$

$$c, k' \geq 0, 1 \geq \ell \geq 0.$$

2. Consider an economy with a continuum $[0,1]$ of consumers of two symmetric types who live forever. Consumers have utility

$$\sum_{t=0}^{\infty} \beta^t \log c_t^i.$$

Consumers of type 1 have an endowment stream of the single good in each period

$$(w_0^1, w_1^1, w_2^1, w_3^1, \dots) = (8, 1, 8, 1, \dots),$$

while consumers of type 2 have

$$(w_0^2, w_1^2, w_2^2, w_3^2, \dots) = (1, 8, 1, 8, \dots).$$

In addition there is one unit of trees that produce $r = 1$ units of the good every period. Each consumer of type i owns $\bar{\theta}_0^i$ of such trees in period 0, $\bar{\theta}_0^i \geq 0$, $\bar{\theta}_0^1 + \bar{\theta}_0^2 = 1$. Trees do not grow or decay.

a) Define an Arrow-Debreu equilibrium for this economy. Find initial asset holdings $\bar{\theta}_0^1$ and $\bar{\theta}_0^2$ such that, in equilibrium, $\hat{c}_t^1 = \hat{c}_t^2$, $t = 0, 1, \dots$

b) Suppose that consumers cannot borrow or lend. Define an equilibrium for this liquidity constrained economy.

c) Consider the function

$$f^l(c^s) = u'(c^s)(c^s - 8) + \beta u'(10 - c^s)(9 - c^s).$$

Show, when $\beta = 0.9$, that $f^l(5) > 0$ and that $\hat{c}_t^i = 5$ satisfies all of the equilibrium conditions for the liquidity constrained economy for the right choice of $\bar{\theta}_0^i$. Show, when $\beta = 0.2$, that the solution to $f^l(c^s) = 0$ and $5 \leq c^s \leq 8$ is such that

$$\hat{c}_t^i = \begin{cases} c^s & \text{if } w_t^i = 8 \\ 10 - c^s & \text{if } w_t^i = 1 \end{cases}$$

is an equilibrium allocation for the right choice of $\bar{\theta}_0^i$. (Hint: you can find the solution of $f^I(c^s) = 0$ by solving a quadratic equation.)

3. Suppose now an economy like that in question 2 in which consumers solve the problem

$$\begin{aligned} & \max \sum_{t=0}^{\infty} \beta^t \log c_t^i \\ \text{s.t. } & \sum_{t=0}^{\infty} p_t c_t^i \leq \sum_{t=0}^{\infty} p_t (w_t^i + \bar{\theta}_0^i r) \\ & \sum_{\tau=t}^{\infty} \beta^\tau \log c_\tau^i \geq \sum_{\tau=t}^{\infty} \beta^\tau \log w_\tau^i \\ & c_t \geq 0. \end{aligned}$$

a) Provide a motivation for this environment. Define an equilibrium for this debt constrained economy.

b) Define a sequential markets equilibrium that corresponds to the Arrow-Debreu equilibrium in part a.

c) Consider the function

$$f^D(c^s) = u(c^s) - u(8) + \beta(u(10 - c^s) - u(1)).$$

Show, when $\beta = 0.9$, that $f^D(5) > 0$ and that $\hat{c}_t^i = 5$ satisfies all of the equilibrium conditions for the debt constrained economy for the right choice of $\bar{\theta}_0^i$. Show, when $\beta = 0.2$, that the solution to $f^D(c^s) = 0$ and $5 < c^s \leq 8$ is such that

$$\hat{c}_t^i = \begin{cases} c^s & \text{if } w_t^i = 8 \\ 10 - c^s & \text{if } w_t^i = 1 \end{cases}$$

is an equilibrium allocation for the right choice of $\bar{\theta}_0^i$. (Hint: you need to find the solution of $f^D(c^s) = 0$ using something like Newton's method..)

d) Unfortunately, the value of $\bar{\theta}_0^i$ that you calculate in part e when $\beta = 0.2$ is negative. Can you think of another way to make the proposed steady state an equilibrium? Explain.

e) Prove that an equilibrium allocation for the debt constrained economy is Pareto efficient among those allocations that satisfy

$$c_t^1 + c_t^2 \leq 0$$

$$\sum_{\tau=t}^{\infty} \beta^\tau \log c_\tau^i \geq \sum_{\tau=t}^{\infty} \beta^\tau \log w_\tau^i, \quad i=1,2; \quad t=0,1,\dots$$

4. Consider now a stochastic version of the economy in question 3. Let $\eta_t \in \{1,2\}$ be the event that occurs in period t . Assume that

$$\text{prob}(\eta_{t+1} = 1 | \eta_t = 2) = \text{prob}(\eta_{t+1} = 2 | \eta_t = 1) = \pi.$$

Assume also that

$$w_t^i = \begin{cases} 8 & \text{if } \eta_t = i \\ 1 & \text{if } \eta_t \neq i \end{cases}$$

for $i=1,2$. Redo the analysis of question 3.