PROBLEM SET #3

1. Consider a two sector growth model in which the representative consumer has the utility function

$$\int_0^\infty e^{-\rho t} \log(a_1 c_1(t)^b + a_2 c_2(t)^b)^{1/b} \, dt$$

The two goods are produced from capital and labor using the production functions

$$y_1(t) = \ell_1(t)$$
$$y_2(t) = k_2(t).$$

The investment good is produced using the production function

$$\dot{k}(t) + \delta k(t) = (a_1 x_1(t)^b + a_2 x_2(t)^b)^{1/b}.$$

The initial value of $k(t)$ is $\bar{k}_0$. $\ell(t)$ is fixed at $\bar{\ell}$. The representative consumer has a utility function of the form

a) Carefully define a competitive equilibrium for this economy. (Feel free to suppress the arguments t to make things less cluttered.)

b) Prove that the equilibrium allocation in this economy solves the one sector optimal growth problem

$$\max \ \int_0^\infty e^{-\rho t} \log c(t) dt$$
$$\text{s.t.} \quad c(t) + \dot{k}(t) + \delta k(t) = (a_1 \ell(t)^b + a_2 k(t)^b)^{1/b}$$
$$k(0) = \bar{k}_0, \ \ell(0) = \bar{\ell}$$
$$c(t) \geq 0,$$

where $c(t) = (a_1 c_1(t)^b + a_2 c_2(t)^b)^{1/b}$.

c) Suppose now that there is a world made up of m different countries, all with the same technologies and preferences, but different endowments, $\bar{\ell}^j$ and $\bar{k}_0^j$. Suppose that there is no international borrowing or lending and no international capital flows. Define an equilibrium for the world economy. Prove that, in this equilibrium, the variables

$c(t) = \sum_{j=1}^{m} (a_1 c_1^j(t)^b + a_2 c_2^j(t)^b)^{1/b}$, $k(t) = \sum_{j=1}^{m} k^j(t)$ solve the optimal growth problem in part b in which $\bar{k}_0 = \sum_{j=1}^{m} \bar{k}_0^j$, $\bar{\ell} = \sum_{j=1}^{m} \bar{\ell}^j$.

d) To make things simple, reinterpret variables like $k^j(t)$ and $k(t)$ to be capital per worker. Derive equations that govern the relationship between $k^j(t)$ and $k(t)$ and that between $y^j(t) = p_1(0) + p_2(0)k^j(t)$ and $y(t)$.

2. Choose two countries that are important (to each other) trading partners, one with higher output per worker than the other. Use Summers-Heston data to try to answer Lucas's question: To what extent can differences in output per worker be explained by differences in capital per worker? Use both Summers-Heston data on capital per worker and IMF real interest rate data to try to make inferences about differences in capital per worker across countries. Discuss what you have learned from this exercise and what more you think would be relevant in explaining differences in output per worker across countries.