INTERNATIONAL TRADE AND PAYMENTS THEORY **T. J. KEHOE**
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PROBLEM SET #3

1. Consider a two sector growth model in which the representative consumer has the utility function

$$\int_0^\infty e^{-\rho t} \log(a_1 c_1^b + a_2 c_2^b)^{1/b} dt$$

and in which investment is produced according to

$$\dot{k} + \delta k = d(a_1 x_1^b + a_2 x_2^b)^{1/b}.$$

Feasible consumption/investment plans satisfy the feasibility constraints

$$c_1 + x_1 = \phi_1(k_1, \ell_1) = k_1$$
$$c_2 + x_2 = \phi_2(k_2, \ell_2) = \ell_2 \; .$$

where

$$k_1 + k_2 = k$$
$$\ell_1 + \ell_2 = \ell \; .$$

Each worker is endowed with one unit of labor in every period. There is a continuum of measure constant $\bar{\ell}$ of workers. The initial value of the capital-labor ratio $k(t)$ is $\bar{k}_0$.

a) Carefully define a competitive equilibrium for this economy. (It is probably easiest to express all variables in per capita terms.)

b) Reduce the equilibrium conditions to two differential equations in k and $z = c/k$ and a transversality condition. Here $c = d(a_1 c_1^b + a_2 c_2^b)^{1/b}$. Draw phase diagrams illustrating the different possible equilibrium paths for any $\bar{k}_0 > 0$.

c) Suppose now that there is a world made up of m different countries all with the same technologies and preferences, but different endowments, $\bar{\ell}^j$ and $\bar{k}_0^j$. Suppose that there is no international borrowing or lending and there are no international capital flows. Define an equilibrium for the world economy. Prove that in this equilibrium the variables $c_i(t) = \sum_{j=1}^m \bar{\ell}^j c_i^j(t) / \sum_{j=1}^m \bar{\ell}^j$, $k(t) = \sum_{j=1}^m \bar{\ell}^j k^j(t) / \sum_{j=1}^m \bar{\ell}^j$, $p_i(t)$, $r(t)$, and $w(t)$ satisfy the equilibrium conditions for the equilibrium in part a where $\bar{k}_0 = \sum_{j=1}^{m} \bar{\ell}^j \bar{k}_0^j / \sum_{j=1}^{m} \bar{\ell}^j$, $\bar{\ell} = \sum_{j=1}^{m} \bar{\ell}^j$.

d) State and prove versions of the factor price equalization theorem, the Stolper-Samuelson theorem, the Rybczynski theorem, and the Heckscher-Ohlin theorem for this particular world economy.

e) Derive equations that govern the relationship between $k^j(t)$ and $k(t)$. Explain how to derive a relationship between $y^i(t) = p_1(t)k^j(t) + p_2(t)\bar{\ell}^j$ and $y(t)$.

f) What effects do the assumptions of no international borrowing and lending and no international capital flows have on your analysis? Try to be precise about which variables are uniquely determined in the equilibrium with borrowing and lending and/or capital flows are which would not be uniquely determined.

2. Consider now a discrete-time version of the model in question 1: The representative consumer has the utility function

$$\sum_{t=0}^{\infty} \beta^t \log(a_1 c_{1t}^b + a_2 c_{2t}^b)^{1/b} .$$

The investment good is produced according to

$$k_{t+1} - (1-\delta)k_t = d(a_1 x_{1t}^b + a_2 x_{2t}^b)^{1/b} .$$

Again, feasible consumption/investment plans satisfy the feasibility constraints

$$\begin{aligned} c_{1t} + x_{1t} &= \phi_1(k_{1t}, \ell_{1t}) = k_{1t} \\ c_{2t} + x_{2t} &= \phi_2(k_{2t}, \ell_{2t}) = \ell_{2t} \ . \end{aligned}$$

where

$$\begin{aligned} k_{1t} + k_{2t} &= k_t \\ \ell_{1t} + \ell_{2t} &= \ell_t \ . \end{aligned}$$

Each worker is endowed with one unit of labor in every period. There is a continuum of measure constant $\bar{\ell}$ of workers. The initial value of the capital-labor ratio k_t is $\bar{k}_0$.

a) Carefully define a competitive equilibrium for this economy. (Once again, it is probably easier to work with per capita variables.)

b) Reduce the equilibrium conditions to two difference equations in k_t and c_t and a transversality condition. Here $c_t = d(a_1 c_{1t}^b + a_2 c_{2t}^b)^{1/b}$.

c) Suppose now that there is a world made up of m different countries all with the same technologies and preferences, but different endowments, $\bar{\ell}^j$ and $\bar{k}_0^j$. Suppose that there is no international borrowing or lending and there are no international capital flows. Define an equilibrium for the world economy. Prove that in this equilibrium the variables $c_{it} = \sum_{j=1}^m \bar{\ell}^j c_{it}^j / \sum_{j=1}^m \bar{\ell}^j$, $k_t = \sum_{j=1}^m \bar{\ell}^j k_t^j / \sum_{j=1}^m \bar{\ell}^j$, p_{it}, r_t, and w_t satisfy the equilibrium conditions for the equilibrium in part a where $\bar{k}_0 = \sum_{j=1}^m \bar{\ell}^j \bar{k}_0^j / \sum_{j=1}^m \bar{\ell}^j$, $\bar{\ell} = \sum_{j=1}^m \bar{\ell}^j$.

d) Consider the case where $\delta = 1$. Let $z_0 = c_0 / (\beta r_0 k_0)$ and $z_t = c_{t-1} / k_t$, $t = 1, 2, \ldots$. Transform the two difference equation in part b into two difference equations in k_t and z_t. Prove that

$$\frac{k_t^i - k_t}{k_t} = \frac{z_t}{z_{t-1}} \left(\frac{k_{t-1}^i - k_{t-1}}{k_{t-1}} \right) = \frac{z_t}{z_0} \left(\frac{\bar{k}_0^i - \bar{k}_0}{\bar{k}_0} \right).$$

e) Again consider the case where $\delta = 1$. Let $s_t = c_t / y_t$ where $y_t = p_{1t} k_t + p_{2t} \bar{\ell} = d(a_1 k_t^b + a_2 \bar{\ell}^b)^{1/b}$. Transform the two difference equation in part b into two difference equations in k_t and s_t. Prove that

$$\frac{y_t^i - y_t}{y_t} = \frac{s_t}{s_{t-1}} \left(\frac{y_{t-1}^i - y_{t-1}}{y_{t-1}} \right) = \frac{s_t}{s_0} \left(\frac{y_0^i - y_0}{y_0} \right).$$

f) Assume that $\delta = 1$, that $c_t = d c_{1t}^{a_1} c_{2t}^{a_2}$ and that $k_{t+1} = d x_{1t}^{a_1} x_{2t}^{a_2}$. (This is, of course, the limiting case where $b = 0$.) Find analytical solutions to parts b, c, d, and e.

3. Suppose again that $\delta = 1$, that $c_t = d c_{1t}^{a_1} c_{2t}^{a_2}$ and that $k_{t+1} = d x_{1t}^{a_1} x_{2t}^{a_2}$. Now suppose that

$$c_{1t} + x_{1t} = \phi_1(k_{1t}, \ell_{1t}) = \theta_1 \ell_{1t}^{1-\alpha_1} k_{1t}^{\alpha_1}$$
$$c_{2t} + x_{2t} = \phi_2(k_{2t}, \ell_{2t}) = \theta_2 \ell_{2t}^{1-\alpha_2} k_{2t}^{\alpha_2}.$$

a) Let $F(k, \ell)$ be the maximum value of

$$\max \ d y_{1t}^{a_1} y_{2t}^{a_2}$$

$$\text{s.t. } y_1 = \theta_1 \ell_1^{1-\alpha_1} k_1^{\alpha_1}$$
$$y_2 = \theta_2 \ell_2^{1-\alpha_2} k_2^{\alpha_2}$$
$$k_1 + k_2 = k$$
$$\ell_1 + \ell_2 = \ell$$
$$k_j, \ell_j \geq 0.$$

Show that $F(k,\ell)$ has the form $Dk^A \ell^{1-A}$.

b) Suppose now that there is a world made up of m different countries all with the same technologies and preferences, but different endowments, $\bar{\ell}^j$ and $\bar{k}_0^j$. Suppose that there is no international borrowing or lending and there are no international capital flows. Define an equilibrium for the world economy.

c) Using the answers to parts a and b, show that necessary and sufficient conditions for the integrated equilibrium approach to work for all $t = T, T+1, \ldots$, is that

$$\kappa_1 (k_t / \bar{\ell}) \geq k_t^i / \bar{\ell}^i \geq \kappa_2 (k_t / \bar{\ell}) \text{ for all } i = 1, \ldots, m \text{ and all } t = T, T+1, \ldots.$$

For some $\kappa_1, \kappa_2 > 0$.

d) Suppose that, in some period T,

$$\kappa_1 (k_T / \bar{\ell}) \geq k_T^i / \bar{\ell}^i \geq \kappa_2 (k_T / \bar{\ell}) \text{ for all } i = 1, \ldots, m.$$

Use the answers to parts a, b, and c and the answer to part f of question 3 to calculate analytical expressions for the equilibrium values of the variables in part b for all $t = T, T+1, \ldots$. [Hint: You can show that $\kappa_1 (k_t / \bar{\ell}) \geq k_t^i / \bar{\ell}^i \geq \kappa_2 (k_t / \bar{\ell})$ for all $i = 1, \ldots, m$ and all $t = T, T+1, \ldots$.]