## **Balanced Growth Paths**

Suppose that we have an economy in which feasible consumption and investment satisfy

$$C_t + I_t = Y_t = A_t K_t^{\alpha} L_t^{1-\alpha}$$
$$K_{t+1} = (1-\delta) K_t + I_t.$$

Here we identify  $Y_t$  with real GDP in the data.

Suppose also that total factor productivity,  $A_t$ , and working age population,  $N_t$ , grow at constant rates:

$$A_{t+1} = g_A A_t = g_A^{t+1} A_0$$
$$N_{t+1} = g_N N_t = g_N^{t+1} N_0.$$

Here, for example,  $g_A$  is the constant growth factor for productivity and  $g_A - 1$  is the constant growth rate.

A **balanced growth path** is a path of  $C_t$ ,  $I_t$ ,  $Y_t$ ,  $K_t$ ,  $L_t$ , t = 0, 1, ..., such that  $C_t$ ,  $I_t$ ,  $Y_t$ ,  $K_t$  grow at the same rate

$$\frac{C_{t+1}}{C_t} = \frac{I_{t+1}}{I_t} = \frac{Y_{t+1}}{Y_t} = \frac{K_{t+1}}{K_t} = g$$

and such that hours worked per working age person are constant

$$\frac{L_{t+1}}{N_{t+1}} = \frac{L_t}{N_t} = \frac{L_0}{N_0} \,.$$

Notice that

$$\frac{Y_{t+1}}{Y_t} = \frac{A_{t+1}K_{t+1}^{\alpha}L_{t+1}^{1-\alpha}}{A_tK_t^{\alpha}L_t^{1-\alpha}}$$
$$g = \left(\frac{A_{t+1}}{A_t}\right) \left(\frac{K_{t+1}}{K_t}\right)^{\alpha} \left(\frac{L_{t+1}}{L_t}\right)^{1-\alpha}$$
$$g = g_A g^{\alpha} \left(\frac{L_{t+1}/N_{t+1}}{L_t/N_t}\frac{N_{t+1}}{N_t}\right)^{1-\alpha}$$
$$g = g_A g^{\alpha} g_N^{1-\alpha}$$
$$g^{1-\alpha} = g_A g_N^{1-\alpha}$$
$$g = g_A^{\frac{1}{1-\alpha}} g_N.$$

That is, the growth factor g is the growth factor for productivity,  $g_A$ , exponentiated to the power  $1/(1-\alpha)$ , multiplied by the growth factor for population  $g_N$ .

Suppose that we divide our variables by the working age population. Then, for example,  $Y_t / N_t$  is real GDP per working age person. Notice that

$$\frac{Y_{t+1} / N_{t+1}}{Y_t / N_t} = \left(\frac{Y_{t+1}}{Y_t}\right) \left(\frac{N_t}{N_{t+1}}\right) = \frac{g}{g_N} = \frac{g_A^{\frac{1}{1-\alpha}}g_N}{g_N} = g_A^{\frac{1}{1-\alpha}}.$$

That is, in the balanced growth path, the growth of real variables per working age person depends only on the growth of productivity.

Notice too that

$$\frac{K_{t+1}}{Y_{t+1}} = \frac{gK_t}{gY_t} = \frac{K_t}{Y_t} = \frac{K_0}{Y_0}$$

In other words, the ratios of real variables, such as the capital-output ratio, stay constant in the balanced growth path.

Let us now rewrite the aggregate production function

$$Y_{t} = A_{t}K_{t}^{\alpha}L_{t}^{1-\alpha}$$

$$Y_{t}^{1-\alpha} = A_{t}\left(\frac{K_{t}}{Y_{t}}\right)^{\alpha}L_{t}^{1-\alpha}$$

$$Y_{t} = A_{t}^{1-\alpha}\left(\frac{K_{t}}{Y_{t}}\right)^{\frac{\alpha}{1-\alpha}}L_{t}$$

$$\frac{Y_{t}}{N_{t}} = A_{t}^{1-\alpha}\left(\frac{K_{t}}{Y_{t}}\right)^{\frac{\alpha}{1-\alpha}}\left(\frac{L_{t}}{N_{t}}\right)$$

We use this equation to do our growth accounting. Notice that, in the balanced growth path, the first term on the right hand side of the equation grows by the factor g and the second two terms are constant.

Our growth accounting decomposes growths and fluctuations in output per working age person  $Y_t / N_t$ , into a productivity factor  $A_t^{1-\alpha}$ , a capital factor  $(K_t / Y_t)^{\frac{\alpha}{1-\alpha}}$ , and a labor factor  $L_t / N_t$ .

A graph of the growth accounting for the United States 1960–2000 shows that business cycles over that period produced only minor deviations from balanced growth: Most of the growth in  $Y_t / N_t$  is accounted for by growth in the productivity factor  $A_t^{1-\alpha}$ , and the capital factor  $(K_t / Y_t)^{\frac{\alpha}{1-\alpha}}$  and the labor factor  $L_t / N_t$  are rightly constant. There are

small, short-term business cycle fluctuations. We can also see a couple of noticeable long-term fluctuations. In particular, productivity growth slowed down during the period 1973–1983. In addition, during the period 1983–2000 output per working age person grew at a higher rate than the rate implied by productivity growth. The additional growth in output was due to growth in the labor factor, that is, to working age people working more hours.



**Growth Accounting for the United States** 

An aside:

Because  $K_t / Y_t$  is constant in a balanced growth path, we ascribe some of the growth in  $Y_t / N_t$  due to increases in the capital stock to productivity. This is how our growth accounting differs from the more traditional growth accounting decomposition

$$\frac{Y_t}{N_t} = A_t \left(\frac{K_t}{N_t}\right)^{\alpha} \left(\frac{L_t}{N_t}\right)^{1-\alpha}.$$

In an economy where  $Y_t / N_t$  grows because of growth in productivity or growth in hours worked per working age person, the capital stock need to grow just to maintain  $K_t / Y_t$ constant. The additional growth in  $Y_t / N_t$  due to this induced growth in  $K_t$  we ascribe to  $A_t$  or  $L_t / N_t$ . This is done by assigning these terms larger exponents in the growth accounting equation: In the case of  $A_t$ , the exponent is  $1/(1-\alpha) > 1$ , and, in the case of  $L_t / N_t$ , it is  $1 > 1 - \alpha$ . Growth in  $K_t$  shows up as a contribution to growth in  $Y_t / N_t$  only if it results in growth in the capital-output ratio  $K_t / Y_t$ .