CENTRO DE ESTUDIOS MONETARIOS Y FINANCIEROS **T. J. KEHOE**
MACROECONOMICS I **WINTER 2011**

PROBLEM SET #5

1. Consider the social planning problem

$$\begin{aligned} &\max \ \textstyle\sum_{t=0}^{\infty} \beta^t \log \ c_t \\ &\text{s.t.} \quad c_t + k_{t+1} \le \theta k_t^{\alpha} \\ &\qquad c_t, k_t \ge 0 \\ &\qquad k_0 \le \bar{k}_0 \,. \end{aligned}$$

a) Write down the Euler conditions and the transversality condition for this problem.

b) Let $v(k_t, k_{t+1})$ be the solution to

$$\begin{aligned} &\max_{c_t} \ \log \ c_t \\ &\text{s.t.} \quad c_t + k_{t+1} \le \theta k_t^{\alpha} \\ &\qquad c_t \ge 0 \end{aligned}$$

for fixed k_t, k_{t+1}. What is $v(k_t, k_{t+1})$? What conditions do k_t and k_{t+1} need to satisfy to ensure $c_t, k_{t+1}, \ge 0$? If we write these conditions as $k_{t+1} \in \Gamma(k_t)$, what is $\Gamma(k_t)$?

c) Write down the Euler conditions and the transversality condition for the problem

$$\begin{aligned} &\max \textstyle\sum_{t=0}^{\infty} \beta^t v(k_t, k_{t+1}) \\ &\text{s.t.} \quad k_{t+1} \in \Gamma(k_t) \\ &\qquad k_0 \le \bar{k}_0 . \end{aligned}$$

d) Prove that a sequence $\hat{k}_0, \hat{k}_1, \ldots$ solves the conditions in part (c) if an only if there exist sequences of Lagrange multipliers $\hat{p}_0, \hat{p}_1, \ldots$ and of consumption $\hat{c}_0, \hat{c}_1, \ldots$ such that $(\hat{c}_0, \hat{c}_1, \ldots, \hat{k}_0, \hat{k}_1, \ldots, \hat{p}_{0,} \hat{p}_1, \ldots)$ satisfy the conditions in part (a).

e) Prove that, if a sequence $\hat{k}_0, \hat{k}_1, \ldots$ satisfies the Euler conditions and transversality condition in part c, then it solves the related planning problem. (Hint: you can adapt the general proof on pp. 98-99 of Stokey, Lucas, and Prescott to these specific functions.)

2. Let $v(k_t, k_{t+1})$ and $\Gamma(k_t)$ be defined as in part b of question 1.

a) Consider the dynamic programming problem with the Bellman equation

$$\begin{gathered} V(k) = \max v(k, k') + \beta V(k') \\ \text{s.t.} \quad k' \in \Gamma(k). \end{gathered}$$

Guess that $V(k)$ has the form $a_0 + a_1 \log k$ and solve for a_0 and a_1.

b) What is the policy function $g(k)$ such that $k' = g(k)$? Verify that $k_{t+1} = g(k_t)$ satisfies the Euler equations and the transversality condition in part c of question 1.

c) Try to approximate $V(k)$: Guess that $V_0(k) = 0$ for all k and use the iterative updating rule

$$\begin{gathered} V_{n+1}(k) = \max \; v(k,k') + \beta V_n(k') \\ \text{s.t.} \quad k' \in \Gamma(k). \end{gathered}$$

Calculate the functions V_1, V_2, V_3, and V_4.

3. Consider the dynamic programming problem where the Bellman equation in question 2 generates a functional equation $T : B(K) \to B(K)$ of the form

$$\begin{gathered} T(V)(k) = \max v(k, k') + \beta V(k') \\ \text{s.t.} \quad k' \in \Gamma(k), \end{gathered}$$

where $B(K)$ is the set of bounded functions $V : K \to R$, $K \subset R$.

a) State Blackwell's sufficient conditions for T to be a contraction. Prove that, if T satisfies these conditions, then T is a contraction.

b) Prove that T satisfies Blackwell's sufficient conditions.

4. Consider an economy specified as in question 1 except that the representative consumer has the utility function

$$\sum_{t=0}^{\infty} \beta^t \left(\gamma \log c_t + (1-\gamma) \log(1-\ell_t)\right)$$

where ℓ_t is the supply of labor.

a) Define a sequential markets equilibrium for this economy.

b) Define an Arrow-Debreu equilibrium for this economy.

c) Let $v(k_t, k_{t+1})$ be the solution to

$$\begin{aligned} \max_{c_t, \ell_t} \quad & \gamma \log c_t + (1-\gamma) \log(1-\ell_t) \\ \text{s.t.} \quad & c_t + k_{t+1} \leq \theta k_t^{\alpha} \ell_t^{1-\alpha} \\ & c_t \geq 0,\ 1 \geq \ell_t \geq 0 \end{aligned}$$

for fixed k_t, k_{t+1}. Explain how you could calculate $v(k_t, k_{t+1})$ on the computer. Explain how you could check that the condition $k_{t+1} \in \Gamma(k_t)$ is satisfied.