An Example of a Model with Debt Constraints

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Consider an economy like that in question 2 in problem set 7 where there is a continuum [0,1] of consumers of two symmetric types who live forever. Consumers have utility

$$\sum_{t=0}^{\infty} \beta^t \log c_t^i .$$

Suppose, as in the example in Kehoe and Levine (2001), that $\beta = 0.5$ and that consumers of type 1 have an endowment stream of the single good in each period

$$(w_0^1, w_1^1, w_2^1, w_3^1, \ldots) = (\omega^g, \omega^b, \omega^g, \omega^b, \ldots) = (24,\ 9,\ 24,\ 9, \ldots),$$

while consumers of type 2 have

$$(w_0^2, w_1^2, w_2^2, w_3^2, \ldots) = (\omega^b, \omega^g, \omega^b, \omega^g, \ldots) = (9,\ 24,\ 9,\ 24, \ldots).$$

In addition there is one unit of trees that produce $r = 1$ units of the good every period. Each consumer of type i owns $\bar{\theta}_0^i$ of such trees in period 0, $\bar{\theta}_0^i \geq 0$, $\bar{\theta}_0^1 + \bar{\theta}_0^2 = 1$. Trees do not grow or decay.

In the sequential markets version of this model, consumers solve the problem

$$\begin{gathered}
\max \sum_{t=0}^{\infty} \beta^t \log c_t^i \\
\text{s.t. } c_t^i + v_t \theta_{t+1}^i \leq w_t^i + \left(v_t + r\right) \theta_t^i \\
\sum_{\tau=t}^{\infty} \beta^{\tau} \log c_{\tau}^i \geq \sum_{\tau=t}^{\infty} \beta^{\tau} \log w_{\tau}^i \\
c_t^i \geq 0 \\
\theta_t^i \geq -\Theta \\
\theta_0^i = \bar{\theta}_0^i .
\end{gathered}$$

An **equilibrium** is sequences $\hat{v}_t$, $\hat{c}_t^i$, $\hat{\theta}_t^i$ such that

1. Given $\hat{v}_t$, the consumers choose $\hat{c}_t^i$, $\hat{\theta}_t^i$ to solve their maximization problems.

2. Goods markets clear:

$$\hat{c}_t^1 + \hat{c}_t^2 = w_t^1 + w_t^2 + r = 24 + 9 + 1 = 34 .$$

3. Asset markets clear:

$$\hat{\theta}_t^1 + \hat{\theta}_t^2 = 1 .$$

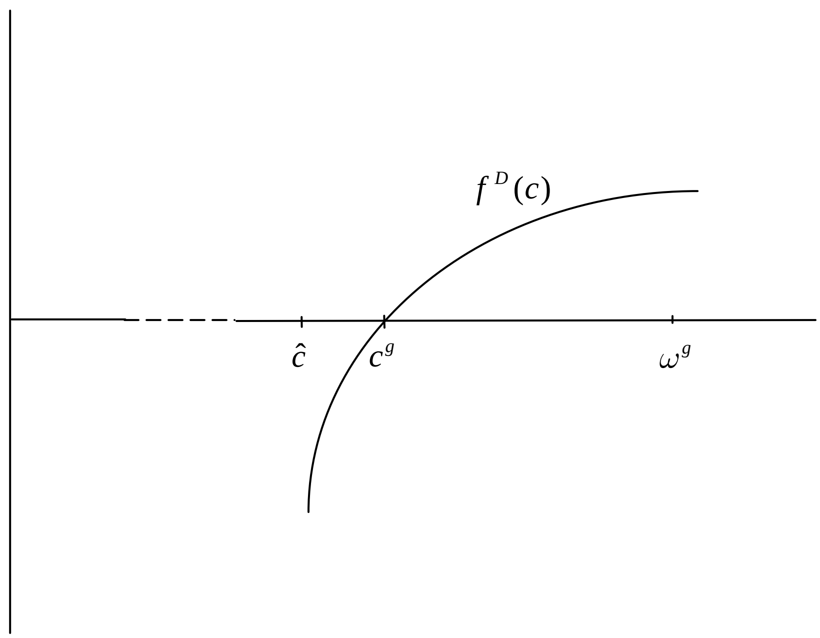
A **symmetric steady state** is v, (c^g, c^b), (θ^g, θ^b) such that $v_t = v$,

$$c_t^i = \begin{cases} c^g & \text{if } w_t^i = 24 \\ c^b & \text{if } w_t^i = 9 \end{cases},$$

and

$$\theta_t^i = \begin{cases} \theta^g & \text{if } w_t^i = 24 \\ \theta^b & \text{if } w_t^i = 9 \end{cases}$$

satisfy the equilibrium conditions for the right choice of $(\bar{\theta}_0^1, \bar{\theta}_0^2)$.



Consider the function

$$f^D(c^g) = \log c^g - \log 24 + \beta(\log(34 - c^g) - \log 9) .$$

Let $\hat{c} = (w_t^1 + w_t^2 + r)/2 = 17$. Notice that

$$f^D(\omega^g) = f^D(24) = \log 24 - \log 24 + \beta(\log(34-24) - \log 9) = \beta(\log 10 - \log 9) > 0 .$$

Notice too that f^D is concave:

$$D^2 f(c) = -\frac{1}{c^2} - \frac{\beta}{(34-c)^2} < 0 .$$

Consequently, if $f^D(\hat{c}) < 0$ as in the diagram, $f^D(c)$ can equal 0 only once in the interval $[\hat{c}, \omega^g]$.

There are two possibilities: Either

$$f^D(\hat{c}) = \log \hat{c} - \log 24 + \beta(\log(34-\hat{c}) - \log 9) \geq 0$$

or there exists $c^g \in [\hat{c}, \omega^g]$ such that

$$f^D(c^g) = \log c^g - \log 24 + \beta(\log(34-c^g) - \log 9) .$$

Since

$$f^D(\hat{c}) = \log 17 - \log 24 + 0.5(\log 17 - \log 9) < 0 ,$$

we look for such a c^g. Setting $f^D(c^g) = 0$, we obtain $c^g = 18$. We want to find values of c^b, θ^g, θ^b, and v such that these variables constitute a symmetric steady state. Obviously, $c^b = 34 - 18 = 16$.

The first order conditions for the consumer's problem are

$$\beta^t \frac{1}{c_t^i} - \lambda_t^i + \beta^t \frac{1}{c_t^i} \sum_{\tau=0}^{t} \mu_t^i = 0$$

$$-\lambda_t^i v_t + \lambda_{t+1}^i \left(v_{t+1} + r\right) = 0 .$$

The individual rationality constraint, if it binds at all, can bind only when $c_t^i = c^g$. (The consumer who receives the bad shock is always happy to receive more that his current income.) In other words, if $c_t^i = c^b$, then $\mu_t^i = 0$. Consider a situation where $c_t^i = c^g$ and $c_{t+1}^i = c^b$. Then the first order conditions with respect to c_t^i and c_{t+1}^i are

$$\beta^t \frac{1}{c_t^i} - \lambda_t^i + \beta^t \frac{1}{c_t^i} \sum_{\tau=0}^{t} \mu_t^i = \beta^t \frac{1}{c_t^i} \left(1 + \sum_{\tau=0}^{t} \mu_t^i\right) - \lambda_t^i = 0$$

$$\beta^{t+1} \frac{1}{c_{t+1}^i} - \lambda_{t+1}^i + \beta^{t+1} \frac{1}{c_{t+1}^i} \sum_{\tau=0}^{t+1} \mu_t^i = \beta^{t+1} \frac{1}{c_{t+1}^i} \left(1 + \sum_{\tau=0}^{t} \mu_t^i\right) - \lambda_{t+1}^i = 0 .$$

Letting $v_t = v$, we can write the first order conditions for the consumer who receives the high consumption level today as

$$\frac{u'(c^g)}{\beta u'(c^b)} = \frac{c^b}{\beta c^g} = \frac{v+r}{v}$$

$$\frac{\frac{1}{18}}{\frac{1}{2}\frac{1}{16}} = \frac{v+1}{v},$$

which can be solved to yield $v = 9/7$. The budget constraint for the consumer who receives the high endowment, $w_t^i = 24$, is

$$c^g + v\theta^g = \omega^g + \left(v+r\right)\theta^b$$

$$18 + \frac{9}{7}\theta^g = 24 + \left(\frac{9}{7}+1\right)\left(1-\theta^g\right),$$

which can be solved to yield $\theta^g = 58/25 = 2.32$. $\theta^b = 1-\theta^g = -1.32$. We can now go back and verify that all of the equilibrium conditions are satisfied for the right choice of $\bar{\theta}_0^1$ and $\bar{\theta}_0^2$.

Now consider the Arrow-Debreu version of this model. The consumers solve

$$\begin{gathered}\max \sum\nolimits_{t=0}^{\infty} \beta^t \log\, c_t^i \\ \text{s.t. } \sum\nolimits_{t=0}^{\infty} p_t c_t^i \le \sum\nolimits_{t=0}^{\infty} p_t \left(w_t^i + r\bar{\theta}_0^i\right) \\ \sum\nolimits_{\tau=t}^{\infty} \beta^{\tau}\, \log\, c_\tau^i \ge \sum\nolimits_{\tau=t}^{\infty} \beta^{\tau}\, \log\, w_\tau^i \\ c_t \ge 0.\end{gathered}$$

An **equilibrium** is sequences $\hat{p}_t$, $\hat{c}_t^i$ such that

1. Given $\hat{p}_t$, the consumers choose $\hat{c}_t^i$ to solve their maximization problems.

2. Goods markets clear:

$$\hat{c}_t^1 + \hat{c}_t^2 = w_t^1 + w_t^2 + r = 24 + 9 + 1 = 34.$$

A **symmetric steady state** is p, (c^g, c^b) such that $p_t = p^t$ and

$$c_t^i = \begin{cases} c^g & \text{if } w_t^i = 24 \\ c^b & \text{if } w_t^i = 9 \end{cases}$$

satisfy the equilibrium conditions for the right choice of $(\bar{\theta}_0^1, \bar{\theta}_0^2)$.

There are a number of ways that we can find the prices $p_t = p^t$ that, together with the allocation

$$c_t^i = \begin{cases} 18 & \text{if } w_t^i = 24 \\ 16 & \text{if } w_t^i = 9 \end{cases},$$

constitute an Arrow-Debreu equilibrium. Consider the first order conditions

$$\beta^t \frac{1}{c_t^i} - \lambda^i p_t + \beta^t \frac{1}{c_t^i} \sum_{\tau=0}^{t} \mu_t^i = 0 .$$

Once again we can show that, if $c_t^i = c^b$, then $\mu_t^i = 0$. Consider a situation where $c_t^i = c^g$ and $c_{t+1}^i = c^b$. Then the first order conditions with respect to c_t^i and c_{t+1}^i are

$$\beta^t \frac{1}{c_t^i} - \lambda^i p_t + \beta^t \frac{1}{c_t^i} \sum_{\tau=0}^{t} \mu_t^i = \beta^t \frac{1}{c_t^i} \left(1 + \sum_{\tau=0}^{t} \mu_t^i\right) - \lambda^i p_t = 0$$

$$\beta^{t+1} \frac{1}{c_{t+1}^i} - \lambda^i p_{t+1} + \beta^{t+1} \frac{1}{c_{t+1}^i} \sum_{\tau=0}^{t+1} \mu_t^i = \beta^{t+1} \frac{1}{c_{t+1}^i} \left(1 + \sum_{\tau=0}^{t} \mu_t^i\right) - \lambda^i p_{t+1} = 0 .$$

These imply that

$$\frac{\beta \frac{1}{c^b}}{\frac{1}{c^g}} = \frac{p_{t+1}}{p_t} .$$

Consequently, $p_t = (9/16)^t$.

Consider now a stochastic version of this economy. Let $\eta_t \in \{1, 2\}$ be the event that occurs in period t. Assume that

$$\text{prob}(\eta_{t+1} = 1 | \eta_t = 2) = \text{prob}(\eta_{t+1} = 2 | \eta_t = 1) = \pi = 1/2 .$$

and that

$$w_t^i = \begin{cases} 24 & \text{if } \eta_t = i \\ 9 & \text{if } \eta_t \neq i \end{cases}$$

for $i = 1, 2$. Once again, there is one unit of trees that produce $r = 1$ units of the good every period.

In the sequential markets version of this model, consumers solve the problem

$$\max \sum_{s\in S} \beta^{t(s)} \pi_s \log\ c_s^i$$
$$\text{s.t. } c_s^i + q_{(s,1)}\theta_{(s,1)}^i + q_{(s,2)}\theta_{(s,2)}^i \le w_s^i + \left(v_s + r\right)\theta_s^i$$
$$\sum_{\sigma\ge s} \beta^{t(\sigma)}\ \pi_\sigma \log\ c_\sigma^i \ge \sum_{\sigma\ge s} \beta^{t(\sigma)}\ \pi_\sigma \log\ w_\sigma^i$$
$$c_s \ge 0$$
$$\theta_s^i \ge -\Theta$$
$$\theta_{\eta_0}^i = \bar{\theta}_0^i .$$

An **equilibrium** is sequences $\hat{v}_s$, $\hat{q}_{(s,1)}$, $\hat{q}_{(s,2)}$ $\hat{c}_s^i$, $\hat{\theta}_{(s,1)}^i$, $\hat{\theta}_{(s,2)}^i$ such that

1. Given $\hat{v}_s$, $\hat{q}_{(s,1)}$, $\hat{q}_{(s,2)}$, the consumers choose $\hat{c}_s^i$, $\hat{\theta}_{(s,1)}^i$, $\hat{\theta}_{(s,2)}^i$ to solve their maximization problems.

2. Goods markets clear:

$$\hat{c}_s^1 + \hat{c}_s^2 = w_s^1 + w_s^2 + r = 24 + 9 + 1 = 34 .$$

3. Asset markets clear:

$$\hat{\theta}_{(s,\eta')}^1 + \hat{\theta}_{(s,\eta')}^2 = 1.$$

A **symmetric stochastic steady state** is v, (q_n, q_r) (c^g, c^b), (θ^g, θ^b) such that $v_t = v$, $q_{(s,\eta')} = q_n$ if $\eta' = \eta_s$, that is, if there is no reversal, $q_{(s,\eta')} = q_r$ if $\eta' \ne \eta_s$, that is, if there is a reversal,

$$c_s^i = \begin{cases} c^g & \text{if } w_s^i = 24 \\ c^b & \text{if } w_s^i = 9 \end{cases},$$

and

$$\theta_s^i = \begin{cases} \theta^g & \text{if } w_s^i = 24 \\ \theta^b & \text{if } w_s^i = 9 \end{cases}$$

satisfy the equilibrium conditions for the right choice of $(\bar{\theta}_0^1, \bar{\theta}_0^2)$.

Let us begin by observing that, if we add together the budget constraints of the two consumer types, we obtain

$$c_s^1 + c_s^2 + q_{(s,1)}\left(\theta_{(s,1)}^1 + \theta_{(s,1)}^2\right) + q_{(s,2)}\left(\theta_{(s,2)}^1 + \theta_{(s,2)}^2\right) = w_s^1 + w_s^2 + \left(v_s + r\right)\left(\theta_s^1 + \theta_s^2\right).$$

Feasibility implies that

$$q_{(s,1)} + q_{(s,2)} = v_s \, .$$

This can be thought of as an arbitrage condition: The price that a consumer receives for selling one unit of trees in state s is v_s. The price that a consumer pays to receive this tree if event 1 occurs is $q_{(s,1)}$, and the price that a consumer pays to receive this tree if no reversal occurs is $q_{(s,2)}$. Since these two events are exhaustive and mutually exclusive, the total amount paid for the tree is $q_{(s,1)} + q_{(s,2)}$.

The first order conditions for the consumer's problem are

$$\beta^{t(s)}\pi_s \frac{1}{c_s^i} - \lambda_s^i + \beta^{t(s)}\pi_s \frac{1}{c_s^i}\sum\nolimits_{\sigma \le s} \mu_\sigma^i = 0$$

$$-\lambda_s^i q_{(s,\eta')} + \lambda_{(s,\eta')}^i \left(v_{(s,\eta')} + r\right) = 0 \, .$$

Once again we can show that, if $c_{(s,\eta')}^i = c^b$, then $\mu_{(s,\eta')}^i = 0$. First, consider the case where $c_s^i = c^g$ and $c_{(s,\eta')}^i$ is c^b. The, since $\mu_{(s,\eta')}^i = 0$, we can write out the first order condition for $c_{(s,\eta)}^i$ as

$$\beta^{t(s)+1}\pi_s\pi \frac{1}{c^b} - \lambda_{(s,\eta')}^i + \beta^{t(s)+1}\pi_s\pi \frac{1}{c^b}\sum\nolimits_{\sigma \le s} \mu_\sigma^i = 0 \, .$$

Combining this with the first order condition for c_s^i, as in the deterministic case, we obtain

$$\frac{u'(c^g)}{\beta\pi u'(c^b)} = \frac{v_{(s,\eta')} + r}{q_{(s,\eta')}} \, .$$

Imposing $v_{(s,\eta')} = v$, this becomes

$$q_{(s,\eta')} = q_r = \frac{\beta\pi u'(c^b)}{u'(c^g)}\left(v + r\right).$$

Here q_r is the price paid for an Arrow security to purchase one unit of the tree in the case of reversal — where $\eta_s = 1$, for example, but $\eta' = 2$. Consider now the case where $c_s^i = c^b$ and $c_{(s,\eta')}^i$ is c^b. (We think of this as the same state s; we are just looking at the other consumer type's first order conditions.) We obtain

$$\frac{u'(c^b)}{\beta(1-\pi)u'(c^b)} = \frac{v+r}{q_n}$$

$$q_n = \beta(1-\pi)(v+1).$$

Here q_n is the price paid for an Arrow security to purchase one unit of the tree in the case of no reversal.

Consider now the function

$$f^D(x^g) = (1-\beta(1-\pi))\left(\log c^g - \log 24\right) + \beta\pi\left(\log(34-c^g) - \log 9\right).$$

Setting $f^D(c^g) = 0$ in the case where $\beta = 0.5$ and $\pi = 0.5$, we obtain $c^g = 21.5252$. We want to find values of c^b, θ^g, θ^b, q_r, q_n, and v such that these variables constitute a symmetric steady state. Obviously, $c^b = 12.4748$.

Plugging these values into the first order conditions that we obtained above, we find that

$$q_r = \frac{\frac{1}{4}\frac{1}{12.4748}}{\frac{1}{21.5252}}(v+1) = 0.4314(v+1)$$

$$q_n = 0.25(v+1).$$

Notice that we can combine these two conditions to obtain

$$q_r + q_n = 0.6814(v+1)$$

$$v = 0.6814(v+1),$$

which implies that $v = 2.1385$, $q_r = 1.3539$, $q_n = 0.7846$. We can plug this into the budget constraint for the consumer with the high endowment,

$$c^g + q_n\theta^g + q_r\theta^b = \omega^g + (v+1)\theta^g$$

$$21.5252 + 0.7846\theta^g + 1.3539(1-\theta^g) = 24 + 3.1385\theta^g,$$

to solve for $\theta^g = -0.3023$, $\theta^b = 1.3023$. We can now go back and verify that all of the equilibrium conditions are satisfied for the right choice of $\bar{\theta}_0^1$ and $\bar{\theta}_0^2$.

Now consider the Arrow-Debreu version of this model. The consumers solve

$$\max \sum\nolimits_{s\in S} \beta^{t(s)} \pi_s \log\, c_s^i$$
$$\text{s.t. } \sum\nolimits_{s\in S} p_s c_s^i \le \sum\nolimits_{s\in S} p_s \left(w_s^i + r\bar{\theta}_0^i \right)$$
$$\sum\nolimits_{\sigma\ge s} \beta^{t(\sigma)}\ \pi_\sigma \log\, c_\sigma^i \ge \sum\nolimits_{\sigma\ge s} \beta^{t(\sigma)}\ \pi_\sigma \log\, w_\sigma^i$$
$$c_s \ge 0\,.$$

An **equilibrium** is sequences $\hat{p}_s$, $\hat{c}_s^i$, such that

1. Given $\hat{p}_s$, the consumers choose $\hat{c}_s^i$ to solve their maximization problems.

2. Goods markets clear:

$$\hat{c}_s^1 + \hat{c}_s^2 = w_s^1 + w_s^2 + r = 24 + 9 + 1 = 34\,.$$

A **symmetric stochastic steady state** is (p_n, p_r), (c^g, c^b) such that $p_{(s,\eta')} = p_n p_s$ if $\eta' = \eta_s$, that is, if there is no reversal, $p_{(s,\eta')} = p_r p_s$ if $\eta' \ne \eta_s$, that is, if there is a reversal and

$$c_s^i = \begin{cases} c^g & \text{if } w_s^i = 24 \\ c^b & \text{if } w_s^i = 9 \end{cases}$$

satisfy the equilibrium conditions for the right choice of $(\bar{\theta}_0^1, \bar{\theta}_0^2)$.

As above, the first order conditions become

$$p_r = \frac{p_{(s,\eta')}}{p_s} = \frac{\beta\pi u'(c^b)}{u'(c^g)}$$

if there is a reversal and

$$p_n = \frac{p_{(s,\eta')}}{p_s} = \beta\left(1-\pi\right)$$

if not.

Letting $p_{\eta_0} = 1$, we can therefore construct Arrow-Debreu prices using the rule

$$p_{(s,\eta')} = \begin{cases} \beta(1-\pi)\, p_s & \text{if } \eta' = \eta_s \\ \beta\pi\left(u'(c^b)/u'(c^g)\right) p_s & \text{if } \eta' \ne \eta_s \end{cases}.$$

In particular,

$$p_{(s,\eta')} = \begin{cases} \frac{1}{2}\frac{1}{2} p_s = 0.25 p_s & \text{if } \eta' = \eta_s \\ \frac{1}{2}\frac{1}{2}\left(\frac{21.5252}{12.4748}\right) p_s = 0.4314 p_s & \text{if } \eta' \neq \eta_s \end{cases}.$$