Equilibrium of the General Equilibrium Growth Model

Consider a model with an infinitely-lived, representative consumer. The production function is $Y_t = A_t K_t^{\alpha} L_t^{1-\alpha}$. The consumer solves the problem

$$\max \sum_{t=t_0}^{\infty} \beta^{t-t_0}[\gamma \log C_t + (1-\gamma)\log(N_t \bar{h} - L_t)]$$
$$\text{s.t. } C_t + K_{t+1} - K_t = w_t L_t + (r_t - \delta)K_t,\ t = t_0, t_0+1, t_0+2, \ldots$$
$$K_{t_0} = \bar{K}_{t_0}.$$

The sequences of total factor productivities $A_{t_0}, A_{t_0+1}, A_{t_0+2}, \ldots$ and of working age populations $N_{t_0}, N_{t_0+1}, N_{t_0+2}, \ldots$ are exogenous.

An **equilibrium** is sequences of consumption $\hat{C}_{t_0}, \hat{C}_{t_0+1}, \hat{C}_{t_0+2}, \ldots$, labor $\hat{L}_{t_0}, \hat{L}_{t_0+1}, \hat{L}_{t_0+2}, \ldots$, capital $\hat{K}_{t_0}, \hat{K}_{t_0+1}, \hat{K}_{t_0+2}, \ldots$, wages $\hat{w}_{t_0}, \hat{w}_{t_0+1}, \hat{w}_{t_0+2}, \ldots$, and rental rates $\hat{r}_{t_0}, \hat{r}_{t_0+1}, \hat{r}_{t_0+2}, \ldots$ such that

1. **Consumer maximization:** Given $\hat{w}_{t_0}, \hat{w}_{t_0+1}, \hat{w}_{t_0+2}, \ldots$ and $\hat{r}_{t_0}, \hat{r}_{t_0+1}, \hat{r}_{t_0+2}, \ldots$, the consumer chooses $\hat{C}_{t_0}, \hat{C}_{t_0+1}, \hat{C}_{t_0+2}, \ldots$, $\hat{L}_{t_0}, \hat{L}_{t_0+1}, \hat{L}_{t_0+2}, \ldots$, and $\hat{K}_{t_0}, \hat{K}_{t_0+1}, \hat{K}_{t_0+2}, \ldots$ to solve

$$\max \sum_{t=t_0}^{\infty} \beta^{t-t_0}[\gamma \log C_t + (1-\gamma)\log(N_t \bar{h} - L_t)]$$
$$\text{s.t. } C_t + K_{t+1} - K_t = \hat{w}_t L_t + (\hat{r}_t - \delta)K_t,\ t = t_0, t_0+1, t_0+2, \ldots$$
$$K_{t_0} = \bar{K}_{t_0}.$$

2. **Firm maximization:** Given $\hat{w}_{t_0}, \hat{w}_{t_0+1}, \hat{w}_{t_0+2}, \ldots$ and $\hat{r}_{t_0}, \hat{r}_{t_0+1}, \hat{r}_{t_0+2}, \ldots$, the firm chooses $\hat{L}_{t_0}, \hat{L}_{t_0+1}, \hat{L}_{t_0+2}, \ldots$, and $\hat{K}_{t_0}, \hat{K}_{t_0+1}, \hat{K}_{t_0+2}, \ldots$ to minimize costs and to earn zero profit. This results in the conditions

$$\hat{r}_t = \alpha A_t \hat{K}_t^{\alpha-1} \hat{L}_t^{1-\alpha}$$
$$\hat{w}_t = (1-\alpha) A_t \hat{K}_t^{\alpha} \hat{L}_t^{-\alpha},\ t = t_0, t_0+1, t_0+2, \ldots.$$

3. **Feasibility:** $\hat{C}_{t_0}, \hat{C}_{t_0+1}, \hat{C}_{t_0+2}, \ldots$, $\hat{L}_{t_0}, \hat{L}_{t_0+1}, \hat{L}_{t_0+2}, \ldots$, and $\hat{K}_{t_0}, \hat{K}_{t_0+1}, \hat{K}_{t_0+2}, \ldots$ are feasible:

$$C_t + K_{t+1} - (1-\delta)K_t = A_t K_t^{\alpha} L_t^{1-\alpha},\ t = t_0, t_0+1, t_0+2, \ldots.$$

Notice that it is possible to define separate $\hat{L}_{t_0}, \hat{L}_{t_0+1}, \hat{L}_{t_0+2}, \ldots$ and $\hat{K}_{t_0}, \hat{K}_{t_0+1}, \hat{K}_{t_0+2}, \ldots$ for the consumer and for the firm and then require that they be equal as part of the feasibility conditions:

$$\hat{L}_t^c = \hat{L}_t^f$$
$$\hat{K}_t^c = \hat{K}_t^f,\ t = t_0, t_0+1, t_0+2, \ldots .$$