

One-Sector Growth Model Notes

1 The representative household's problem

The representative household enjoys consumption and leisure time in all periods $t = 0, 1, \dots, \infty$. The household has no exogenous income, but it has fixed time endowments in each period, \bar{h} , and an initial capital endowment, K_0 .

Assume that the household's utility function is time-separable, and takes the form

$$U(C_0, C_1, \dots, L_0, L_1, \dots) = \sum_{t=0}^{\infty} \beta^t u(C_t, \bar{h} - L_t)$$

where C_t and $\bar{h} - L_t$ denote consumption and leisure in period t , $u(\cdot, \cdot)$ is a period utility function which is strictly increasing, strictly concave and satisfies the Inada Conditions ($\lim_{C_t \rightarrow 0} u_C(C_t, \bar{h} - L_t) = \infty$, $\lim_{L_t \rightarrow \bar{h}} u_L(C_t, \bar{h} - L_t) = -\infty$; these help ensure that the solutions are interior). $\beta \in (0, 1)$ is the household's discount factor.

1.1 The household's investment decision and its budget constraints

The household makes investment decisions to increase its initial capital stock over time. The law of motion for capital is given by:

$$K_{t+1} = (1 - \delta)K_t + I_t, \quad t = 0, 1, \dots, \infty$$

where $\delta \in (0, 1)$ is the depreciation rate, and I_t denotes the household's investment in period t .

In all periods, the household has capital K_t and rents it to the firm at a rental rate r_t . After the firm uses the capital, it returns the remaining stock $(1 - \delta)K_t$ to the household. Once we include investment I_t , the household gets the next period's capital stock K_{t+1} .

The household's budget constraint is given by

$$C_t + I_t = w_t L_t + r_t K_t, \quad t = 0, 1, \dots, \infty$$

We can then summarize the household's utility maximization problem as

$$\begin{aligned} \max_{\{C_t, L_t, I_t\}_{t=0}^{\infty}} & \sum_{t=0}^{\infty} \beta^t u(C_t, \bar{h} - L_t) \\ \text{s.t.} & C_t + I_t = w_t L_t + r_t K_t \quad t = 0, 1, \dots, \infty \end{aligned}$$

$$\begin{aligned}
K_{t+1} &= (1-\delta)K_t + I_t, \quad t = 0, 1, \dots, \infty \\
K_0 &\text{ given} \\
C_t &\geq 0, 0 \leq L_t \leq \bar{h}
\end{aligned}$$

Under our assumptions over $u(\cdot, \cdot)$, one can prove a solution exists. (Technically, we need a transversality condition. Instead we use an assumption that after some T , $K_{t+1} = \gamma K_t \quad \forall t \geq T$)

1.2 Characterizing the solution

Substituting the law of motion for capital into the budget constraint we get

$$C_t + K_{t+1} - (1-\delta)K_t = w_t L_t + r_t K_t, \quad t = 0, 1, \dots, \infty$$

We can form the Lagrangian as

$$L = \sum_{t=0}^{\infty} \beta^t u(C_t, \bar{h} - L_t) + \sum_{t=0}^{\infty} \lambda_t [w_t L_t + (1-\delta + r_t)K_t - C_t - K_{t+1}]$$

We can take first-order conditions with respect to the decision variables C_t, L_t, K_{t+1}

These are:

$$\begin{aligned}
C_t : \beta^t u_C(C_t, \bar{h} - L_t) - \lambda_t &= 0 \\
L_t : -\beta^t u_L(C_t, \bar{h} - L_t) - w_t \lambda_t &= 0 \\
K_{t+1} : -\lambda_t + (1-\delta + r_{t+1})\lambda_{t+1} &= 0
\end{aligned}$$

Combining equations (1) and (2) gives us the *intratemporal Euler Equation*:

$$-u_L(C_t, \bar{h} - L_t) = w_t u_C(C_t, \bar{h} - L_t), \quad t = 0, 1, \dots, \infty$$

Combining equations (2) and (3) gives us the *intertemporal Euler Equation*:

$$u_C(C_t, \bar{h} - L_t) - \beta(1-\delta + r_{t+1})u_C(C_{t+1}, \bar{h} - L_{t+1}) = 0, \quad t = 0, 1, \dots, \infty$$

So given prices, these two sets of euler equations together with the budget constraints and the condition for $\{K_t\}_{t=T}^{\infty}$ completely characterize the solution to the household's problem.

2 The representative firm's problem

We assume that for each period, our representative firm operates a constant returns-to-scale technology represented by the production function

$$Y_t = A_t F(K_{Ft}, L_{Ft})$$

where for each period, Y_t denotes the firm's output, A_t represents total factor productivity (TFP), and $F(\cdot, \cdot)$ is a homogeneous function of degree one (i.e. $F(\lambda K_{Ft}, \lambda L_{Ft}) = \lambda F(K_{Ft}, L_{Ft})$) which combines the firm's capital input, K_{Ft} , and the firm's labor input, N_{Ft} . (We also assume $F(\cdot, \cdot)$ is strictly increasing, strictly concave and satisfies

the Inada Conditions.)

Again, the firm does not own any of the production inputs and only rents capital and hires labor from households.

As before, the firm's optimization problem simplifies to a period-by-period profit maximization problem. If we assume that the firm is a price-taker, then for each $t=0,1,\dots,\infty$ the firm solves

$$\begin{aligned} \max_{\{K_{Ft}, N_{Ft}\}} & A_t F(K_{Ft}, L_{Ft}) - r_t K_{Ft} - w_t L_{Ft} \\ \text{s.t. } & K_{Ft} \geq 0, N_{Ft} \geq 0 \end{aligned}$$

The first-order conditions for the representative firm's problem give the familiar equating of marginal products and their prices.

$$A_t F_K(K_{Ft}, L_{Ft}) = r_t$$

$$A_t F_L(K_{Ft}, L_{Ft}) = w_t$$

Now we proceed to characterize the competitive equilibrium (CE) below.

3 Competitive equilibrium

Definition: Given the initial capital stock K_0 and a sequence of exogenous TFP shocks $\{A_t\}_{t=0}^\infty$, a competitive equilibrium consists of a sequence of prices $\{r_t, w_t\}_{t=0}^\infty$, and a sequence of quantities $\{C_t, L_t, K_t, I_t, Y_t\}_{t=0}^\infty$

such that:

1. The representative household chooses $\{C_t, L_t, K_t, I_t\}_{t=0}^\infty$ to solve

$$\begin{aligned} \max_{\{C_t, L_t, I_t\}_{t=0}^\infty} & \sum_{t=0}^{\infty} \beta^t u(C_t, \bar{h} - L_t) \\ \text{s.t. } & C_t + I_t = w_t L_t + r_t K_t \quad t = 0, 1, \dots, \infty \\ & K_{t+1} = (1 - \delta) K_t + I_t, \quad t = 0, 1, \dots, \infty \\ & K_0 \text{ given} \\ & C_t \geq 0, 0 \leq L_t \leq \bar{h} \end{aligned}$$

2. For $t = 0, 1, \dots, \infty$ the representative firm chooses $\{L_{Ft}, K_{Ft}\}$ to solve

$$\begin{aligned} \max_{\{K_{Ft}, N_{Ft}\}} & A_t F(K_{Ft}, L_{Ft}) - r_t K_{Ft} - w_t N_{Ft} \\ \text{s.t. } & K_{Ft} \geq 0, N_{Ft} \geq 0 \\ \text{where } & Y_t = A_t F(K_{Ft}, L_{Ft}). \end{aligned}$$

3. For $t = 0, 1, \dots, \infty$, all markets clear:

$$C_t + K_{t+1} - (1 - \delta)K_t = A_t F(K_{Ft}, L_{Ft})$$

$$K_t = K_{Ft}$$

$$L_t = L_{Ft}$$

3.1 Characterizing the competitive equilibrium

Recall that from the household's problem we get the Euler equations and the budget constraints:

$$-u_L(C_t, \bar{h} - L_t) = w_t u_C(C_t, \bar{h} - L_t), \quad t = 0, 1, \dots, \infty$$

$$u_C(C_t, \bar{h} - L_t) = \beta(1 - \delta + r_{t+1})u_C(C_{t+1}, \bar{h} - L_{t+1}), \quad t = 0, 1, \dots, \infty$$

$$C_t + K_{t+1} - (1 - \delta)K_t = w_t L_t + r_t K_t, \quad t = 0, 1, \dots, \infty$$

Combining with equations (4) and (5) which determine factor prices, we get:

$$-\frac{u_L(C_t, \bar{h} - L_t)}{u_C(C_t, \bar{h} - L_t)} = A_t F_L(K_{Ft}, L_{Ft}), \quad t = 0, 1, \dots, \infty$$

$$\frac{u_C(C_t, \bar{h} - L_t)}{u_C(C_{t+1}, \bar{h} - L_{t+1})} = \beta(1 - \delta + A_{t+1} F_K(K_{Ft+1}, L_{Ft+1})), \quad t = 0, 1, \dots, \infty$$

$$C_t + K_{t+1} - (1 - \delta)K_t = A_t F_L(K_{Ft}, L_{Ft})L_t + A_t F_K(K_{Ft}, L_{Ft})K_t, \quad t = 0, 1, \dots, \infty$$

Together with the initial capital stock K_0 and the assumption that after some T , $K_{t+1} = \gamma K_t \quad \forall t \geq T$, we are able to solve for the full sequence of allocations.