HECKSCHER-OHLIN MODEL

Fixed Proportions Model

Production technology:

$$y_1 = \min[k_1,\ \ell_1/2] = k_1 = \ell_1/2$$
$$y_2 = \min[k_2/2,\ \ell_2] = k_2/2 = \ell_2 .$$

Endowments:

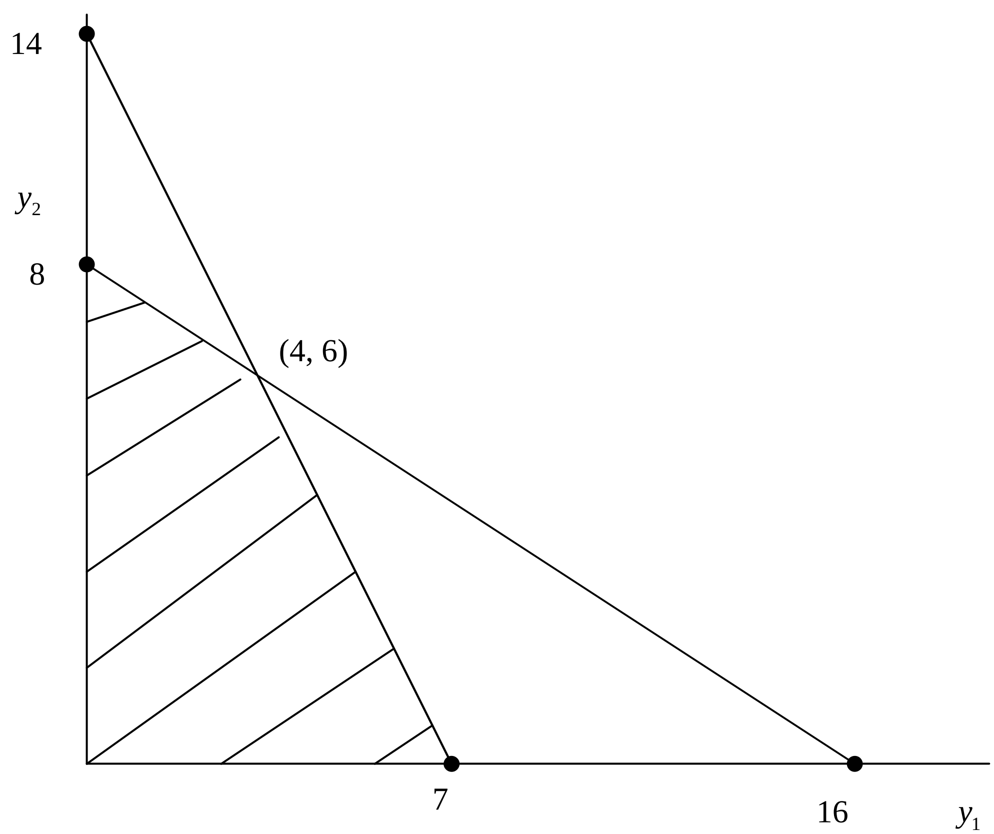
$$\bar{k} = 16,\ \bar{\ell} = 14.$$

Utility of the representative consumer/worker:

$$u(c_1, c_2) = \log\ c_1 + \log\ c_2 .$$

Production Possibility Set

$$k_1 + k_2 \le \bar{k} \qquad \ell_1 + \ell_2 \le \bar{\ell}$$
$$y_1 + 2y_2 \le 16 \qquad 2y_1 + y_2 \le 14$$



An **autarky equilibrium** is a set of
goods prices $\hat{p}_1$, $\hat{p}_2$,
factor prices $\hat{r}$, $\hat{w}$,
a consumption plan $\hat{c}_1$, $\hat{c}_2$,
and production plans $\hat{y}_1$, $\hat{y}_2$, $\hat{k}_1$, $\hat{k}_2$, $\hat{\ell}_1$, $\hat{\ell}_2$
such that

- Given $\hat{p}_1$, $\hat{p}_2$, $\hat{r}$, $\hat{w}$, the consumer chooses $\hat{c}_1$, $\hat{c}_2$ to solve

$$\begin{aligned} &\max \quad \log\ c_1 + \log\ c_2 \\ &\text{s. t.} \quad \hat{p}_1 c_1 + \hat{p}_2 c_2 = \hat{r}\bar{k} + \hat{w}\bar{\ell}. \end{aligned}$$

- $\hat{p}_1 - \hat{r} - 2\hat{w} \le 0$, $= 0$ if $\hat{y}_1 > 0$,
  $\hat{p}_2 - 2\hat{r} - \hat{w} \le 0$, $= 0$ if $\hat{y}_2 > 0$.

- $\hat{c}_1 = \hat{y}_1$,
  $\hat{c}_2 = \hat{y}_2$.

- $\hat{y}_1 = \min[\hat{k}_1, \ \hat{\ell}_1 / 2]$,
  $\hat{y}_2 = \min[\hat{k}_2 / 2, \ \hat{\ell}_2]$.

- $\hat{k}_1 + \hat{k}_2 \le \bar{k}$,
  $\hat{\ell}_1 + \hat{\ell}_2 \le \bar{\ell}$.

Solving the consumer's problem, we obtain

$$\hat{c}_1 = \frac{\hat{r}\bar{k} + \hat{w}\bar{\ell}}{2\hat{p}_1}, \ \hat{c}_2 = \frac{\hat{r}\bar{k} + \hat{w}\bar{\ell}}{2\hat{p}_2},$$

which imply

$$\frac{\hat{c}_2}{\hat{c}_1} = \frac{\hat{p}_1}{\hat{p}_2}.$$

Guess that $\hat{c}_1 = \hat{y}_1 = 4$, $\hat{c}_2 = \hat{y}_2 = 6$. This implies that

$$\frac{\hat{p}_1}{\hat{p}_2} = \frac{6}{4} = \frac{3}{2}.$$

Set $\hat{w}=1$ (numeraire). We can use the zero profit conditions to solve for $\hat{p}_2$, $\hat{w}$, $\hat{r}$:

$$\hat{p}_1 - \hat{r} - 2 = 0\,,$$
$$\hat{p}_2 - 2\hat{r} - 1 = 0 \iff \frac{2}{3}\hat{p}_1 - 2r - 1 = 0\ .$$

We can solve to obtain $\hat{p}_1 = 9/4 = 2.2500$, $\hat{p}_2 = 6/4 = 1.5000$, $\hat{r} = 1/4 = 0.2500$.

Autarky Equilibrium

	$\hat{p}_j$	$\hat{c}_j$	$\hat{y}_j$	$\hat{k}_j$	$\hat{\ell}_j$	$\hat{r}$	$\hat{w}$
good 1	9/4	4	4	4	8	1/4	1
good 2	6/4	6	6	12	6		

$$u(\hat{c}_1,\hat{c}_2) = \log\ 4 + \log\ 6 = 3.1781$$

Real income index $e^{1/2(u(\hat{c}_1,\hat{c}_2))} = \hat{c}_1^{1/2}\hat{c}_2^{1/2} = 4^{1/2}6^{1/2} = 4.8990$.

Trade Equilibrium for a Small Open Economy

Terms of trade $\dfrac{\hat{p}_1}{\hat{p}_2}$ are determined in the rest of the world.

A **trade equilibrium** is a set of
goods prices $\hat{p}_1$, $\hat{p}_2$,
factor prices $\hat{r}$, $\hat{w}$,
a consumption plan $\hat{c}_1$, $\hat{c}_2$,
and production plans $\hat{y}_1$, $\hat{y}_2$, $\hat{k}_1$, $\hat{k}_2$, $\hat{\ell}_1$, $\hat{\ell}_2$
such that

- Given $\hat{p}_1$, $\hat{p}_2$, $\hat{r}$, $\hat{w}$, the consumer chooses $\hat{c}_1$, $\hat{c}_2$ to solve

$$\begin{aligned} &\max\ \ \log\ c_1 + \log\ c_2 \\ &\text{s. t.}\quad \hat{p}_1 c_1 + \hat{p}_2 c_2 = \hat{r}\bar{k} + \hat{w}\bar{\ell}. \end{aligned}$$

- $\hat{p}_1 - \hat{r} - 2\hat{w} \le 0$, $=0$ if $\hat{y}_1 > 0$,
  $\hat{p}_2 - 2\hat{r} - \hat{w} \le 0$, $=0$ if $\hat{y}_2 > 0$.

- $\hat{y}_1 = \min[\hat{k}_1,\ \hat{\ell}_1/2]$,

$\hat{y}_2 = \min[\hat{k}_2 / 2,\ \hat{\ell}_2]$.

- $\hat{k}_1 + \hat{k}_2 \le \bar{k}$,
  $\hat{\ell}_1 + \hat{\ell}_2 \le \bar{\ell}$.

- $\dfrac{\hat{p}_1}{\hat{p}_2} = \dfrac{\bar{p}_1}{\bar{p}_2}$ exogenously given.

(There are no longer conditions that $\hat{c}_1 = \hat{y}_1$, $\hat{c}_2 = \hat{y}_2$. Now the international terms of trade $\hat{p}_1 / \hat{p}_2$ are exogenously given. In a two country trade model, $\hat{p}_1 / \hat{p}_2$ would be determined by the conditions for equilibrium in the market for goods, $\hat{c}_1^1 + \hat{c}_1^2 = \hat{y}_1^1 + \hat{y}_1^2$ and $\hat{c}_2^1 + \hat{c}_2^2 = \hat{y}_2^1 + \hat{y}_2^2$. Here we are assuming that the country is too small to affect $\hat{p}_1 / \hat{p}_2$.)

Suppose that, in the rest of the world

$$\frac{\hat{p}_1}{\hat{p}_2} = 1.$$

Set $\hat{w} = 1$ (numeraire). We can use the zero profit conditions to solve for $\hat{p}_2$, $\hat{w}$, $\hat{r}$:

$$\hat{p}_1 - 2\hat{r} - 1 = 0,$$
$$\hat{p}_2 - \hat{r} - 2 = 0 \iff \hat{p}_1 - \hat{r} - 2 = 0 \ .$$

We can solve to obtain $\hat{p}_1 = 3$, $\hat{p}_2 = 3$, $\hat{r} = 1$.

Solving the consumer's problem, we obtain

$$\hat{c}_1 = \frac{\hat{r}\bar{k} + \hat{w}\bar{\ell}}{2\hat{p}_1} = \frac{16+14}{2 \cdot 3} = 5,$$
$$\hat{c}_2 = \frac{\hat{r}\bar{k} + \hat{w}\bar{\ell}}{2\hat{p}_2} = \frac{16+14}{2 \cdot 3} = 5.$$

Small Open Economy Trade Equilibrium

	$\hat{p}_j$	$\hat{c}_j$	$\hat{y}_j$	$\hat{k}_j$	$\hat{\ell}_j$	$\hat{r}$	$\hat{w}$
good 1	3	5	4	4	8	1	1
good 2	3	5	6	12	6		

$$u(\hat{c}_1, \hat{c}_2) = \log\ 5 + \log\ 5 = \log 25$$

Real income index $e^{1/2(u(\hat{c}_1,\hat{c}_2))} = \hat{c}_1^{1/2}\hat{c}_2^{1/2} = 5^{1/2}5^{1/2} = 5.$ Real income increases by a factor of $5.0000 / 4.8990 = 1.0206$, a little more than 2 percent.

Who would be opposed to this?

Let us suppose that capitalists and workers are separate people. In autarky, the consumption of capitalists is

$$\hat{c}_1^K = \frac{\hat{r}\bar{k}}{2\hat{p}_1} = \frac{4}{2\cdot 9/4} = 0.8889$$
$$\hat{c}_2^K = \frac{\hat{r}\bar{k}}{2\hat{p}_2} = \frac{4}{2\cdot 6/4} = 1.3333\,.$$

The consumption of workers is $\hat{c}_1^L = \frac{\hat{w}\bar{\ell}}{2\hat{p}_1} = \frac{14}{2\cdot 9/4} = 3.1111$, $\hat{c}_2^L = \frac{\hat{w}\bar{\ell}}{2\hat{p}_2} = \frac{14}{2\cdot 6/4} = 4.6667$.

In trade,

$$\hat{c}_1^K = \hat{c}_2^K = \frac{16}{2\cdot 3} = 2.6667$$

$$\hat{c}_1^L = \hat{c}_2^L = \frac{14}{2\cdot 3} = 2.3333.$$

The real income of capitalists goes from $(0.8889)^{1/2}(1.3333)^{1/2} = 1.0887$ to $(2.6667)^{1/2}(2.6667)^{1/2} = 2.6667$, an increase of 145 percent.

The real income of workers goes from $(3.1111)^{1/2}(4.6667)^{1/2} = 3.8103$ to $(2.6667)^{1/2}(2.6667)^{1/2} = 2.6667$, a decrease of 39 percent.

Specific Factors Model

Production technology:

$$y_1 = \ell_1^{1/2}k_1^{1/2}$$
$$y_2 = \ell_2^{1/2}t_2^{1/2}\,.$$

Endowments:

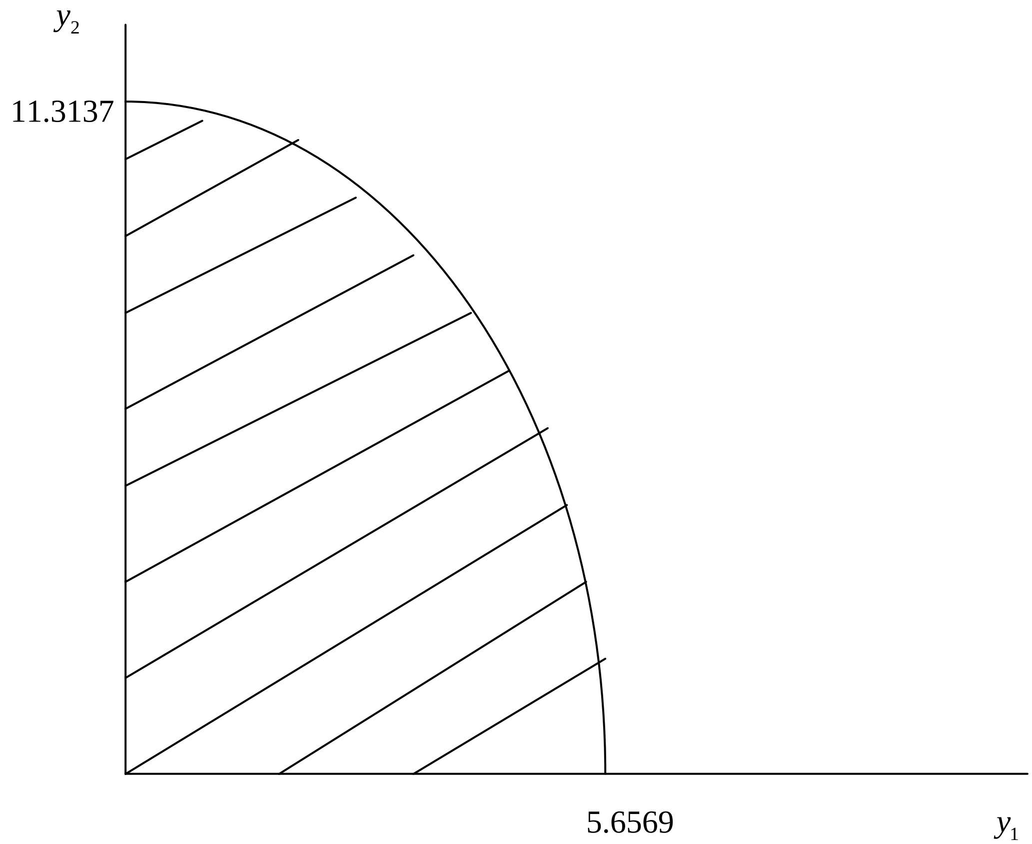
$$\bar{k} = 1,\ \bar{t} = 4,\ \bar{\ell} = 32\,.$$

Utility of the representative consumer/worker:

$$u(c_1, c_2) = \log\ c_1 + \log\ c_2 .$$

Production Possibility Set

$$\left.\begin{array}{ll} \ell_1 + \ell_2 & \leq 32 \\ k_1 & \leq 1 \\ t_2 & \leq 4 \end{array}\right\} \; y_1^2 + y_2^2 / 4 \leq 32 .$$



An **autarky equilibrium** is a set of
goods prices $\hat{p}_1$, $\hat{p}_2$,
factor prices $\hat{r}$, $\hat{q}$, $\hat{w}$,
a consumption plan $\hat{c}_1$, $\hat{c}_2$,
and production plans $\hat{y}_1$, $\hat{y}_2$, $\hat{k}_1$, $\hat{t}_2$, $\hat{\ell}_1$, $\hat{\ell}_2$
such that

- Given $\hat{p}_1$, $\hat{p}_2$, $\hat{r}$, $\hat{q}$, $\hat{w}$, the consumer chooses $\hat{c}_1$, $\hat{c}_2$ to solve

$$\begin{aligned} &\max \quad \log\ c_1 + \log\ c_2 \\ &\text{s. t.} \quad \hat{p}_1 c_1 + \hat{p}_2 c_2 = \hat{r}\overline{k} + \hat{q}\overline{t} + \hat{w}\overline{\ell}. \end{aligned}$$

- $\hat{r} = \hat{p}_1(1/2)\hat{\ell}_1^{1/2}\hat{k}_1^{-1/2}$, $\hat{w} = \hat{p}_1(1/2)\hat{\ell}_1^{-1/2}\hat{k}_1^{1/2}$,
  $\hat{q} = \hat{p}_2(1/2)\hat{\ell}_2^{1/2}\hat{t}_2^{-1/2}$, $\hat{w} = \hat{p}_2(1/2)\hat{\ell}_2^{-1/2}\hat{t}_2^{1/2}$.

- $\hat{c}_1 = \hat{y}_1$,
  $\hat{c}_2 = \hat{y}_2$.

- $\hat{y}_1 = \hat{\ell}_1^{1/2}\hat{k}_1^{1/2}$,
  $\hat{y}_2 = \hat{\ell}_2^{1/2}\hat{t}_2^{1/2}$.

- $\hat{\ell}_1 + \hat{\ell}_2 \le \overline{\ell}$,
  $\hat{k}_1 \le \overline{k}$,
  $\hat{t}_2 \le \overline{t}$.

Solving the consumer's problem, we obtain

$$\hat{c}_1 = \frac{\hat{r}\overline{k} + \hat{q}\overline{t} + \hat{w}\overline{\ell}}{2\hat{p}_1},\ \hat{c}_2 = \frac{\hat{r}\overline{k} + \hat{q}\overline{t} + \hat{w}\overline{\ell}}{2\hat{p}_2},$$

which imply

$$\frac{\hat{c}_2}{\hat{c}_1} = \frac{\hat{p}_1}{\hat{p}_2}.$$

From the profit maximization conditions (factor prices equal marginal revenue product), we know that

$$w = p_1(1/2)\ell_1^{-1/2}k_1^{1/2} = p_2(1/2)\ell_2^{-1/2}t_2^{1/2}$$

$$\frac{p_1}{p_2} = \frac{\ell_2^{-1/2}t_2^{1/2}}{\ell_1^{-1/2}k_1^{1/2}} = \left(\frac{\ell_1^{1/2}k_1^{1/2}}{\ell_2^{1/2}t_2^{1/2}}\right)\left(\frac{t_2}{k_1}\right) = \left(\frac{y_1}{y_2}\right)\left(\frac{t_2}{k_1}\right) = \left(\frac{y_1}{y_2}\right)\left(\frac{4}{1}\right).$$

Setting $c_1 = y_1$, $c_2 = y_2$, we obtain

$$\frac{y_2}{y_1} = \frac{p_1}{p_2} = \left(\frac{y_1}{y_2}\right)\left(\frac{4}{1}\right)$$

MRS = price ratio = MRT

$$\left(\frac{y_2}{y_1}\right)^2 = 4 \quad \Rightarrow \quad y_2 = 2y_1 .$$

Plugging this into the production possibility frontier, we obtain

$$y_1^2 + y_2^2 / 4 = 32$$
$$y_1^2 + \left(2y_1\right)^2 / 4 = 32$$
$$2y_1^2 = 32$$
$$y_1 = 4 , \quad y_1 = 8 .$$

Autarky Equilibrium

	$\hat{p}_j$	$\hat{c}_j$	$\hat{y}_j$	$\hat{k}_j$	$\hat{t}_j$	$\hat{\ell}_j$	$\hat{r}$	$\hat{q}$	$\hat{w}$
good 1	8	4	4	1	0	16	16	4	1
good 2	4	8	8	0	4	16			

$$u(\hat{c}_1, \hat{c}_2) = \log\ 4 + \log\ 8 = 3.4657$$

Real income index $e^{1/2(u(\hat{c}_1,\hat{c}_2))} = \hat{c}_1^{1/2}\hat{c}_2^{1/2} = 4^{1/2}8^{1/2} = 5.6569$.

Trade Equilibrium for a Small Open Economy

Terms of trade $\dfrac{\hat{p}_1}{\hat{p}_2}$ are determined by the rest of the world.

An **trade equilibrium** is a set of
goods prices $\hat{p}_1$, $\hat{p}_2$,
factor prices $\hat{r}$, $\hat{q}$, $\hat{w}$,
a consumption plan $\hat{c}_1$, $\hat{c}_2$,
and production plans $\hat{y}_1$, $\hat{y}_2$, $\hat{k}_1$, $\hat{t}_2$, $\hat{\ell}_1$, $\hat{\ell}_2$
such that

- Given $\hat{p}_1$, $\hat{p}_2$, $\hat{r}$, $\hat{q}$, $\hat{w}$, the consumer chooses $\hat{c}_1$, $\hat{c}_2$ to solve

$$\begin{aligned} &\max \ \ \log\ c_1 + \log\ c_2 \\ &\text{s. t.} \quad \hat{p}_1 c_1 + \hat{p}_2 c_2 = \hat{r}\overline{k} + \hat{q}\overline{t} + \hat{w}\overline{\ell}. \end{aligned}$$

- $\hat{r} = \hat{p}_1 (1/2) \hat{\ell}_1^{1/2} \hat{k}_1^{-1/2}$, $\hat{w} = \hat{p}_1 (1/2) \hat{\ell}_1^{-1/2} \hat{k}_1^{1/2}$,
  $\hat{q} = \hat{p}_2 (1/2) \hat{\ell}_2^{1/2} \hat{t}_2^{-1/2}$, $\hat{w} = \hat{p}_2 (1/2) \hat{\ell}_2^{-1/2} \hat{t}_2^{1/2}$.

- $\hat{y}_1 = \hat{\ell}_1^{1/2} \hat{k}_1^{1/2}$,
  $\hat{y}_2 = \hat{\ell}_2^{1/2} \hat{t}_2^{1/2}$.

- $\hat{\ell}_1 + \hat{\ell}_2 \le \overline{\ell}$,
  $\hat{k}_1 \le \overline{k}$,
  $\hat{t}_2 \le \overline{t}$.

- $\dfrac{\hat{p}_1}{\hat{p}_2} = \dfrac{\overline{p}_1}{\overline{p}_2}$ exogenously given.

(There are no longer conditions that $\hat{c}_1 = \hat{y}_1$, $\hat{c}_2 = \hat{y}_2$.)

Suppose that, in the rest of the world

$$\frac{\hat{p}_1}{\hat{p}_2} = 1.$$

It is still the case that profit maximization implies

$$w = p_1 (1/2) \ell_1^{-1/2} k_1^{1/2} = p_2 (1/2) \ell_2^{-1/2} t_2^{1/2}$$

$$\frac{p_1}{p_2} = \frac{\ell_2^{-1/2} t_2^{1/2}}{\ell_1^{-1/2} k_1^{1/2}} = \left(\frac{\ell_1^{1/2} k_1^{1/2}}{\ell_2^{1/2} t_2^{1/2}}\right)\left(\frac{t_2}{k_1}\right) = \left(\frac{y_1}{y_2}\right)\left(\frac{t_2}{k_1}\right) = \left(\frac{y_1}{y_2}\right)\left(\frac{4}{1}\right).$$

Since $p_1 / p_2 = 1$, this implies that

$$y_2 = 4 y_1.$$

Plugging this into the production possibility frontier, we obtain

$$y_1^2 + y_2^2 / 4 = 32$$

$$y_1^2 + (4y_1)^2/4 = 32$$
$$5y_1^2 = 32$$
$$y_1 = (32/5)^{1/2} = 2.5298, \;\; y_2 = 4(32/5)^{1/2} = 10.1193.$$

To obtain factor inputs, we plug into the production function

$$y_1 = \ell_1^{1/2} k_1^{1/2}$$
$$2.5298 = \ell_1^{1/2}(1)^{1/2}$$
$$\ell_1 = 32/5 = 6.4000$$
$$\ell_2 = 32 - 6.4000 = 25.6000.$$

To obtain goods prices and factor prices, we set $\hat{w} = 1$ (numeraire) and then plug into the profit maximization conditions,

$$1 = p_1(1/2)(6.4)^{1/2}(1)^{1/2} \quad \Rightarrow \quad p_1 = 5.0596, p_2 = 5.0596$$
$$r = p_1(1/2)(6.4)^{1/2}(1)^{-1/2} = 5.0596(1/2)(6.4)^{1/2} = 6.4000$$
$$q = p_2(1/2)(25.6)^{1/2}(4)^{-1/2} = 5.0596(1/2)(25.6)^{1/2}(4)^{-1/2} = 6.4000.$$

To obtain consumption levels, we plug into the demand functions,

$$\hat{c}_1 = \frac{\hat{r}\overline{k} + \hat{q}\overline{t} + \hat{w}\overline{\ell}}{2\hat{p}_1} = 6.3246$$
$$\hat{c}_2 = \frac{\hat{r}\overline{k} + \hat{q}\overline{t} + \hat{w}\overline{\ell}}{2\hat{p}_2} = 6.3246.$$

Small Open Economy Trade Equilibrium

	$\hat{p}_j$	$\hat{c}_j$	$\hat{y}_j$	$\hat{k}_j$	$\hat{t}_j$	$\hat{\ell}_j$	$\hat{r}$	$\hat{q}$	$\hat{w}$
good 1	5.0596	6.3246	2.5298	1	0	6.4	6.4	6.4	1
good 2	5.0596	6.3246	10.1193	0	4	25.6			

$$u(\hat{c}_1, \hat{c}_2) = \log\ 6.3246 + \log\ 6.3246 = 3.6889$$

Real income index $e^{1/2(u(\hat{c}_1,\hat{c}_2))} = \hat{c}_1^{1/2}\hat{c}_2^{1/2} = 6.3246^{1/2}6.3246^{1/2} = 6.3246$. Real income increases by a factor of $6.3246/5.6569 = 1.1180$, almost 12 percent.

Who would be opposed to this?

Let us suppose that capitalists, landowners, and workers are separate people.

In autarky, the consumption of capitalists is $\hat{c}_1^K = \frac{\hat{r}\bar{k}}{2\hat{p}_1} = \frac{16\cdot 1}{2\cdot 8} = 1$, $\hat{c}_2^K = \frac{\hat{r}\bar{k}}{2\hat{p}_2} = \frac{16\cdot 1}{2\cdot 4} = 2$.

The consumption of landowners is $\hat{c}_1^T = \frac{\hat{q}\bar{t}}{2\hat{p}_1} = \frac{4\cdot 4}{2\cdot 8} = 1$, $\hat{c}_2^T = \frac{\hat{q}\bar{t}}{2\hat{p}_2} = \frac{4\cdot 4}{2\cdot 4} = 2$.

The consumption of workers is $\hat{c}_1^L = \frac{\hat{w}\bar{\ell}}{2\hat{p}_1} = \frac{1\cdot 32}{2\cdot 8} = 2$, $\hat{c}_2^L = \frac{\hat{w}\bar{\ell}}{2\hat{p}_2} = \frac{1\cdot 32}{2\cdot 4} = 4$.

In trade,

$$
\begin{aligned}
\hat{c}_1^K = \hat{c}_2^K &= \frac{6.4\cdot 1}{2\cdot 5.0596} = 0.6325 \\
\hat{c}_1^T = \hat{c}_2^T &= \frac{6.4\cdot 4}{2\cdot 5.0596} = 2.5298 \\
\hat{c}_1^L = \hat{c}_2^L &= \frac{1\cdot 32}{2\cdot 5.0596} = 3.1623 .
\end{aligned}
$$

The real income of capitalists goes from $(1)^{1/2}(2)^{1/2} = 1.4142$ to $(0.6325)^{1/2}(0.6325)^{1/2} = 0.6325$, an decrease of 55 percent.

The real income of landowners goes from $(1)^{1/2}(2)^{1/2} = 1.4142$ to $(2.5298)^{1/2}(2.5298)^{1/2}$ $= 2.5298$, an increase of 79 percent.

The real income of workers goes from $(2)^{1/2}(4)^{1/2} = 2.8284$ to $(3.1623)^{1/2}(3.1623)^{1/2} = 3.1623$, an increase of 12 percent.

Equilibrium in a World Trade Model

We define an equilibrium for a world economy with m countries. We consider only the case of a world with identical, fixed proportions production functions in each country. The definition for a world with identical, specific factors production technologies should be obvious.

Production technology:

$$
\begin{aligned}
y_1 &= \min[k_1,\ \ell_1/2] = k_1 = \ell_1/2 \\
y_2 &= \min[k_2/2,\ \ell_2] = k_2/2 = \ell_2 .
\end{aligned}
$$

Endowments:

$$\bar{k}^i,\ \bar{\ell}^i \text{ in each country, } i = 1, 2, ..., m.$$

Utility of the representative consumer/worker in country i, $i = 1, 2, ..., m$:

$$u(c_1^i, c_2^i) = \log\ c_1^i + \log\ c_2^i\ .$$

A **trade equilibrium** is a set of
goods prices $\hat{p}_1$, $\hat{p}_2$,
factor prices $\hat{r}^i$, $\hat{w}^i$, $i = 1, 2, ..., m$,
consumption plans $\hat{c}_1^i$, $\hat{c}_2^i$, $i = 1, 2, ..., m$,
and production plans $\hat{y}_1^i$, $\hat{y}_2^i$, $\hat{k}_1^i$, $\hat{k}_2^i$, $\hat{\ell}_1^i$, $\hat{\ell}_2^i$, $i = 1, 2, ..., m$,
such that

- Given $\hat{p}_1$, $\hat{p}_2$, $\hat{r}^i$, $\hat{w}^i$, the consumer in country i chooses $\hat{c}_1^i$, $\hat{c}_2^i$ to solve

$$\begin{aligned} &\max\quad \log\ c_1^i + \log\ c_2^i \\ &\text{s. t.}\quad \hat{p}_1 c_1^i + \hat{p}_2 c_2^i = \hat{r}^i \bar{k}^i + \hat{w}^i \bar{\ell}^i . \end{aligned}$$

- $\hat{p}_1 - \hat{r}^i - 2\hat{w}^i \le 0$, $= 0$ if $\hat{y}_1^i > 0$,
  $\hat{p}_2 - 2\hat{r}^i - \hat{w}^i \le 0$, $= 0$ if $\hat{y}_2^i > 0$.

- $\hat{y}_1^i = \min[\hat{k}_1^i,\ \hat{\ell}_1^i / 2]$,
  $\hat{y}_2^i = \min[\hat{k}_2^i / 2,\ \hat{\ell}_2^i]$.

- $\hat{c}_1^1 + \hat{c}_1^2 + ... + \hat{c}_1^m = \hat{y}_1^1 + \hat{y}_1^2 + ... + \hat{y}_1^m$,
  $\hat{c}_2^1 + \hat{c}_2^2 + ... + \hat{c}_2^m = \hat{y}_2^1 + \hat{y}_2^2 + ... + \hat{y}_2^m$.

- $\hat{k}_1^i + \hat{k}_2^i \le \bar{k}^i$, $i = 1, 2, ..., m$,
  $\hat{\ell}_1^i + \hat{\ell}_2^i \le \bar{\ell}^i$, $i = 1, 2, ..., m$.

Points to notice:

1. We usually consider the case where there are two countries, $m = 2$, in examples.

2. If there are two countries, i and j, that both produce both of the two goods,

$$\hat{y}_1^i > 0,\ \hat{y}_2^i > 0 \text{ and } \hat{y}_1^j > 0,\ \hat{y}_2^j > 0,$$

   then the profit maximization conditions imply that the factor prices in the two countries are equal, prices $\hat{r}^i = \hat{r}^j, \hat{w}^i = \hat{w}^j$.

3. If $\hat{y}_j^i - \hat{c}_j^i$ is positive, then $\hat{y}_j^i - \hat{c}_j^i$ is the amount of good j exported by country i. If $\hat{y}_j^i - \hat{c}_j^i$ is negative, then $\hat{c}_j^i - \hat{y}_j^i$ is the amount of good j imported by country i.

A Note on Real Income

We calculate real income using a monotonic transformation of the utility function that is homogenous of degree one:

$$r(c_1, c_2) = e^{(1/2)(\log c_1 + \log c_2)} = c_1^{1/2} c_2^{1/2}.$$

The monotonic transformation $v(u) = e^{(1/2)u}$ ensures that $c_1^{1/2} c_2^{1/2}$ represents the same consumer preferences (that is, has the same indifference curves) as $\log c_1 + \log c_2$. It is easy to verify that $r(c_1, c_2) = c_1^{1/2} c_2^{1/2}$ is homogenous of degree one:

$$r(\theta c_1, \theta c_2) = \left(\theta c_1\right)^{1/2} \left(\theta c_2\right)^{1/2} = \theta c_1^{1/2} c_2^{1/2} = \theta r(c_1, c_2).$$

This means that we can meaningfully talk about percent changes in real income.

Our concept of changes in real income is what is traditionally known as the equivalent variation: In measuring the change in real income between situation 1 and situation 2, we ask by how much would we need to change income in situation 1, keeping prices fixed at situation 1 prices, to make a consumer indifferent between his or her consumption bundle in situation 1 and his or her consumption bundle in situation 2.

In the autarky equilibrium of our economy in the fixed proportions model real income is

$$r(4,6) = 4^{1/2} 6^{1/2} = 4.8990.$$

In the trade equilibrium, real income rises to

$$r(5,5) = 5^{1/2} 5^{1/2} = 5.0000.$$

The increase in real income is 2.06 percent.

Let us verify that this is indeed the equivalent variation: In autarky, prices are $\hat{p}_1 = 9/4$, $\hat{p}_1 = 6/4$, and income is $\hat{r}\overline{k} + \hat{w}\overline{\ell} = (1/4)16 + (1)14 = 18$. Suppose instead, income were 2.06 percent higher,

$$(1.0206)18 = 18.3711.$$

Then consumer demands would be

$$\hat{c}_1 = \frac{18.3711}{2\hat{p}_1} = \frac{18.3711}{2(9/4)} = 4.0825$$
$$\hat{c}_2 = \frac{18.3711}{2\hat{p}_2} = \frac{18.3711}{2(6/4)} = 6.1237.$$

Notice that, as we claimed, $r(5,5) = r(4.0825, 6.1237) = 5.0000$ (and that, of course, $\log 5 + \log 5 = \log 4.0825 + \log 6.1237 = 3.2189$).