

MONOPOLISTIC COMPETITION WITH HETEROGENEOUS FIRMS

There is a continuum of firms that produce differentiated products.

Consumers have utility functions that exhibit love for variety and solve the maximization problem

$$\begin{aligned} \max \quad & (1 - \mu) \log c_0 + \frac{\mu}{\rho} \log \int_0^n c(z)^\rho dz \\ \text{s.t.} \quad & p_0 c_0 + \int_0^n p(z) c(z) dz = w \bar{\ell} + \pi \\ & c(z) \geq 0. \end{aligned}$$

Here π are profits of the firms, which are owned by the consumers. The solution to this problem is

$$\begin{aligned} c_0 &= (1 - \mu) \frac{w \bar{\ell} + \pi}{p_0} \\ c(z) &= \frac{\mu(w \bar{\ell} + \pi)}{p(z)^{\frac{1}{1-\rho}} \int_0^n p(z')^{\frac{-\rho}{1-\rho}} dz'} = \frac{\mu(w \bar{\ell} + \pi)}{p(z)^{\frac{1}{1-\rho}} P^{\frac{-\rho}{1-\rho}}}, \end{aligned}$$

where

$$P = \left(\int_0^n p(z')^{\frac{-\rho}{1-\rho}} dz' \right)^{\frac{-(1-\rho)}{\rho}}.$$

Good 0 is produced with the constant returns production function $y_0 = \ell_0$ and sold in a competitive market. We set $p_0 = w = 1$ as numeraire.

Firm i has the production function

$$y(z) = \max \left[x(z) (\ell(z) - f), 0 \right].$$

Notice that firms have potentially different productivity levels $x(z)$. The firm solves the profit maximization problem

$$\max p(z) c(z) - \frac{c(z)}{x(z)} - f = p(z) \frac{\mu(\bar{\ell} + \pi)}{p(z)^{\frac{1}{1-\rho}} P^{\frac{-\rho}{1-\rho}}} - \frac{\mu(\bar{\ell} + \pi)}{x(z) p(z)^{\frac{1}{1-\rho}} P^{\frac{-\rho}{1-\rho}}} - f$$

taking P as given. The solution is

$$p(z) = \frac{1}{\rho x(z)}.$$

A model with a finite number of productivity levels

Suppose that there are 3 different productivity levels, $x_3 > x_2 > x_1 > 0$ and that there is a measure n_j of potential firms of each productivity level x_j . Suppose too that firms exit until remaining firms all earn nonnegative profits. Depending on parameters, there are 6 different possibilities:

1. A subset of firms with productivity x_3 produces and earns 0 profits. This subset has measure $\hat{n}_3 \leq n_3$.
2. All firms with productivity x_3 produce and earn nonnegative profits. No firm with productivity x_2 can earn nonnegative profit.
3. All firms with productivity x_3 produce and earn positive profits. A subset of firms with productivity x_2 produces and earns 0 profits. This subset has measure $\hat{n}_2 \leq n_2$.
4. All firms with productivity x_3 produce and earn positive profits. All firms with productivity x_2 produce and earn nonnegative profits. No firm with productivity x_1 can earn nonnegative profit.
5. All firms with productivities x_2 and x_3 produce and earn positive profits. A subset of firms with productivity x_1 produces and earns 0 profits. This subset has measure $\hat{n}_1 \leq n_1$.
6. All firms with productivities x_2 and x_3 produce and earn positive profits. All firms with productivity x_1 produce and earn nonnegative profits.

Notice that possibility 1 is just the Dixit-Stiglitz model with homogenous firms.

To illustrate how to compute equilibria, we suppose that we are in case 5.

$$P = \left(\int_0^{\hat{n}_1} \left(\frac{1}{\rho x_1} \right)^{\frac{-\rho}{1-\rho}} dz + \int_0^{n_2} \left(\frac{1}{\rho x_2} \right)^{\frac{-\rho}{1-\rho}} dz + \int_0^{n_3} \left(\frac{1}{\rho x_3} \right)^{\frac{-\rho}{1-\rho}} dz \right)^{\frac{-(1-\rho)}{\rho}}$$

$$P = \frac{1}{\rho} \left(\hat{n}_1 x_1^{\frac{\rho}{1-\rho}} + n_2 x_2^{\frac{\rho}{1-\rho}} + n_3 x_3^{\frac{\rho}{1-\rho}} \right)^{\frac{-(1-\rho)}{\rho}} .$$

To determine \hat{n}_1 , we solve

$$p_1 c_1 - \frac{c_1}{x_1} - f = \frac{\mu(\bar{\ell} + \pi) \left(\frac{1}{\rho x_1} - \frac{1}{x_1} \right)}{\frac{1}{\rho} x_1^{-\frac{1}{1-\rho}} \left(\hat{n}_1 x_1^{\frac{\rho}{1-\rho}} + n_2 x_2^{\frac{\rho}{1-\rho}} + n_3 x_3^{\frac{\rho}{1-\rho}} \right)} - f = 0$$

$$\frac{\mu(\bar{\ell} + \pi)(1-\rho)x_1^{\frac{\rho}{1-\rho}}}{\left(\hat{n}_1 x_1^{\frac{\rho}{1-\rho}} + n_2 x_2^{\frac{\rho}{1-\rho}} + n_3 x_3^{\frac{\rho}{1-\rho}} \right)} - f = 0$$

$$\hat{n}_1 = \frac{(1-\rho)\mu(\bar{\ell} + \pi)}{f} - \left(n_2 \left(\frac{x_2}{x_1} \right)^{\frac{\rho}{1-\rho}} + n_3 \left(\frac{x_3}{x_1} \right)^{\frac{\rho}{1-\rho}} \right).$$

To determine π , we solve

$$\pi = n_2 \left(p_2 c_2 + \frac{c_2}{x_2} - f \right) + n_3 \left(p_3 c_3 + \frac{c_3}{x_3} - f \right)$$

$$\pi = \hat{n}_1 \left(p_1 c_1 + \frac{c_1}{x_1} - f \right) + n_2 \left(p_2 c_2 + \frac{c_2}{x_2} - f \right) + n_3 \left(p_3 c_3 + \frac{c_3}{x_3} - f \right)$$

$$\pi = \frac{\mu(\bar{\ell} + \pi)(1-\rho) \left(\hat{n}_1 x_1^{\frac{\rho}{1-\rho}} + n_2 x_2^{\frac{\rho}{1-\rho}} + n_3 x_3^{\frac{\rho}{1-\rho}} \right)}{\left(\hat{n}_1 x_1^{\frac{\rho}{1-\rho}} + n_2 x_2^{\frac{\rho}{1-\rho}} + n_3 x_3^{\frac{\rho}{1-\rho}} \right)} - (\hat{n}_1 + n_2 + n_3)f = \mu(\bar{\ell} + \pi)(1-\rho) - (\hat{n}_1 + n_2 + n_3)f$$

$$\pi = \mu(\bar{\ell} + \pi)(1-\rho) - (\hat{n}_1 + n_2 + n_3)f$$

Plugging in the expression for \hat{n}_1 , we obtain

$$\pi = f \left[n_2 \left[\left(\frac{x_2}{x_1} \right)^{\frac{\rho}{1-\rho}} - 1 \right] + n_3 \left[\left(\frac{x_3}{x_1} \right)^{\frac{\rho}{1-\rho}} - 1 \right] \right]$$

and

$$\hat{n}_1 = \frac{(1-\rho)\mu\bar{\ell}}{f} - (1-(1-\rho)\mu) \left(n_2 \left(\frac{x_2}{x_1} \right)^{\frac{\rho}{1-\rho}} + n_3 \left(\frac{x_3}{x_1} \right)^{\frac{\rho}{1-\rho}} \right) - (1-\rho)\mu(n_2 + n_3)$$

$$\hat{n}_1 = \frac{(1-\rho)\mu\bar{\ell}}{f} - (1-(1-\rho)\mu) \left(n_2 \left[\left(\frac{x_2}{x_1} \right)^{\frac{\rho}{1-\rho}} + \frac{(1-\rho)\mu}{1-(1-\rho)\mu} \right] + n_3 \left[\left(\frac{x_3}{x_1} \right)^{\frac{\rho}{1-\rho}} + \frac{(1-\rho)\mu}{1-(1-\rho)\mu} \right] \right).$$

Notice that, if $n_1 = n_2 = n_3 = n/3$ and $x_1 = x_2 = x_3$, this collapses to the usual formula for homogenous firms.

A model with a continuum of productivity levels

Suppose that there is a measure n of potential firms. Firm productivities are distributed on the interval $x \geq 1$ according to the Pareto distribution with distribution function

$$F(x) = 1 - x^{-\gamma},$$

which has the density function

$$dF(x) = \gamma x^{-\gamma-1}.$$

Notice that the mean of x is

$$E(x) = \int_1^{\infty} x dF(x) = \int_1^{\infty} x \gamma x^{-\gamma-1} dx = -\frac{\gamma x^{-(\gamma-1)}}{\gamma-1} \Big|_1^{\infty} = \frac{\gamma}{\gamma-1}$$

and the variance is

$$\begin{aligned} E(x^2) - (E(x))^2 &= \int_1^{\infty} \left(x - \frac{\gamma}{\gamma-1}\right)^2 dF(x) = \int_1^{\infty} \left(x - \frac{\gamma}{\gamma-1}\right)^2 \gamma x^{-\gamma-1} dx = -\frac{\gamma x^{-(\gamma-2)}}{\gamma-2} \Big|_1^{\infty} - \left(\frac{\gamma}{\gamma-1}\right)^2 \\ E(x^2) - (E(x))^2 &= \frac{\gamma}{(\gamma-2)(\gamma-1)^2}. \end{aligned}$$

For the variance to be finite, we require that $\gamma > 2$. As we will see, we also require that $\gamma > \rho/(1-\rho)$.

We can think of restricting productivities to satisfy $x \geq 1$ as a normalization of units relating labor to consumption of differentiated goods by fixing the minimum productivity. If we want to normalize units in some other way, we could replace the distribution function with

$$F(x) = 1 - \theta^\gamma x^{-\gamma},$$

for $x \geq \theta$, which has the density function

$$dF(x) = \gamma \theta^\gamma x^{-\gamma-1}.$$

There are now only two possibilities:

1. There is a level of productivity $\bar{x} > 1$ for which firms earn 0 profits. The set of firms with productivities $x \geq \bar{x}$ produce. This set has measure $n\bar{x}^{-\gamma}$.
2. All firms produce and earn nonnegative profits.

Case 1:

We start by supposing that there is a cutoff productivity \bar{x} where firms earn 0 profits and calculate

$$P^{1-\rho} = n \int_{\bar{x}}^{\infty} p(x)^{\frac{-\rho}{1-\rho}} dF(x) = n \int_{\bar{x}}^{\infty} (\rho x)^{\frac{\rho}{1-\rho}} \gamma x^{-\gamma-1} dx$$

$$P^{1-\rho} = - \frac{n \rho^{\frac{\rho}{1-\rho}} (1-\rho) \gamma x^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}}{\gamma(1-\rho)-\rho} \Bigg|_{\bar{x}}^{\infty} = \frac{n \rho^{\frac{\rho}{1-\rho}} (1-\rho) \gamma \bar{x}^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}}{\gamma(1-\rho)-\rho}.$$

Notice that we require $\gamma(1-\rho) > \rho$ for P to be finite. The demand for goods produced by a firm with productivity x is

$$c(x) = \frac{\mu(\bar{\ell} + \pi)}{p(x)^{\frac{1}{1-\rho}} P^{\frac{-\rho}{1-\rho}}} = \frac{(\gamma(1-\rho)-\rho)\mu(\bar{\ell} + \pi)}{(\rho x)^{\frac{-1}{1-\rho}} n \rho^{\frac{\rho}{1-\rho}} (1-\rho) \gamma \bar{x}^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}} = \frac{\rho(\gamma(1-\rho)-\rho)\mu(\bar{\ell} + \pi)x^{\frac{1}{1-\rho}}}{n(1-\rho)\gamma \bar{x}^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}}.$$

We calculate the cutoff productivity \bar{x}

$$p(\bar{x})c(\bar{x}) - \frac{c(\bar{x})}{\bar{x}} - f = \frac{\rho(\gamma(1-\rho)-\rho)\mu(\bar{\ell} + \pi)\bar{x}^{\frac{1}{1-\rho}}}{n(1-\rho)\gamma \bar{x}^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}} \left(\frac{1}{\rho \bar{x}} - \frac{1}{\bar{x}} \right) - f = 0$$

$$\frac{(\gamma(1-\rho)-\rho)\mu(\bar{\ell} + \pi)\bar{x}^{\gamma}}{n\gamma} - f = 0$$

$$\bar{x} = \left(\frac{n\gamma f}{(\gamma(1-\rho)-\rho)\mu(\bar{\ell} + \pi)} \right)^{\frac{1}{\gamma}}.$$

Notice that this expression depends on profits π , which we can calculate as

$$\pi = n \int_{\bar{x}}^{\infty} \left(p(x)c(x) - \frac{c(x)}{x} - f \right) dF(x) = n \int_{\bar{x}}^{\infty} \left(\frac{(\gamma(1-\rho)-\rho)\mu(\bar{\ell} + \pi)x^{\frac{\rho}{1-\rho}}}{n\gamma \bar{x}^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}} - f \right) \gamma x^{-\gamma-1} dx$$

$$\pi = n \frac{(\gamma(1-\rho)-\rho)\mu(\bar{\ell} + \pi)}{n\gamma \bar{x}^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}} \int_{\bar{x}}^{\infty} x^{\frac{\rho}{1-\rho}} \gamma x^{-\gamma-1} dx - n f \int_{\bar{x}}^{\infty} \gamma x^{-\gamma-1} dx$$

$$\pi = (1-\rho)\mu(\bar{\ell} + \pi) - n\bar{x}^{-\gamma} f$$

$$\pi = \frac{(1-\rho)\mu\bar{\ell} - n\bar{x}^{-\gamma}f}{1-(1-\rho)\mu}.$$

Notice how similar this expression is to the analogous expression for the model with a finite number of productivity levels.

$$\pi = \frac{(1-\rho)\mu\bar{\ell}}{1-(1-\rho)\mu} - \frac{nf}{1-(1-\rho)\mu} \frac{(\gamma(1-\rho)-\rho)\mu(\bar{\ell}+\pi)}{n\gamma f} = \frac{(1-\rho)\mu\bar{\ell}}{1-(1-\rho)\mu} - \frac{(\gamma(1-\rho)-\rho)\mu(\bar{\ell}+\pi)}{\gamma(1-(1-\rho)\mu)}$$

$$\pi = \frac{\rho\mu\bar{\ell}}{\gamma - \rho\mu},$$

which implies that

$$\bar{x}^{-\gamma} = \frac{(\gamma(1-\rho)-\rho)\mu\left(\bar{\ell} + \frac{\rho\mu\bar{\ell}}{\gamma - \rho\mu}\right)}{n\gamma f} = \frac{(\gamma(1-\rho)-\rho)\mu\bar{\ell}}{(\gamma - \rho\mu)nf}.$$

Case 2:

Notice that we are wrong to guess that there is a cutoff productivity \bar{x} where firms earn 0 profits if the value that we calculate for \bar{x} is less than 1:

$$\bar{x} = \left(\frac{(\gamma - \rho\mu)nf}{(\gamma(1-\rho) - \rho)\mu\bar{\ell}} \right)^{\frac{1}{\gamma}} < 1$$

$$nf < \frac{(\gamma(1-\rho) - \rho)\mu\bar{\ell}}{\gamma - \rho\mu},$$

that is, if the fixed costs of having all potential firms produce is sufficiently low. In this case,

$$p^{\frac{-\rho}{1-\rho}} = \frac{n\rho^{\frac{\rho}{1-\rho}}(1-\rho)\gamma}{\gamma(1-\rho) - \rho}$$

$$c(x) = \frac{\mu(\bar{\ell} + \pi)}{p(x)^{\frac{1}{1-\rho}} P^{\frac{-\rho}{1-\rho}}} = \frac{(\gamma(1-\rho) - \rho)\mu(\bar{\ell} + \pi)}{(\rho x)^{\frac{-1}{1-\rho}} n\rho^{\frac{\rho}{1-\rho}}(1-\rho)\gamma} = \frac{\rho(\gamma(1-\rho) - \rho)\mu(\bar{\ell} + \pi)x^{\frac{1}{1-\rho}}}{n(1-\rho)\gamma}$$

The calculation of total profits becomes

$$\pi = n \int_1^\infty \left(p(x)c(x) - \frac{c(x)}{x} - f \right) dF(x) = n \int_1^\infty \left(\frac{(\gamma(1-\rho) - \rho)\mu(\bar{\ell} + \pi)x^{\frac{\rho}{1-\rho}}}{n\gamma} - f \right) \gamma x^{-\gamma-1} dx$$

$$\pi = n \frac{(\gamma(1-\rho) - \rho)\mu(\bar{\ell} + \pi)}{n\gamma} \int_1^\infty x^{\frac{\rho}{1-\rho}} \gamma x^{-\gamma-1} dx - nf \int_1^\infty \gamma x^{-\gamma-1} dx$$

$$\pi = (1-\rho)\mu(\bar{\ell} + \pi) - nf$$

$$\pi = \frac{(1-\rho)\mu\bar{\ell} - nf}{1 - (1-\rho)\mu}.$$

Notice that the profits of a firm with productivity $x=1$ are

$$p(1)c(1) - c(1) - f = \frac{\rho(\gamma(1-\rho) - \rho)\mu(\bar{\ell} + \pi)}{n(1-\rho)\gamma} \left(\frac{1}{\rho} - 1 \right) - f = \frac{(\gamma(1-\rho) - \rho)\mu \left(\bar{\ell} + \frac{(1-\rho)\mu\bar{\ell} - nf}{1 - (1-\rho)\mu} \right)}{n\gamma} - f$$

$$p(1)c(1) - c(1) - f = \frac{(\gamma(1-\rho) - \rho)\mu\bar{\ell}}{n\gamma(1 - (1-\rho)\mu)} - \frac{(\gamma - \rho\mu)f}{\gamma(1 - (1-\rho)\mu)}$$

$$p(1)c(1) - c(1) - f = \frac{\gamma - \rho\mu}{n\gamma(1 - (1-\rho)\mu)} \left(\frac{(\gamma(1-\rho) - \rho)\mu\bar{\ell}}{\gamma - \rho\mu} - nf \right) > 0.$$

A two-country model with a continuum of productivity levels

Suppose now that there are two countries, $i = 1, 2$. Let each country has a population of $\bar{\ell}_i$ and a measure of potential firms of n_i . Firms' productivities are distributed according to the Pareto distribution, $F(x) = 1 - x^{-\gamma}$.

A firm in country i faces a fixed cost of exporting to country j , $j \neq i$, of f_e where $f_e > f_d = f$ and an iceberg transportation cost of $\tau_i^j - 1 \geq 0$. The solution to the firm's profit maximization problem is to set

$$p_i^j(x) = \frac{\tau_i^j}{\rho x}.$$

In each country there are three possibilities:

1. There are two cutoff levels of productivity $\bar{x}_{ie} > \bar{x}_{id} > 1$. Firms with \bar{x}_{ie} earn 0 profits exporting. Firms with \bar{x}_{id} earn 0 profits producing for the domestic market. The set of firms with $x \geq \bar{x}_{ie}$ produce for the domestic market and for export. The set of firms with $\bar{x}_{ie} \geq x \geq \bar{x}_{id}$ produce for the domestic market only. The set of firms with $x < \bar{x}_{id}$ cannot earn nonnegative profits and do not produce.
2. There is one cutoff level of productivity $\bar{x}_{ie} > 1$. Firms with \bar{x}_{ie} earn 0 profits exporting. The set of firms with $x \geq \bar{x}_{ie}$ produce for the domestic market and for export. The set of firms with $\bar{x}_{ie} \geq x \geq 1$ produce for the domestic market only and earn nonnegative profits.
3. All firms produce for the domestic market and for export. They earn nonnegative profits doing both.

Suppose that we are in case 1. We calculate the price index in country 1:

$$\begin{aligned} (P^1)^{\frac{-\rho}{1-\rho}} &= n_1 \int_{\bar{x}_{id}}^{\infty} p_1^1(x) dF(x) + n_2 \int_{\bar{x}_{2e}}^{\infty} p_2^1(x) dF(x) \\ (P^1)^{\frac{-\rho}{1-\rho}} &= n_1 \int_{\bar{x}_{id}}^{\infty} (\rho x)^{\frac{\rho}{1-\rho}} \gamma x^{-\gamma-1} dx + n_2 \int_{\bar{x}_{2e}}^{\infty} \left(\frac{\tau_2^1}{\rho x} \right)^{\frac{\rho}{1-\rho}} \gamma x^{-\gamma-1} dx \\ (P^1)^{\frac{-\rho}{1-\rho}} &= - \frac{n_1 \rho^{\frac{\rho}{1-\rho}} (1-\rho) \gamma x^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}}{\gamma(1-\rho) - \rho} \Bigg|_{\bar{x}_{id}}^{\infty} + \frac{n_2 (\tau_2^1)^{\frac{-\rho}{1-\rho}} \rho^{\frac{\rho}{1-\rho}} (1-\rho) \gamma x^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}}{\gamma(1-\rho) - \rho} \Bigg|_{\bar{x}_{2e}}^{\infty} \end{aligned}$$

$$\begin{aligned} (P^1)^{\frac{-\rho}{1-\rho}} &= \frac{n_1 \rho^{\frac{\rho}{1-\rho}} (1-\rho) \gamma \bar{x}_{1d}^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}}{\gamma(1-\rho)-\rho} + \frac{n_2 (\tau_2^1)^{\frac{-\rho}{1-\rho}} \rho^{\frac{\rho}{1-\rho}} (1-\rho) \gamma \bar{x}_{2e}^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}}{\gamma(1-\rho)-\rho} \\ (P^1)^{\frac{-\rho}{1-\rho}} &= \frac{\rho^{\frac{\rho}{1-\rho}} (1-\rho) \gamma \left(n_1 \bar{x}_{1d}^{\frac{\rho-\gamma(1-\rho)}{1-\rho}} + n_2 (\tau_2^1)^{\frac{-\rho}{1-\rho}} \bar{x}_{2e}^{\frac{\rho-\gamma(1-\rho)}{1-\rho}} \right)}{\gamma(1-\rho)-\rho} \end{aligned}$$

The demand in country 1 for goods produced by a firm in country 1 with productivity $x \geq x_{1d}$ is

$$c_1^1(x) = \frac{\mu(\bar{\ell}_1 + \pi_1)}{p_1^1(x)^{\frac{1}{1-\rho}} (P^1)^{\frac{-\rho}{1-\rho}}} = \frac{\rho(\gamma(1-\rho)-\rho) \mu(\bar{\ell}_1 + \pi_1) x^{\frac{1}{1-\rho}}}{(1-\rho) \gamma \left(n_1 \bar{x}_{1d}^{\frac{\rho-\gamma(1-\rho)}{1-\rho}} + n_2 (\tau_2^1)^{\frac{-\rho}{1-\rho}} \bar{x}_{2e}^{\frac{\rho-\gamma(1-\rho)}{1-\rho}} \right)}.$$

We calculate an expression for the cutoff productivity \bar{x}_{1d} :

$$p_1^1(\bar{x}_{1d}) c_1^1(\bar{x}_{1d}) - \frac{c_1^1(\bar{x}_{1d})}{\bar{x}_{1d}} - f_d = \frac{\rho(\gamma(1-\rho)-\rho) \mu(\bar{\ell}_1 + \pi_1) \bar{x}_{1d}}{(1-\rho) \gamma \left(n_1 \bar{x}_{1d}^{\frac{\rho-\gamma(1-\rho)}{1-\rho}} + n_2 (\tau_2^1)^{\frac{-\rho}{1-\rho}} \bar{x}_{2e}^{\frac{\rho-\gamma(1-\rho)}{1-\rho}} \right)} \left(\frac{1}{\rho \bar{x}_{1d}} - \frac{1}{\bar{x}_{1d}} \right) - f_d = 0$$

$$p_1^1(\bar{x}_{1d}) c_1^1(\bar{x}_{1d}) - \frac{c_1^1(\bar{x}_{1d})}{\bar{x}_{1d}} - f_d = \frac{(\gamma(1-\rho)-\rho) \mu(\bar{\ell}_1 + \pi_1) \bar{x}_{1d}^{\frac{\rho}{1-\rho}}}{\gamma \left(n_1 \bar{x}_{1d}^{\frac{\rho-\gamma(1-\rho)}{1-\rho}} + n_2 (\tau_2^1)^{\frac{-\rho}{1-\rho}} \bar{x}_{2e}^{\frac{\rho-\gamma(1-\rho)}{1-\rho}} \right)} - f_d = 0.$$

Similarly, we calculate an expression for the cutoff productivity \bar{x}_{1e} :

$$p_1^2(\bar{x}_{1e}) c_1^2(\bar{x}_{1e}) - \frac{c_1^2(\bar{x}_{1e})}{\bar{x}_{1e}} - f_e = \frac{(\gamma(1-\rho)-\rho) \mu(\bar{\ell}_2 + \pi_2) (\tau_1^2)^{\frac{-\rho}{1-\rho}} \bar{x}_{1e}^{\frac{\rho}{1-\rho}}}{\gamma \left(n_2 \bar{x}_{2d}^{\frac{\rho-\gamma(1-\rho)}{1-\rho}} + n_1 (\tau_1^2)^{\frac{-\rho}{1-\rho}} \bar{x}_{1e}^{\frac{\rho-\gamma(1-\rho)}{1-\rho}} \right)} - f_e = 0.$$

The expression for π_1 is

$$\pi_1 = n_1 \int_{\bar{x}_{1d}}^{\infty} \left(p_1^1(x) c_1^1(x) - \frac{c_1^1(x)}{x} - f_d \right) dF(x) + n_1 \int_{\bar{x}_{1e}}^{\infty} \left(p_1^2(x) c_1^2(x) - \frac{c_1^2(x)}{x} - f_e \right) dF(x)$$

$$\pi_1 = n_1 \int_{\bar{x}_{1d}}^{\infty} \left(\frac{(\gamma(1-\rho) - \rho) \mu(\bar{\ell}_1 + \pi_1) x^{\frac{\rho}{1-\rho}}}{\gamma \left(n_1 \bar{x}_{1d}^{\frac{\rho-\gamma(1-\rho)}{1-\rho}} + n_2 (\tau_2^1)^{\frac{-\rho}{1-\rho}} \bar{x}_{2e}^{\frac{\rho-\gamma(1-\rho)}{1-\rho}} \right)} - f_d \right) \gamma x^{-\gamma-1} dx$$

$$+ n_1 \int_{\bar{x}_{1e}}^{\infty} \left(\frac{(\gamma(1-\rho) - \rho) \mu(\bar{\ell}_2 + \pi_2) (\tau_1^2)^{\frac{-\rho}{1-\rho}} x^{\frac{\rho}{1-\rho}}}{\gamma \left(n_2 \bar{x}_{2d}^{\frac{\rho-\gamma(1-\rho)}{1-\rho}} + n_1 (\tau_1^2)^{\frac{-\rho}{1-\rho}} \bar{x}_{1e}^{\frac{\rho-\gamma(1-\rho)}{1-\rho}} \right)} - f_e \right) \gamma x^{-\gamma-1} dx$$

$$\pi_1 = \frac{(1-\rho) \mu(\bar{\ell}_1 + \pi_1) n_1 \bar{x}_{1d}^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}}{\frac{\rho-\gamma(1-\rho)}{1-\rho} + n_2 (\tau_2^1)^{\frac{-\rho}{1-\rho}} \bar{x}_{2e}^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}} + \frac{(1-\rho) \mu(\bar{\ell}_2 + \pi_2) n_1 (\tau_1^2)^{\frac{-\rho}{1-\rho}} \bar{x}_{1e}^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}}{\frac{\rho-\gamma(1-\rho)}{1-\rho} + n_1 (\tau_1^2)^{\frac{-\rho}{1-\rho}} \bar{x}_{1e}^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}} - n_1 (\bar{x}_{1d}^{-\gamma} f_d + \bar{x}_{1e}^{-\gamma} f_e).$$

There are analogous expressions for \bar{x}_{2d} , \bar{x}_{1e} , and π_2 . This provides us with a system of 6 equations in 6 unknowns to be solved for \bar{x}_{1d} , \bar{x}_{1e} , π_1 , \bar{x}_{2d} , \bar{x}_{2e} , and π_2 . To simplify the calculations, let us consider the symmetric case where $\bar{\ell}_1 = \bar{\ell}_2 = \bar{\ell}$, $n_1 = n_2 = n$, and $\tau_2^1 = \tau_1^2 = \tau$. In this case,

$$\frac{(\gamma(1-\rho) - \rho) \mu(\bar{\ell} + \pi) \bar{x}_d^{\frac{\rho}{1-\rho}}}{n \gamma \left(\bar{x}_d^{\frac{\rho-\gamma(1-\rho)}{1-\rho}} + \tau^{\frac{-\rho}{1-\rho}} \bar{x}_e^{\frac{\rho-\gamma(1-\rho)}{1-\rho}} \right)} - f_d = 0$$

$$\frac{(\gamma(1-\rho) - \rho) \mu(\bar{\ell} + \pi) \tau^{\frac{-\rho}{1-\rho}} \bar{x}_e^{\frac{\rho}{1-\rho}}}{n \gamma \left(\bar{x}_d^{\frac{\rho-\gamma(1-\rho)}{1-\rho}} + \tau^{\frac{-\rho}{1-\rho}} \bar{x}_e^{\frac{\rho-\gamma(1-\rho)}{1-\rho}} \right)} - f_e = 0$$

$$\pi = (1-\rho) \mu(\bar{\ell} + \pi) - n (\bar{x}_d^{-\gamma} f_d + \bar{x}_e^{-\gamma} f_e).$$

Notice that

$$\frac{\bar{x}_e}{\bar{x}_d} = \tau \left(\frac{f_e}{f_d} \right)^{\frac{1-\rho}{\rho}}$$

and that

$$\tau^{\frac{-\rho}{1-\rho}} = \frac{\rho^{\frac{\rho}{1-\rho}} (1-\rho) \gamma n \left(\bar{x}_d^{\frac{\rho-\gamma(1-\rho)}{1-\rho}} + \tau^{\frac{-\rho}{1-\rho}} \bar{x}_e^{\frac{\rho-\gamma(1-\rho)}{1-\rho}} \right)}{\gamma(1-\rho) - \rho}.$$

The equation for π can be rewritten as

$$\pi = (1-\rho)\mu(\bar{\ell} + \pi) - n \left(\bar{x}_d^{-\gamma} f_d + \left(\tau \left(\frac{f_e}{f_d} \right)^{\frac{1-\rho}{\rho}} \bar{x}_d \right)^{-\gamma} f_e \right)$$

$$\pi = (1-\rho)\mu(\bar{\ell} + \pi) - n \left(f_d^{\frac{\rho-\gamma(1-\rho)}{\rho}} + \tau^{-\gamma} f_e^{\frac{\rho-\gamma(1-\rho)}{\rho}} \right) f_d^{\frac{\gamma(1-\rho)}{\rho}} \bar{x}_d^{-\gamma}.$$

Notice the similarity between this expression and the analogous expression for the closed economy model.

The equation for \bar{x}_d can be rewritten as

$$\frac{(\gamma(1-\rho) - \rho)\mu(\bar{\ell} + \pi)\bar{x}_d^{\frac{\rho}{1-\rho}}}{n\gamma \left(\bar{x}_d^{\frac{\rho-\gamma(1-\rho)}{1-\rho}} + \tau^{\frac{-\rho}{1-\rho}} \left(\tau \left(\frac{f_e}{f_d} \right)^{\frac{1-\rho}{\rho}} \bar{x}_d \right)^{\frac{\rho-\gamma(1-\rho)}{1-\rho}} \right)} - f_d = 0$$

$$\frac{(\gamma(1-\rho) - \rho)\mu(\bar{\ell} + \pi)\bar{x}_d^{\gamma}}{n\gamma \left(f_d^{\frac{\rho-\gamma(1-\rho)}{\rho}} + \tau^{-\gamma} f_e^{\frac{\rho-\gamma(1-\rho)}{\rho}} \right)} - f_d^{\frac{\gamma(1-\rho)}{\rho}} = 0$$

$$\bar{x}_d^{-\gamma} = \frac{(\gamma(1-\rho) - \rho)\mu(\bar{\ell} + \pi)f_d^{\frac{-\gamma(1-\rho)}{\rho}}}{n\gamma \left(f_d^{\frac{\rho-\gamma(1-\rho)}{\rho}} + \tau^{-\gamma} f_e^{\frac{\rho-\gamma(1-\rho)}{\rho}} \right)}.$$

Plugging this expression into the expression for π , we obtain

$$\pi = \frac{\rho\mu(\bar{\ell} + \pi)}{\gamma}$$

$$\pi = \frac{\rho\mu\bar{\ell}}{\gamma - \rho\mu},$$

which implies that

$$\bar{x}_d^{-\gamma} = \frac{(\gamma(1-\rho) - \rho)\mu\bar{\ell}f_d^{\frac{-\gamma(1-\rho)}{\rho}}}{n(\gamma - \rho\mu) \left(f_d^{\frac{\rho-\gamma(1-\rho)}{\rho}} + \tau^{-\gamma} f_e^{\frac{\rho-\gamma(1-\rho)}{\rho}} \right)}.$$

A model with costly entry

Consider the closed economy case. The entry condition is

$$\int_{\bar{x}}^{\infty} \left(p(x)c(x) - \frac{c(x)}{x} - f \right) dF(x) = \phi$$

where ϕ is the entry cost.

In this formulation,

$$\pi = 0;$$

that is, since expected profits are equal to the cost of entry, there are no profits net of entry costs in equilibrium.

As before, we can obtain an expression for the cutoff productivity \bar{x} :

$$\bar{x} = \left(\frac{n\gamma f}{(\gamma(1-\rho) - \rho)\mu\bar{\ell}} \right)^{\frac{1}{\gamma}}.$$

We now calculate an expression for n :

$$\int_{\bar{x}}^{\infty} \left(\frac{\rho(\gamma(1-\rho) - \rho)\mu\bar{\ell}x^{\frac{1}{1-\rho}}}{n(1-\rho)\gamma\bar{x}^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}} \left(\frac{1}{\rho x} - \frac{1}{x} \right) - f \right) \gamma x^{-\gamma-1} dx = \phi$$

$$\frac{(1-\rho)\mu\bar{\ell}\bar{x}^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}}{n\bar{x}^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}} - f\bar{x}^{-\gamma} = \phi$$

$$\bar{x}^{-\gamma} = \frac{(1-\rho)\mu\bar{\ell} - n\phi}{nf} = \frac{(\gamma(1-\rho) - \rho)\mu\bar{\ell}}{n\gamma f}$$

$$n = \frac{\rho\mu\bar{\ell}}{\gamma\phi}.$$

Notice that this implies that

$$\bar{x} = \left(\frac{\rho f}{(\gamma(1-\rho) - \rho)\phi} \right)^{\frac{1}{\gamma}}.$$