Monopolistic competition with heterogeneous firms and trade (Melitz-Chaney)

Consumers:

$$\max\ (1-\alpha)\log c_0 + \frac{\alpha}{\rho}\log\int_Z c(z)^\rho\, dz$$

$$\text{s.t. } p_0 c_0 + \int_0^\mu p(z)c(z)dz = w\bar{\ell} + \pi$$

$$c(z) \geq 0.$$

Firms:

$$y_0 = \ell_0$$

$$y(z) = \max\left[x(z)\left(\ell(z) - f\right), 0\right].$$

where $x(z) > 0$ is the firm's productivity level.

The solution to the representative consumer's problem is

$$c_0 = (1-\alpha)\frac{w\bar{\ell}+\pi}{p_0}$$

$$c(z) = \frac{\alpha(w\bar{\ell}+\pi)}{p(z)^{\frac{1}{1-\rho}}\int_Z p(z')^{\frac{-\rho}{1-\rho}}dz'} = \frac{\alpha(w\bar{\ell}+\pi)}{p(z)^{\frac{1}{1-\rho}}P^{\frac{-\rho}{1-\rho}}},$$

where

$$P = \left(\int_Z p(z')^{\frac{-\rho}{1-\rho}}dz'\right)^{\frac{-(1-\rho)}{\rho}}.$$

Firm z solves

$$\max\ p(z)c(z) - \frac{wc(z)}{x(z)} - wf = p(z)\frac{\alpha(w\bar{\ell}+\pi)}{p(z)^{\frac{1}{1-\rho}}P^{\frac{-\rho}{1-\rho}}} - \frac{w\alpha(w\bar{\ell}+\pi)}{x(z)\,p(z)^{\frac{1}{1-\rho}}P^{\frac{-\rho}{1-\rho}}} - wf\ ,$$

taking P as given.

$$p(z) = \frac{w}{\rho x(z)}.$$

There is a measure μ of potential firms. Firm productivities are distributed on the interval $x \geq 1$ according to the Pareto distribution

$$F(x) = 1 - x^{-\gamma}.$$

Definition of equilibrium: An equilibrium is
price functions $\hat{p}_0,\ \hat{p}(z)$,
a wage rates $\hat{w}$,
profits $\hat{\pi}$,
consumption functions $\hat{c}_0,\ \hat{c}(z)$,
production plans $\hat{y}_0,\ \hat{\ell}_0,\ \hat{y}(z),\ \hat{\ell}(z)$,
and a set of firms that produce positive amounts $\hat{Z} \subset [0, \mu]$
such that

- Given $\hat{p}_0,\ \hat{p}(z),\ \hat{w},\ \hat{\pi}$, the consumer chooses $\hat{c}_0,\ \hat{c}(z)$ to solve

$$\max\ (1-\alpha)\log c_0 + \frac{\alpha}{\rho}\log \int_{\hat{Z}} c(z)^{\rho}\, dz$$
$$\text{s.t. } \hat{p}_0 c_0 + \int_{\hat{Z}} \hat{p}(z)c(z)dz = \hat{w}\bar{\ell} + \hat{\pi}$$
$$c(z) \geq 0.$$

- $\hat{p}_0 - \hat{w} \leq 0, = 0 \text{ if } \hat{y}_0 > 0$.

- $\hat{p}(z) = \dfrac{\hat{w}}{\rho x(z)}$.

- $\hat{p}(z)\hat{y}(z) - \dfrac{\hat{y}(z)}{x(z)} - f \geq 0 \text{ for all } z \in \hat{Z},\ \hat{p}(z)\hat{y}(z) - \dfrac{\hat{y}(z)}{x(z)} - f \leq 0 \text{ for all } z \notin \hat{Z}$

- $\hat{c}_0 = \hat{y}_0$.

- $\hat{c}(z) = \hat{y}(z),\ z \in [0, \mu]$.

- $\hat{\ell}_0 + \int_0^{\mu} \hat{\ell}(z)\, dz = \overline{\ell}$.

- $\hat{\pi} = \hat{w} \int_{\hat{Z}} (\hat{p}(z)\hat{y}(z) - \hat{y}(z) / x(z) - f) dz$.

Alternatively, we could index firms by x, letting $\hat{X}$ be the set of productivities for which firms produce positive amounts and writing the consumer's problem as

$$\max \ (1-\alpha)\log c_0 + \frac{\alpha\mu}{\rho}\log \int_{\hat{X}} c(x)^{\rho}\, dF(x)$$

$$\text{s.t. } \hat{p}_0 c_0 + \int_{\hat{X}} \hat{p}(x)c(x)\,dF(x) = \hat{w}\bar{\ell} + \hat{\pi}$$

$$c(x) \geq 0.$$

Suppose that, in equilibrium, not all firms produce.

Then there is $\bar{x} > 1$ such that no firm with $x(z) < \bar{x}$ produces and all firms with $x(z) \geq \bar{x}$ produce.

Set $\hat{w} = 1$.

$$P^{\frac{-\rho}{1-\rho}} = \mu \int_{\bar{x}}^{\infty} p(x)^{\frac{-\rho}{1-\rho}} dF(x) = \mu \int_{\bar{x}}^{\infty} (\rho x)^{\frac{\rho}{1-\rho}} \gamma x^{-\gamma-1} dx$$

$$P^{\frac{-\rho}{1-\rho}} = -\frac{\mu \rho^{\frac{\rho}{1-\rho}} (1-\rho) \gamma x^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}}{\gamma(1-\rho)-\rho} \Bigg|_{\bar{x}}^{\infty} = \frac{\mu \rho^{\frac{\rho}{1-\rho}} (1-\rho) \gamma \bar{x}^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}}{\gamma(1-\rho)-\rho}.$$

Notice that we require $\alpha(1-\rho) > \rho$ for P to be finite. The demand for goods produced by a firm with productivity x is

$$c(x) = \frac{\alpha(\bar{\ell} + \pi)}{p(x)^{\frac{1}{1-\rho}} P^{\frac{-\rho}{1-\rho}}} = \frac{\big(\gamma(1-\rho)-\rho\big)\alpha(\bar{\ell}+\pi)}{(\rho x)^{\frac{-1}{1-\rho}} \mu\rho^{\frac{\rho}{1-\rho}}(1-\rho)\gamma\bar{x}^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}}.$$

$$c(x) = \frac{\rho\big(\gamma(1-\rho)-\rho\big)\alpha(\bar{\ell}+\pi)x^{\frac{1}{1-\rho}}}{\mu(1-\rho)\gamma\bar{x}^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}}$$

We calculate the cutoff productivity $\bar{x}$:

$$p(\bar{x})c(\bar{x}) - \frac{c(\bar{x})}{\bar{x}} - f = \frac{\rho\big(\gamma(1-\rho)-\rho\big)\alpha(\bar{\ell}+\pi)\bar{x}^{\frac{1}{1-\rho}}}{\mu(1-\rho)\gamma\bar{x}^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}}\left(\frac{1}{\rho\bar{x}} - \frac{1}{\bar{x}}\right) - f = 0$$

$$\frac{\big(\gamma(1-\rho)-\rho\big)\alpha(\bar{\ell}+\pi)\bar{x}^{\gamma}}{\mu\gamma} - f = 0$$

$$\bar{x} = \left(\frac{\mu\gamma f}{\big(\gamma(1-\rho)-\rho\big)\alpha(\bar{\ell}+\pi)}\right)^{\frac{1}{\gamma}}.$$

Notice that this expression depends on profits π, which we can calculate as

$$\pi = \mu \int_{\bar{x}}^{\infty} \left(p(x)c(x) - \frac{c(x)}{x} - f \right) dF(x)$$

$$\pi = \mu \int_{\bar{x}}^{\infty} \left(\frac{\left(\gamma(1-\rho) - \rho\right)\alpha(\bar{\ell} + \pi) x^{\frac{\rho}{1-\rho}}}{\mu\gamma\bar{x}^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}} - f \right) \gamma x^{-\gamma-1} dx$$

$$\pi = \mu \frac{\left(\gamma(1-\rho) - \rho\right)\alpha(\bar{\ell} + \pi)}{\mu\gamma\bar{x}^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}} \int_{\bar{x}}^{\infty} x^{\frac{\rho}{1-\rho}} \gamma x^{-\gamma-1} dx - \mu f \int_{\bar{x}}^{\infty} \gamma x^{-\gamma-1} dx$$

$$\pi = (1-\rho)\alpha(\bar{\ell} + \pi) - \mu\bar{x}^{-\gamma} f$$

$$\pi = \frac{(1-\rho)\alpha\bar{\ell} - \mu\bar{x}^{-\gamma} f}{1-(1-\rho)\alpha}$$

$$\pi = \frac{(1-\rho)\alpha\overline{\ell}}{1-(1-\rho)\alpha} - \frac{\mu f}{1-(1-\rho)\alpha}\,\frac{\left(\gamma(1-\rho)-\rho\right)\alpha(\overline{\ell}+\pi)}{\mu\gamma f}$$

$$\pi = \frac{(1-\rho)\alpha\overline{\ell}}{1-(1-\rho)\alpha} - \frac{\left(\gamma(1-\rho)-\rho\right)\alpha(\overline{\ell}+\pi)}{\gamma\left(1-(1-\rho)\alpha\right)}$$

$$\pi = \frac{\rho\alpha\overline{\ell}}{\gamma-\rho\alpha},$$

which implies that

$$\overline{x}^{-\gamma} = \frac{\left(\gamma(1-\rho)-\rho\right)\alpha\left(\overline{\ell}+\dfrac{\rho\alpha\overline{\ell}}{\gamma-\rho\alpha}\right)}{\mu\gamma f} = \frac{\left(\gamma(1-\rho)-\rho\right)\alpha\overline{\ell}}{(\gamma-\rho\alpha)\mu f}.$$

Suppose now that there are two symmetric countries that engage in free trade.

Each country i, $i = 1,2$, has a population of $\overline{\ell}$ and a measure of potential firms of μ.

Firms' productivities are again distributed according to the Pareto distribution, $F(x) = 1 - x^{-\gamma}$.

A firm in country i faces a fixed cost of exporting to country j, $j \neq i$, of f_e where $f_e > f_d = f$ and an iceberg transportation cost of $\tau - 1 \geq 0$.

Definition of equilibrium: An equilibrium is
price functions $\hat{p}_0$, $\hat{p}_j^i(z)$, $i, j = 1, 2$,
wage rates $\hat{w}_j$, $j = 1, 2$,
profits $\hat{\pi}_j$, $j = 1, 2$,
consumption functions $\hat{c}_j^0$, $\hat{c}_j^i(z)$, $i, j = 1, 2$,
production plans $\hat{y}_{j0}$, $\hat{\ell}^i_{j0}$, $\hat{y}_j^i(z)$, $\hat{\ell}^i_j(z)$, $i, j = 1, 2$,
a set of firms that produce for domestic consumption in country j,
$\hat{Z}_{jd} \subset [0, \mu]$,
and a set of firms that produce for export in country j, $\hat{Z}_{je} \subset [0, \mu]$,
$\hat{Z}_{je} \subset \hat{Z}_{jd}$,
such that

- Given $\hat{p}_0,\ \hat{p}_1^1(z),\ \hat{p}_2^1(z),\ \hat{w}_1,\ \hat{\pi}_1$, the consumer in country 1 chooses $\hat{c}_1^1(z),\ \hat{c}_2^1(z)$ to solve the problem

$$\max\ (1-\alpha)\log c_0^1 + \frac{\alpha}{\rho}\left(\log\int_{\hat{Z}_{1d}} c_1^1(z)^\rho dz + \log\int_{\hat{Z}_{2e}} c_2^1(z)^\rho dz\right)$$
$$\text{s.t. } \hat{p}_0 c_0^1 + \int_{\hat{Z}_{1d}} \hat{p}_1^1(z) c_1^1(z) dz + \int_{\hat{Z}_{2e}} \hat{p}_2^1 c_2^1(z) dz = \hat{w}_1 \overline{\ell} + \hat{\pi}_1$$
$$c(z) \geq 0,$$

and similarly for the consumer in country 2.

- $\hat{p}_0 - \hat{w}_j \leq 0,\ = 0 \text{ if } \hat{y}_{j0} > 0.$

- $\hat{p}_j^j(z) = \dfrac{\hat{w}_j}{\rho x(z)}.$

- $\hat{p}_j^i(z) = \dfrac{\tau \hat{w}_j}{\rho x(z)}$.

- $\hat{p}_j^j(z)\hat{y}_j^j(z) - \dfrac{\hat{y}_j^j(z)}{x(z)} - f_d \geq 0$ for all $z \in \hat{Z}_{jd}$, $\hat{p}_j^j(z)\hat{y}_j^j(z) - \dfrac{\hat{y}_j^j(z)}{x(z)} - f_d \leq 0$ for all $z \notin \hat{Z}_{jd}$

- $\hat{p}_j^i(z)\hat{y}_j^i(z) - \dfrac{\tau \hat{y}_j^i(z)}{x(z)} - f_e \geq 0$ for all $z \in \hat{Z}_{je}$,

  $\hat{p}_j^i(z)\hat{y}_j^i(z) - \dfrac{\tau \hat{y}_j^i(z)}{x(z)} - f_e \leq 0$ for all $z \notin \hat{Z}_{je}$

- $\hat{c}_0^1 + \hat{c}_0^2 = \hat{y}_{10} + \hat{y}_{20}$.

- $\hat{c}_j^i(z) = \hat{y}_j^i(z),\ z \in [0, \mu].$

- $\hat{\ell}_{j0} + \int_0^{\mu} (\hat{\ell}_j^1(z) + \hat{\ell}_j^2(z))\, dz = \overline{\ell}.$

- $$\hat{\pi}_j = \hat{w}_j \int_{\hat{Z}_{jd}} (\hat{p}_j^j(z)\hat{y}_j^j(z) - \hat{y}_j^j(z)/x(z) - f_d)dz$$
$$+\hat{w}_j \int_{\hat{Z}_{je}} (\hat{p}_j^i(z)\hat{y}_j^i(z) - \tau\hat{y}_j^i(z)/x(z) - f_e)dz.$$

Suppose again that, in equilibrium, not all firms produce.

To simplify, we impose symmetry:

$$P^{\frac{-\rho}{1-\rho}} = \mu\int_{\bar{x}_d}^{\infty} p_1^1(x)^{\frac{-\rho}{1-\rho}} dF(x) + \mu\int_{\bar{x}_e}^{\infty} p_2^1(x)^{\frac{-\rho}{1-\rho}} dF(x)$$

$$P^{\frac{-\rho}{1-\rho}} = \mu\int_{\bar{x}_d}^{\infty} (\rho x)^{\frac{\rho}{1-\rho}} \gamma x^{-\gamma-1} dx + \mu\int_{\bar{x}_e}^{\infty} \left(\frac{\tau}{\rho x}\right)^{\frac{-\rho}{1-\rho}} \gamma x^{-\gamma-1} dx$$

$$P^{\frac{-\rho}{1-\rho}} = -\left.\frac{\mu\rho^{\frac{\rho}{1-\rho}}(1-\rho)\gamma x^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}}{\gamma(1-\rho)-\rho}\right|_{\bar{x}_d}^{\infty} + \left.\frac{\mu\tau^{\frac{-\rho}{1-\rho}}\rho^{\frac{\rho}{1-\rho}}(1-\rho)\gamma x^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}}{\gamma(1-\rho)-\rho}\right|_{\bar{x}_e}^{\infty}$$

$$P^{\frac{-\rho}{1-\rho}} = \frac{\mu\rho^{\frac{\rho}{1-\rho}}(1-\rho)\gamma\overline{x}_d^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}}{\gamma(1-\rho)-\rho} + \frac{\mu\tau^{\frac{-\rho}{1-\rho}}\rho^{\frac{\rho}{1-\rho}}(1-\rho)\gamma\overline{x}_e^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}}{\gamma(1-\rho)-\rho}$$

$$P^{\frac{-\rho}{1-\rho}} = \frac{\rho^{\frac{\rho}{1-\rho}}(1-\rho)\gamma\left(\mu\overline{x}_d^{\frac{\rho-\gamma(1-\rho)}{1-\rho}} + \mu\tau^{\frac{-\rho}{1-\rho}}\overline{x}_e^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}\right)}{\gamma(1-\rho)-\rho}$$

The demand in country 1 for goods produced by a firm in country 1 with productivity $x \geq \overline{x}_d$ is

$$c_1^1(x) = \frac{\alpha(\overline{\ell}+\pi)}{p_1^1(x)^{\frac{1}{1-\rho}}P^{\frac{-\rho}{1-\rho}}} = \frac{\rho\big(\gamma(1-\rho)-\rho\big)\alpha(\overline{\ell}+\pi)x^{\frac{1}{1-\rho}}}{(1-\rho)\gamma\left(\mu\overline{x}_d^{\frac{\rho-\gamma(1-\rho)}{1-\rho}} + \mu\tau^{\frac{-\rho}{1-\rho}}\overline{x}_e^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}\right)}.$$

We calculate an expression for the cutoff productivity $\overline{x}_d$:

$$p_1^1(\overline{x}_d)c_1^1(\overline{x}_d) - \frac{c_1^1(\overline{x}_d)}{\overline{x}_d} - f_d = 0$$

$$\frac{\rho\left(\gamma(1-\rho)-\rho\right)\alpha(\overline{\ell}+\pi)\overline{x}_d}{(1-\rho)\gamma\left(\mu\overline{x}_d^{\frac{\rho-\gamma(1-\rho)}{1-\rho}} + \mu\tau^{\frac{-\rho}{1-\rho}}\overline{x}_e^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}\right)}\left(\frac{1}{\rho\overline{x}_d} - \frac{1}{\overline{x}_d}\right) - f_d = 0$$

$$p_1^1(\overline{x}_d)c_1^1(\overline{x}_d) - \frac{c_1^1(\overline{x}_d)}{\overline{x}_d} - f_d = \frac{\left(\gamma(1-\rho)-\rho\right)\alpha(\overline{\ell}+\pi)\overline{x}_d^{\frac{\rho}{1-\rho}}}{\gamma\left(\mu\overline{x}_d^{\frac{\rho-\gamma(1-\rho)}{1-\rho}} + \mu\tau^{\frac{-\rho}{1-\rho}}\overline{x}_e^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}\right)} - f_d = 0.$$

Similarly, we calculate an expression for the cutoff productivity $\bar{x}_e$:

$$p_1^2(\bar{x}_e)c_1^2(\bar{x}_e) - \frac{\tau c_1^2(\bar{x}_e)}{\bar{x}_e} - f_e = \frac{\big(\gamma(1-\rho)-\rho\big)\alpha(\bar{\ell}+\pi)\tau^{\frac{-\rho}{1-\rho}}\bar{x}_e^{\frac{\rho}{1-\rho}}}{\gamma\left(\mu\bar{x}_d^{\frac{\rho-\gamma(1-\rho)}{1-\rho}} + \mu\tau^{\frac{-\rho}{1-\rho}}\bar{x}_e^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}\right)} - f_e = 0.$$

The expression for π is

$$\pi = \mu\int_{\bar{x}_d}^{\infty}\left(p_1^1(x)c_1^1(x) - \frac{c_1^1(x)}{x} - f_d\right)dF(x) + \mu\int_{\bar{x}_e}^{\infty}\left(p_1^2(x)c_1^2(x) - \frac{\tau c_1^2(x)}{x} - f_e\right)dF(x)$$

$$\pi = \mu\int_{\bar{x}_d}^{\infty}\left(\frac{\left(\gamma(1-\rho)-\rho\right)\alpha(\bar{\ell}+\pi)x^{\frac{\rho}{1-\rho}}}{\gamma\left(\mu\bar{x}_d^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}+\mu\tau^{\frac{-\rho}{1-\rho}}\bar{x}_e^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}\right)}-f_d\right)\gamma x^{-\gamma-1}dx$$

$$+\mu\int_{\bar{x}_e}^{\infty}\left(\frac{\left(\gamma(1-\rho)-\rho\right)\alpha(\bar{\ell}+\pi)\tau^{\frac{-\rho}{1-\rho}}x^{\frac{\rho}{1-\rho}}}{\gamma\left(\mu\bar{x}_d^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}+\mu\tau^{\frac{-\rho}{1-\rho}}\bar{x}_e^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}\right)}-f_e\right)\gamma x^{-\gamma-1}dx$$

$$\pi = \frac{(1-\rho)\alpha(\bar{\ell}+\pi)\mu\bar{x}_d^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}}{\mu\bar{x}_d^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}+\mu\tau^{\frac{-\rho}{1-\rho}}\bar{x}_e^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}}+\frac{(1-\rho)\alpha(\bar{\ell}+\pi)\mu\tau^{\frac{-\rho}{1-\rho}}\bar{x}_e^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}}{\mu\bar{x}_d^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}+\mu\tau^{\frac{-\rho}{1-\rho}}\bar{x}_e^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}}-\mu\left(\bar{x}_d^{-\gamma}f_d+\bar{x}_e^{-\gamma}f_e\right)$$

.

This provides us with a system of 3 equations is 3 unknowns to be solved for $\bar{x}_d$, $\bar{x}_e$, and π:

$$\frac{\left(\gamma(1-\rho)-\rho\right)\alpha(\bar{\ell}+\pi)\bar{x}_d^{\frac{\rho}{1-\rho}}}{\mu\gamma\left(\bar{x}_d^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}+\tau^{\frac{-\rho}{1-\rho}}\bar{x}_e^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}\right)}-f_d=0$$

$$\frac{\left(\gamma(1-\rho)-\rho\right)\alpha(\bar{\ell}+\pi)\tau^{\frac{-\rho}{1-\rho}}\bar{x}_e^{\frac{\rho}{1-\rho}}}{\mu\gamma\left(\bar{x}_d^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}+\tau^{\frac{-\rho}{1-\rho}}\bar{x}_e^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}\right)}-f_e=0$$

$$\pi=(1-\rho)\alpha(\bar{\ell}+\pi)-\mu\left(\bar{x}_d^{-\gamma}f_d+\bar{x}_e^{-\gamma}f_e\right)$$

Notice that

$$\frac{\overline{x}_e}{\overline{x}_d} = \tau \left(\frac{f_e}{f_d} \right)^{\frac{1-\rho}{\rho}}$$

and that

$$P^{\frac{-\rho}{1-\rho}} = \frac{\rho^{\frac{\rho}{1-\rho}}(1-\rho)\gamma\mu \left(\overline{x}_d^{\frac{\rho-\gamma(1-\rho)}{1-\rho}} + \tau^{\frac{-\rho}{1-\rho}} \overline{x}_e^{\frac{\rho-\gamma(1-\rho)}{1-\rho}} \right)}{\gamma(1-\rho)-\rho}.$$

The equation for π can be rewritten as

$$\pi = (1-\rho)\alpha(\bar{\ell}+\pi) - \mu\left(\bar{x}_d^{-\gamma} f_d + \left(\tau\left(\frac{f_e}{f_d}\right)^{\frac{1-\rho}{\rho}} \bar{x}_d\right)^{-\gamma} f_e\right)$$

$$\pi = (1-\rho)\alpha(\bar{\ell}+\pi) - \mu\left(f_d^{\frac{\rho-\gamma(1-\rho)}{\rho}} + \tau^{-\gamma} f_e^{\frac{\rho-\gamma(1-\rho)}{\rho}}\right) f_d^{\frac{\gamma(1-\rho)}{\rho}} \bar{x}_d^{-\gamma}.$$

The equation for $\bar{x}_d$ can be rewritten as

$$\frac{\left(\gamma(1-\rho)-\rho\right)\alpha(\bar{\ell}+\pi)\bar{x}_d^{\frac{\rho}{1-\rho}}}{\mu\gamma\left(\bar{x}_d^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}+\tau^{\frac{-\rho}{1-\rho}}\left(\tau\left(\frac{f_e}{f_d}\right)^{\frac{1-\rho}{\rho}}\bar{x}_d\right)^{\frac{\rho-\gamma(1-\rho)}{1-\rho}}\right)}-f_d=0$$

$$\frac{\left(\gamma(1-\rho)-\rho\right)\alpha(\bar{\ell}+\pi)\bar{x}_d^{\gamma}}{\mu\gamma\left(f_d^{\frac{\rho-\gamma(1-\rho)}{\rho}}+\tau^{-\gamma}f_e^{\frac{\rho-\gamma(1-\rho)}{\rho}}\right)}-f_d^{\frac{\gamma(1-\rho)}{\rho}}=0$$

$$\bar{x}_d^{-\gamma}=\frac{\left(\gamma(1-\rho)-\rho\right)\alpha(\bar{\ell}+\pi)f_d^{\frac{-\gamma(1-\rho)}{\rho}}}{\mu\gamma\left(f_d^{\frac{\rho-\gamma(1-\rho)}{\rho}}+\tau^{-\gamma}f_e^{\frac{\rho-\gamma(1-\rho)}{\rho}}\right)}.$$

Plugging this expression into the expression for π, we obtain

$$\pi = \frac{\rho\alpha(\overline{\ell} + \pi)}{\gamma}$$

$$\pi = \frac{\rho\alpha\overline{\ell}}{\gamma - \rho\alpha},$$

which implies that

$$\overline{x}_d^{-\gamma} = \frac{\left(\gamma(1-\rho)-\rho\right)\alpha\,\overline{\ell} f_d^{\frac{-\gamma(1-\rho)}{\rho}}}{\mu(\gamma-\rho\alpha)\left(f_d^{\frac{\rho-\gamma(1-\rho)}{\rho}}+\tau^{-\gamma} f_e^{\frac{\rho-\gamma(1-\rho)}{\rho}}\right)}$$

$$\overline{x}_e^{-\gamma} = \tau^{-\gamma}\left(\frac{f_e}{f_d}\right)^{\frac{-\gamma(1-\rho)}{\rho}} \overline{x}_d^{-\gamma} = \frac{\left(\gamma(1-\rho)-\rho\right)\alpha\,\overline{\ell}\,\tau^{-\gamma} f_e^{\frac{-\gamma(1-\rho)}{\rho}}}{\mu(\gamma-\rho\alpha)\left(f_d^{\frac{\rho-\gamma(1-\rho)}{\rho}}+\tau^{-\gamma} f_e^{\frac{\rho-\gamma(1-\rho)}{\rho}}\right)}.$$