

## A Ricardian model with a continuum of goods

Consider an economy in which there are two countries and a continuum of goods indexed  $z \in [0,1]$ . Goods are produced using labor:

$$y_j(z) = \ell_j(z) / a_j(z).$$

where

$$\begin{aligned} a_1(z) &= e^{\alpha z} \\ a_2(z) &= e^{\alpha(1-z)}. \end{aligned}$$

Here  $y_j(z)$  is the production of good  $z$  in country  $j$  and  $\ell_j(z)$  is the input of labor. The stand-in consumer in each country has the utility function

$$\int_0^1 \log c_j(z) dz.$$

This consumer is endowed with  $\bar{\ell}_j$  unites of labor where  $\bar{\ell}_1 = \bar{\ell}_2 = \bar{\ell}$ .

**Definition of equilibrium:** An equilibrium is

a price function  $\hat{p}(z)$ ,

wage rates  $\hat{w}_1, \hat{w}_2$ ,

consumption functions  $\hat{c}_1(z), \hat{c}_2(z)$ ,

and production plans  $\hat{y}_1(z), \hat{\ell}_1(z), \hat{y}_2(z), \hat{\ell}_2(z)$

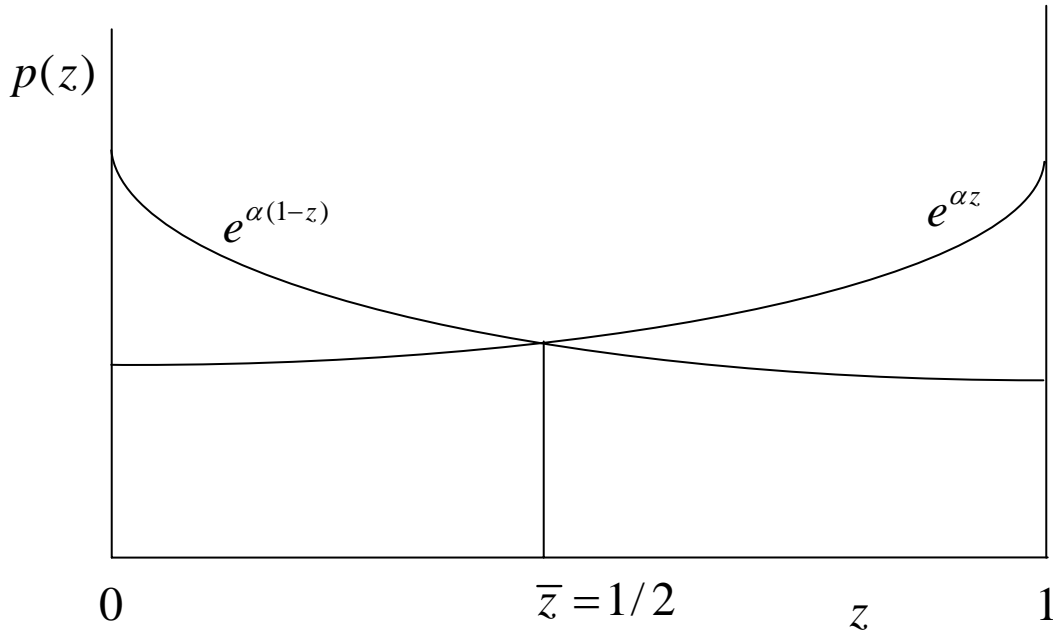
such that

- Given  $\hat{p}(z), \hat{w}_j$ , the consumer in country  $j, j=1,2$ , chooses  $\hat{c}^j(z)$  to solve

$$\begin{aligned} \max \quad & \int_0^1 \log c_j(z) dz \\ \text{s.t.} \quad & \int_0^1 \hat{p}(z)c_j(z) dz \leq \hat{w}_j \bar{\ell}_j \\ & c_j(z) \geq 0. \end{aligned}$$

- $\hat{p}(z) - a_j(z)\hat{w}_j \leq 0, = 0$  if  $\hat{y}_j(z) > 0, j=1,2, z \in [0,1]$
- $\hat{c}_1(z) + \hat{c}_2(z) = \hat{y}_1(z) + \hat{y}_2(z), z \in [0,1]$ .
- $\int_0^1 \hat{\ell}_j(z) dz = \bar{\ell}, j=1,2$ .

Because of symmetry, we know that there is an equilibrium in which  $\hat{w}_1 = \hat{w}_2 = 1$ . This implies that the pattern of production, trade, and specialization is



Country 1 produces and exports the goods in the interval  $[0, \bar{z}]$  while country 2 produces and exports the goods in the interval  $(\bar{z}, 1]$ .

The prices of the goods are

$$\hat{p}(z) = \begin{cases} e^{\alpha z} & z \in [0, \bar{z}] \\ e^{\alpha(1-z)} & z \in (\bar{z}, 1] \end{cases}.$$

The consumption levels are

$$\hat{c}_1(z) = \hat{c}_2(z) = \frac{\bar{\ell}}{\hat{p}(z)}.$$

The production plans are

$$\hat{y}_1(z) = \frac{2\bar{\ell}}{\hat{p}(z)}, \hat{\ell}_1(z) = 2\bar{\ell}, \hat{y}_2(z) = \hat{\ell}_2(z) = 0, z \in [0, \bar{z}]$$

$$\hat{y}_1(z) = \hat{\ell}_1(z) = 0, \hat{y}_2(z) = \frac{2\bar{\ell}}{\hat{p}(z)}, \hat{\ell}_2(z) = 2\bar{\ell}, z \in (\bar{z}, 1].$$

## Model with tariffs

Suppose that each country imposes an ad valorem tariff  $\tau$  on imports from the other country. Suppose too that tariff revenues are rebated in a lump sum form to the representative consumer.

**Definition of equilibrium:** An equilibrium is  
 producer price functions  $\hat{p}_1(z), \hat{p}_2(z)$   
 consumer price functions  $\hat{q}_1(z), \hat{q}_2(z)$   
 wage rates  $\hat{w}_1, \hat{w}_2$ ,  
 consumption functions  $\hat{c}_1(z), \hat{c}_2(z)$ ,  
 production plans  $\hat{y}_1(z), \hat{\ell}_1(z), \hat{y}_2(z), \hat{\ell}_2(z)$ ,  
 and tariff revenues  $\hat{T}_1, \hat{T}_2$   
 such that

- $\hat{q}_1(z) = \min[a_1(z)\hat{w}_1, (1+\tau)a_2(z)\hat{w}_2]$   
 $\hat{q}_2(z) = \min[(1+\tau)a_1(z)\hat{w}_1, a_2(z)\hat{w}_2]$ .
- Given  $\hat{q}_j(z), \hat{w}_j$ , the consumer in country  $j, j=1,2$ , chooses  $\hat{c}_j(z)$  to solve

$$\begin{aligned} & \max \int_0^1 \log c_j(z) dz \\ \text{s. t. } & \int_0^1 \hat{q}_j(z)c_j(z) dz \leq \hat{w}_j \bar{\ell}_j + \hat{T}_j \\ & c_j(z) \geq 0. \end{aligned}$$

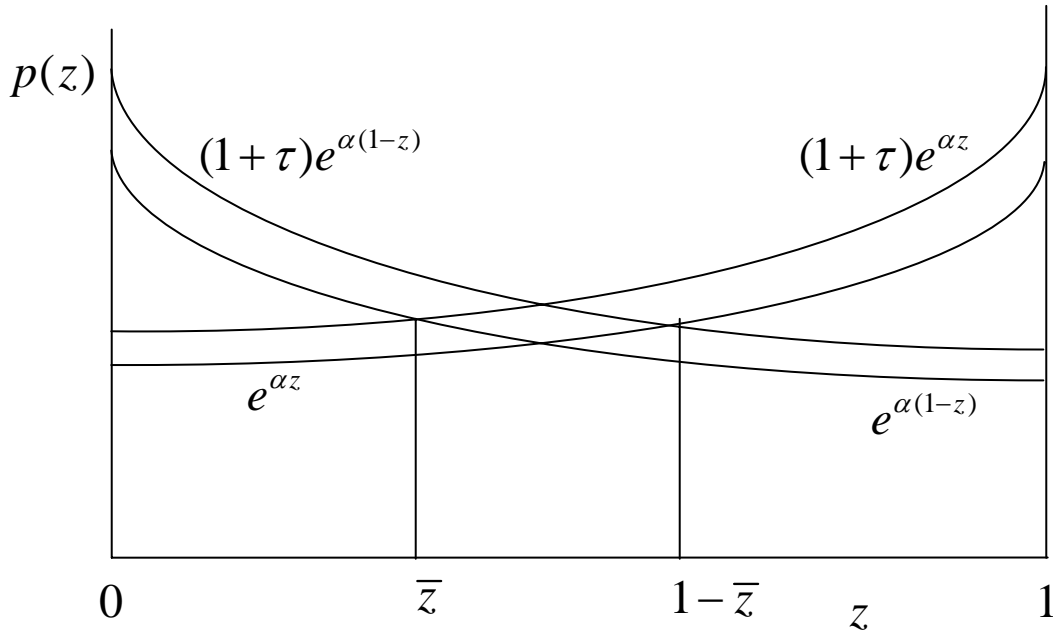
- $\hat{p}_j(z) - a_j(z)\hat{w}_j \leq 0, = 0$  if  $\hat{y}_j(z) > 0, j=1,2, z \in [0,1]$
- $\hat{c}_1(z) + \hat{c}_2(z) = \hat{y}_1(z) + \hat{y}_2(z), z \in [0,1]$
- $\hat{T}_1 = \int_{a_1(z)\hat{w}_1 > (1+\tau)a_2(z)\hat{w}_2} \tau \hat{p}_2(z)\hat{c}_1(z) dz$   
 $\hat{T}_2 = \int_{a_2(z)\hat{w}_2 > (1+\tau)a_1(z)\hat{w}_1} \tau \hat{p}_1(z)\hat{c}_2(z) dz.$
- $\int_0^1 \hat{\ell}_j(z) dz = \bar{\ell}, j=1,2.$

Once again, because of symmetry, we know that there is an equilibrium in which  $\hat{w}_1 = \hat{w}_2 = 1$ .

There are two possibilities: either there is no trade in equilibrium or there is trade in equilibrium.

First,  $\tau$  is so large and/or  $\alpha$  is so small that there is no trade in equilibrium because  $(1+\tau) > e^\alpha$ , which implies that  $a_1(z)\hat{w}_1 < (1+\tau)a_2(z)\hat{w}_2$  and  $(1+\tau)a_1(z)\hat{w}_1 > a_2(z)\hat{w}_2$  for all  $z \in [0,1]$ .

Second, if  $(1+\tau) < e^\alpha$ , then the pattern of production, trade, and specialization looks something like this:



Country 1 produces the goods in the interval  $[0, 1 - \bar{z}]$  and exports the goods in the interval  $[0, \bar{z}]$ . Country 2 produces the goods in the interval  $[0, \bar{z}]$  and exports the goods in the interval  $[0, 1 - \bar{z}]$ . The goods in the interval  $[\bar{z}, 1 - \bar{z}]$  are not traded.

$$\begin{aligned} (1+\tau)e^{\alpha\bar{z}} &= e^{\alpha(1-\bar{z})} \\ \log(1+\tau) + \alpha\bar{z} &= \alpha(1-\bar{z}) \\ \bar{z} &= \frac{1}{2} - \frac{\log(1+\tau)}{2\alpha}. \end{aligned}$$

To find  $\hat{T}_1 = \hat{T}_2 = T$ , we use the solution to the consumer's problem

$$c_j(z) = \frac{w_j \bar{\ell} + T_j}{q_j(z)}$$

to obtain

$$T = \int_0^{\bar{z}} \tau p(z) \frac{\bar{\ell} + T}{(1 + \tau)p(z)} dz$$

$$\hat{T}_1 = \hat{T}_2 = \hat{T} = \frac{\tau \bar{z} \bar{\ell}}{1 + \tau(1 - \bar{z})}.$$

The producer prices goods are

$$\hat{p}_1(z) = e^{\alpha z}, \quad z \in [0, 1 - \bar{z}]$$

$$\hat{p}_2(z) = e^{\alpha(1-z)}, \quad z \in (\bar{z}, 1].$$

The consumer prices are

$$\hat{q}_1(z) = \begin{cases} e^{\alpha z} & z \in [0, 1 - \bar{z}] \\ (1 + \tau)e^{\alpha(1-z)} & z \in (1 - \bar{z}, 1] \end{cases}$$

$$\hat{q}_2(z) = \begin{cases} (1 + \tau)e^{\alpha z} & z \in [0, \bar{z}] \\ e^{\alpha(1-z)} & z \in (\bar{z}, 1] \end{cases}.$$

The consumption levels are

$$\hat{c}_1(z) = \frac{\bar{\ell} + \hat{T}}{\hat{q}_1(z)}$$

$$\hat{c}_2(z) = \frac{\bar{\ell} + \hat{T}}{\hat{q}_2(z)}.$$

The production plans are

$$\hat{y}_1(z) = \frac{(2 + \tau)(\bar{\ell} + \hat{T})}{(1 + \tau)\hat{p}_1(z)}, \quad \hat{\ell}_1(z) = \frac{(2 + \tau)(\bar{\ell} + \hat{T})}{(1 + \tau)}, \quad \hat{y}_2(z) = \hat{\ell}_2(z) = 0, \quad z \in [0, \bar{z}]$$

$$\hat{y}_1(z) = \frac{\bar{\ell} + \hat{T}}{\hat{p}_1(z)}, \quad \hat{\ell}_1(z) = \bar{\ell} + \hat{T}, \quad \hat{y}_2(z) = \frac{\bar{\ell} + \hat{T}}{\hat{p}_2(z)}, \quad \hat{\ell}_2(z) = \bar{\ell} + \hat{T}, \quad z \in (\bar{z}, 1 - \bar{z}]$$

$$\hat{y}_1(z) = \hat{\ell}_1(z) = 0, \quad \hat{y}_2(z) = \frac{(2 + \tau)(\bar{\ell} + \hat{T})}{(1 + \tau)\hat{p}_2(z)}, \quad \hat{\ell}_2(z) = \frac{(2 + \tau)(\bar{\ell} + \hat{T})}{(1 + \tau)}, \quad z \in (1 - \bar{z}, 1].$$

## A Heckscher-Ohlin Model with a Continuum of Goods

Suppose now that goods are produced using both capital and labor:

$$y_j(z) = k_j(z)^{\alpha(z)} \ell_j(z)^{1-\alpha(z)},$$

where  $\alpha(z) = z$ ,  $z \in [0,1]$ . Notice that production technologies are now identical across countries. Endowments, however, are different. Specifically,

$$\bar{\ell}_1 = \bar{k}_2 > \bar{\ell}_2 = \bar{k}_1.$$

**Definition of equilibrium:** An equilibrium is

a price function  $\hat{p}(z)$ ,

factor prices  $\hat{r}_1, \hat{w}_1, \hat{r}_2, \hat{w}_2$ ,

consumption functions  $\hat{c}_1(z), \hat{c}_2(z)$ ,

and production plans  $\hat{y}_1(z), \hat{k}_1(z), \hat{\ell}_1(z), \hat{y}_2(z), \hat{k}_2(z), \hat{\ell}_2(z)$

such that

- Given  $\hat{p}(z), \hat{w}_j$ , the consumer in country  $j$ ,  $j = 1, 2$ , chooses  $\hat{c}_j(z)$  to solve

$$\begin{aligned} & \max \int_0^1 \log c_j(z) dz \\ \text{s.t. } & \int_0^1 \hat{p}(z) c_j(z) dz \leq \hat{r}_j \bar{k}_j + \hat{w}_j \bar{\ell}_j \\ & c_j(z) \geq 0. \end{aligned}$$

- $\hat{p}(z) \alpha(z) \hat{k}_j^{\alpha(z)-1} \hat{\ell}_j^{1-\alpha(z)} - \hat{r}_j \leq 0$ ,  $= 0$  if  $\hat{y}_j(z) > 0$ ,  $j = 1, 2$ ,  $z \in [0,1]$   
 $\hat{p}(z) (1 - \alpha(z)) \hat{k}_j^{\alpha(z)} \hat{\ell}_j^{-\alpha(z)} - \hat{w}_j \leq 0$ ,  $= 0$  if  $\hat{y}_j(z) > 0$ ,  $j = 1, 2$ ,  $z \in [0,1]$
- $\hat{c}_1(z) + \hat{c}_2(z) = \hat{y}_1(z) + \hat{y}_2(z)$ ,  $z \in [0,1]$ .
- $\int_0^1 \hat{\ell}_j(z) dz = \bar{\ell}$ ,  $j = 1, 2$ .

Because of symmetry, we know that there is an equilibrium on which  $\hat{r}_1 = \hat{w}_2$  and  $\hat{w}_1 = \hat{r}_2$ .

There are two possibilities: either  $\hat{r}_1 = \hat{w}_2 > \hat{w}_1 = \hat{r}_2 = 1$  or  $\hat{r}_1 = \hat{w}_2 = \hat{w}_1 = \hat{r}_2 = 1$ . If

$\hat{r}_1 = \hat{w}_2 > \hat{w}_1 = \hat{r}_2 = 1$ , then country 1 specializes in all of the goods less capital intensive than a specific level  $\bar{z}$ , that is all  $z \leq \bar{z}$ , and country 2 specializes in all goods more capital intensive than the same  $\bar{z}$ , that is all  $z > \bar{z}$ . Because of symmetry,  $\bar{z} = 1/2$ . The

graph is the same as for part (a). On the other hand, if  $\hat{r}_1 = \hat{w}_2 = \hat{w}_1 = \hat{r}_2 = 1$ , then the structure of production and trade is indeterminate.

We use the first-order conditions for firm  $z$  in country  $j$  to obtain

$$\begin{aligned}\ell_j(z) &= \left( \frac{r_j(1-z)}{w_j z} \right)^z y_j(z) \\ k_j(z) &= \left( \frac{w_j z}{r_j(1-z)} \right)^{1-z} y_j(z) \\ p(z) &= \frac{r_j^z w_j^{1-z}}{z^z (1-z)^{1-z}}.\end{aligned}$$

To see which of the two cases that we are in, we suppose that  $\hat{r}_1 = \hat{w}_2 = \hat{w}_1 = \hat{r}_2 = 1$ . Let us calculate the demand for labor in country 1 under the assumption that that country 1 produces all of the goods  $z \leq 1/2$ . If this amount of labor is less than  $\bar{\ell}$ , then we know that we are in the other case, where  $\hat{r}_1 = \hat{w}_2 > \hat{w}_1 = \hat{r}_2 = 1$ .

$$\begin{aligned}\ell_1(z) &= \left( \frac{(1-z)}{z} \right)^z y_1(z) \\ p(z) &= \frac{1}{z^z (1-z)^{1-z}} \\ c_1(z) &= \frac{\bar{k}_1 + \bar{\ell}_1}{p(z)} \\ c_2(z) &= \frac{\bar{k}_2 + \bar{\ell}_2}{p(z)},\end{aligned}$$

which imply that

$$\begin{aligned}y_1(z) &= c_1(z) + c_2(z) = \frac{\bar{k}_1 + \bar{\ell}_1 + \bar{k}_2 + \bar{\ell}_2}{p(z)} = z^z (1-z)^{1-z} (\bar{k}_1 + \bar{\ell}_1 + \bar{k}_2 + \bar{\ell}_2) \\ \ell_1(z) &= \left( \frac{(1-z)}{z} \right)^z z^z (1-z)^{1-z} (\bar{k}_1 + \bar{\ell}_1 + \bar{k}_2 + \bar{\ell}_2) = (1-z) (\bar{k}_1 + \bar{\ell}_1 + \bar{k}_2 + \bar{\ell}_2).\end{aligned}$$

The total demand for labor in country 1 is

$$\int_0^{1/2} (1-z) (\bar{k}_1 + \bar{\ell}_1 + \bar{k}_2 + \bar{\ell}_2) dz = (\bar{k}_1 + \bar{\ell}_1 + \bar{k}_2 + \bar{\ell}_2) \left( z - \frac{z^2}{2} \right) \Big|_0^{1/2} = \frac{3}{8} (\bar{k}_1 + \bar{\ell}_1 + \bar{k}_2 + \bar{\ell}_2).$$

If

$$\bar{\ell}_1 < \frac{3}{8}(\bar{k}_1 + \bar{\ell}_1 + \bar{k}_2 + \bar{\ell}_2),$$

then we know that we are in the case where  $\hat{r}_1 = \hat{w}_2 > \hat{w}_1 = \hat{r}_2 = 1$ . Since  $\bar{k}_2 = \bar{\ell}_1$  and  $\bar{\ell}_2 = \bar{k}_1$ , this condition is

$$\begin{aligned}\bar{\ell}_1 &< \frac{3}{4}(\bar{k}_1 + \bar{\ell}_1) \\ \bar{\ell}_1 &< \frac{1}{3}\bar{k}_1.\end{aligned}$$

If  $\bar{\ell}_1 < \frac{1}{3}\bar{k}_1$ , let us solve for  $\hat{w}_2 = \hat{r}_1 = r$

$$\begin{aligned}p(z) &= \frac{r^z}{z^z(1-z)^{1-z}} \\ y_1(z) = c_1(z) + c_2(z) &= \frac{2(r\bar{k}_1 + \bar{\ell}_1)}{p(z)} = \frac{2z^z(1-z)^{1-z}(r\bar{k}_1 + \bar{\ell}_1)}{r^z} \\ \ell_1(z) &= \left(\frac{r(1-z)}{z}\right)^z \frac{2z^z(1-z)^{1-z}(r\bar{k}_1 + \bar{\ell}_1)}{r^z} \\ \ell_1(z) &= 2(1-z)(r\bar{k}_1 + \bar{\ell}_1) \\ \int_0^{1/2} 2(1-z)(r\bar{k}_1 + \bar{\ell}_1) dz &= 2(r\bar{k}_1 + \bar{\ell}_1) \left(z - \frac{z^2}{2}\right) \Big|_0^{1/2} = \frac{3}{4}(r\bar{k}_1 + \bar{\ell}_1).\end{aligned}$$

Solving for  $\hat{w}_2 = \hat{r}_1 = r$ , we obtain

$$\begin{aligned}\frac{3}{4}(r\bar{k}_1 + \bar{\ell}_1) &= \bar{\ell}_1 \\ \hat{w}_2 = \hat{r}_1 = r &= \frac{\bar{\ell}_1}{3\bar{k}_1}.\end{aligned}$$