General Equilibrium with Time and Uncertainty

1. Pure exchange, m consumers, one good per state

a) Primitive concepts

Events $\eta_t = 1,...,n$ (finite number)

Stationary Markov chain $\pi_{ij} = \text{prob}(\eta_t = j \mid \eta_{t-1} = i)$

An event history, or state, is a node on the event tree $s = (\eta_0, \eta_1, ..., \eta_t)$

$t(s)$ is length of s minus one, the time period in which s occurs

η_s and $\eta_{t(s)}$, last event in history

S is set of all states (countable)

$\pi(s) = \pi_{\eta_0\eta_1}\pi_{\eta_1\eta_2}...\pi_{\eta_{t-1}\eta_t}$

Preferences $\sum_{s\in S}\beta_i^{t(s)}\pi(s)u_i(c_s^i,\eta_s)$ ($u_i(\cdot,\eta)$ can depend on event — allows for demand shocks)

$0 > \beta_i > 1$

$u_i(\cdot,\eta)$ strictly concave, motonically increasing

Endowment $w^i(\eta_s) > 0$ (depends on event)

b) Arrow-Debreu market structure

One set of markets — a market for c_s at each state $s \in S$ — at $t = 0$ where η_0 is known

An **equilibrium** is sequences $\hat{p}_s$ and $\hat{c}_s^i$, $i = 1,...,m$, $s \in S$, such that

- $\hat{c}_s^i$, $s \in S$, solves

$$\max \sum_{s\in S}\beta_i^{t(s)}\pi(s)u_i(c_s^i,\eta_s)$$

$$\text{s.t. } \sum_{s\in S}\hat{p}_s c_s^i \le \sum_{s\in S}\hat{p}_s w^i(\eta_s)$$

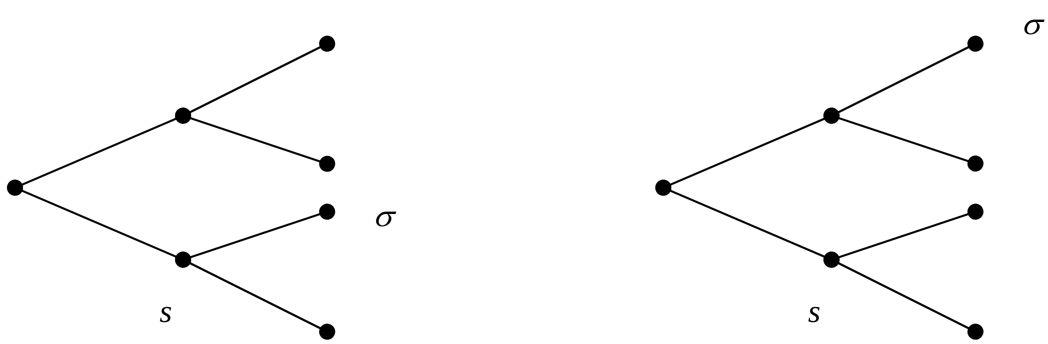
$$c_s^i \ge 0$$

- $\sum_{i=1}^{m} \hat{c}_s^i \le \sum_{i=1}^{m} w^i(\eta_s)$, $s \in S$.

c) Sequential markets market structure

$n+1$ market at every node $s \in S$: one for the consumption good c_s and one for each of n Arrow securities, $b_{(s,j)}$, that pay one unit of consumption in period $t(s)+1$ if event j occurs, where history would then be $(s, \eta_{t(s)+1}) = (s, j)$.

Let $\sigma \ge s$, $\sigma \in S$ and $s \in S$, mean that, if $s = (\eta_0, ..., \eta_s)$, then $\sigma = (\eta_0, ..., \eta_s, \eta_{t(s)+1}, ..., \eta_{\sigma.})$ — in other words, s is an earlier node in the same path as σ.



$\sigma \ge s$ $\qquad\qquad$ $\sigma \not\ge s$

An e**quilibrium** is sequences $\hat{q}_s$, $\hat{c}_s^i$, and $\hat{b}_s^i$, $i = 1, ..., m$, $s \in S$, such that

- $\hat{c}_s^i$, $\hat{b}_s^i$, $s \in S$, solve

$$\max \sum_{s \in S} \beta_i^{t(s)} \pi(s) u_i(c_s^i, \eta_s)$$

$$\text{s.t. } c_s^i + \sum_{j=1}^{n} \hat{q}_{(s,j)} b_{(s,j)}^i \le w^i(\eta_s) + b_s^i$$

$$c_s^i \ge 0, \; b_s^i \ge -B$$

$$b_{\eta_0}^i = 0$$

(Here, as usual, B is a positive constant that prevents Ponzi schemes but is large enough so that the constraint does not otherwise bind in equilibrium.)

- $\sum_{i=1}^{m} \hat{c}_s^i \leq \sum_{i=1}^{m} w^i(\eta_s)$, $s \in S$

- $\sum_{i=1}^{m} \hat{b}_s^i = 0$, $s \in S$.

It is easy to show that $\hat{c}_\sigma^i$, $\hat{b}_\sigma^i$, $\sigma \geq s$, solve

$$\max \sum_{\sigma \geq s} \beta^{t(\sigma)-t(s)} (\pi(\sigma)/\pi(s)) u_i(x_\sigma^i, \eta_\sigma)$$

$$\text{s.t. } c_\sigma^i + \sum_{j=1}^{n} \hat{q}_{(\sigma,j)} b_{(\sigma,j)}^i \leq w^i(\eta_\sigma) + b_\sigma^i,\ \sigma \geq s$$

$$c_\sigma^i \geq 0,\ b_\sigma^i \geq -B$$

$$b_s^i \text{ given.}$$

In other words, the consumer does not want to change his plan if he resolves his problem at every node.

2. Production, representative consumer, one good per node

a) Primitive concepts

Events, histories, probabilities as before

Preferences $\sum_{s \in S} \beta^{t(s)} \pi(s) u(c_s, \overline{\ell}(\eta_s) - \ell_s, \eta_s)$

Endowment of labor $\overline{\ell}(\eta_s) > 0$

Endowment of capital k_0 at $s = \eta_0$

Production function $f(k, \ell, \eta)$

$f(\cdot, \cdot, \eta)$ is concave and homogeneous of degree one (continuously differentiable for convenience)

Let $s+1$ be any state of the form (s, j), $j = 1, \ldots, n$

Feasibility

$$c_s + k_{s+1} - (1-\delta) k_s \leq f(k_s, \ell_s, \eta_s)$$

Production set

$$Y=\left\{(k_s,\ell_s,c_s),\ s\in S\,\middle|\,c_s+k_{s+1}-(1-\delta)k_s\le f(k_s,\ell_s,\eta_s);\ k_s,\ell_s,c_s\ge 0,\ k_{(s,j)}=k_{s+1},\ j=1,\dots,n\right\}.$$

b) Arrow-Debreu market structure

One set of markets — markets for c_s, k_{s+1}, and ℓ_s at each state $s\in S$ and a market for k_{η_0} — at $t=0$ where η_0 is known

An e**quilibrium** is sequences $\hat{p}_s$, $\hat{w}_s$, $\hat{c}_s$, $\hat{k}_s$, $\hat{\ell}_s$, $s\in S$, and $\hat{v}_0$, such that

- $\hat{c}_s$, $\hat{\ell}_s$, $s\in S$, solve

$$\max \sum\nolimits_{s\in S}\beta^{t(s)}\pi(s)u(c_s,\overline{\ell}(\eta_s)-\ell_s,\eta_s)$$
$$\text{s.t. } \sum\nolimits_{s\in S}\hat{p}_s c_s\le\sum\nolimits_{s\in S}\hat{w}_s\ell_s+\hat{v}_0\overline{k}_0$$
$$c_s,\ell_s,(\overline{\ell}(\eta_s)-\ell_s)\ge 0.$$

- $(\hat{k}_s,\ \hat{\ell}_s,\ \hat{c}_s)\in Y$ where the consumer and the firm choose the same $\hat{\ell}_s$ and $\hat{k}_{\eta_0}=\overline{k}_0$

  (We could define $\hat{\ell}^c_s$ and $\hat{\ell}^f_s$ separately and require that $\hat{\ell}^c_s=\hat{\ell}^f_s$.)

- $$\hat{p}_{\eta_0}\left(f(\hat{k}_{\eta_0},\hat{\ell}_{\eta_0},\eta_0)+(1-\delta)\hat{k}_{\eta_0}\right)+\sum\nolimits_{s\in S}\sum\nolimits_{j=1}^{n}\hat{p}_{(s,j)}\left(f(\hat{k}_{s+1},\hat{\ell}_{(s,j)},j)+(1-\delta)\hat{k}_{s+1}\right)$$
  $$-\sum\nolimits_{s\in S}\hat{w}_s\hat{\ell}_s-\sum\nolimits_{s\in S}\hat{p}_s\hat{k}_{s+1}-\hat{v}_0\overline{k}_0=0,$$
  $$\hat{p}_{\eta_0}\left(f(k_{\eta_0},\ell_{\eta_0},\eta_0)+(1-\delta)k_{\eta_0}\right)+\sum\nolimits_{s\in S}\sum\nolimits_{j=1}^{n}\hat{p}_{(s,j)}\left(f(k_{s+1},\ell_{(s,j)},j)+(1-\delta)k_{s+1}\right)$$
  $$-\sum\nolimits_{s\in S}\hat{w}_s\ell_s-\sum\nolimits_{s\in S}\hat{p}_s k_{s+1}-\hat{v}_0 k_{\eta_0}\le 0$$

  for all $(k_s,\ \ell_s,\ c_s)\in Y$.

  First order conditions for the firm:

$$\sum_{j=1}^{n} \hat{p}_{(s,j)} \left(\frac{\partial f}{\partial k}(\hat{k}_{s+1}, \hat{\ell}_{(s,j)}, \eta_{(s,j)}) + 1 - \delta \right) - \hat{p}_s = 0$$

$$\hat{p}_{\eta_0} \left(\frac{\partial f}{\partial k}(\hat{k}_{\eta_0}, \hat{\ell}_{\eta_0}, \eta_0) + 1 - \delta \right) - \hat{v}_0 = 0$$

$$\hat{p}_s \frac{\partial f}{\partial \ell}(\hat{k}_s, \hat{\ell}_s, \eta_s) - \hat{w}_s = 0\,.$$

c) Sequential markets market structure

Market at every node $s \in S$ in consumption c_s, next period capital k_{s+1}, labor ℓ_s, and n securities, $b_{(s,j)}$, $j = 1,...,n$

An e**quilibrium** is sequences $\hat{r}_s$, $\hat{w}_s$, $\hat{q}_s$, $\hat{c}_s$, $\hat{k}_{s+1}$, $\hat{\ell}_s$, $\hat{b}_s$, $s \in S$, such that

- $\hat{c}_s$, $\hat{k}_{s+1}$, $\hat{\ell}_s$, $\hat{b}_s$, $s \in S$, solve

$$\max \sum_{s \in S} \beta^{t(s)} \pi(s) u(c_s, \overline{\ell}(\eta_s) - \ell_s, \eta_s)$$

$$\text{s.t. } c_s + k_{s+1} + \sum_{j=1}^{n} \hat{q}_{(s,j)} b_{(s,j)} \le \hat{w}_s \ell_s + (1 + \hat{r}_s - \delta) k_s + b_s$$

$$c_s, k_s, \ell_s, (\overline{\ell}(\eta_s) - \ell_s) \ge 0,\ b_s \ge -B$$

$$k_{\eta_0} = \overline{k}_0,\ b_{\eta_0} = 0$$

- $\hat{r}_s = \dfrac{\partial f}{\partial k}(\hat{k}_s,\ \hat{\ell}_s,\ \eta_s)$

  $\hat{w}_s = \dfrac{\partial f}{\partial k}(\hat{k}_s,\ \hat{\ell}_s,\ \eta_s)$

- $\hat{c}_s + \hat{k}_{s+1} - (1 - \delta)\hat{k}_s \le f(\hat{k}_s,\ \hat{\ell}_s,\ \eta_s)$

  $\hat{b}_s = 0\,.$

First order conditions for the consumer:

$$\beta^{t(s)}\pi(s)\frac{\partial u}{\partial c}(\hat{c}_s, \overline{\ell}(\eta_s) - \hat{\ell}_s, \eta_s) - p_s = 0$$

$$\sum_{j=1}^{n} p_{(s,j)}\left(1 + \hat{r}_{(s,j)} - \delta\right) - \hat{p}_s = 0$$

$$p_{(s,j)} - p_s \hat{q}_{(s,j)} = 0$$

Combining these conditions, we obtain the asset pricing equations

$$\hat{q}_{(s,j)} = \beta\pi_{\eta_s j}\frac{\frac{\partial u}{\partial c}(\hat{c}_{(s,j)}, \overline{\ell}(\eta_s) - \hat{\ell}_s, j)}{\frac{\partial u}{\partial c}(\hat{c}_s, \overline{\ell}(\eta_s) - \hat{\ell}_s, \eta_s)}$$

and the arbitrage conditions

$$\sum_{j=1}^{n} \hat{q}_{(s,j)}(1 + \hat{r}_{(s,j)} - \delta) = 1 .$$

d) Recursive equilibrium

The concept is like that of sequential markets equilibrium, but the idea of state is very different.

An e**quilibrium** is functions $k'(k,\eta)$, $r(k,\eta)$, $w(k,\eta)$, $q_{\eta'}(k,\eta)$, $c(k,\eta)$, $\ell(k,\eta)$ such that the sequences generated by the rules

$$\hat{k}_{s+1} = k'(\hat{k}_s, \eta_s)\,,\ \hat{k}_{\eta_0} = \overline{k}_0$$

$$\hat{r}_s = r(\hat{k}_s, \eta_s)$$

$$\hat{w}_s = w(\hat{k}_s, \eta_s)$$

$$\hat{q}_{(s,j)} = q_j(\hat{k}_s, \eta_s)$$

$$\hat{c}_s = c(\hat{k}_s, \eta_s)$$

$$\hat{\ell}_s = \ell(\hat{k}_s, \eta)$$

is a sequential market equilibrium.

More directly:

An **equilibrium** is functions $V(k,\eta)$, $k'(k,\eta)$, $r(k,\eta)$, $w(k,\eta)$, $q_{\eta'}(k,\eta)$, $c(k,\eta)$, $\ell(k,\eta)$ such that

- given $r(k,\eta)$, $w(k,\eta)$, $q_{\eta'}(k,\eta)$, the function $V(k,\eta)$ is the value function $V(k,0,\eta)$ that satisfies the functional equation

$$V(k,b,\eta) = \max u(c, \overline{\ell}(\eta) - \ell, \eta) + \beta \sum\nolimits_{\eta'=1}^{n} \pi_{\eta\eta'} V(k', b'_{\eta'}, \eta')$$

$$\text{s.t. } c + k' + \sum\nolimits_{\eta'=1}^{n} q_{\eta'}(k,\eta) b'_{\eta'} \leq w(k,\eta)\ell + (l + r(k,\eta) - \delta)k + b$$

$$c, k' \geq 0, \; b'_{\eta'} \geq -B$$

$$k, \; b \text{ given}$$

and $k'(k,\eta) = k'(k,0,\eta)$, $c(k,\eta) = c(k,0,\eta)$, $\ell(k,\eta) = \ell(k,0,\eta)$, $b'_{\eta'}(k,\eta) = b'_{\eta'}(k,0,\eta) = 0$, $\eta' = 1,...,n$, are the corresponding policy functions

- $r(k,\eta) = \dfrac{\partial f}{\partial k}(k, \ell(\eta), \eta)$

  $w(k,\eta) = \dfrac{\partial f}{\partial k}(k, \ell(\eta), \eta)$

  $$\hat{q}_{\eta'}(k,\eta) = \beta \pi_{\eta\eta'} \frac{\dfrac{\partial u}{\partial c}(c(k'(k,\eta),\eta'), \overline{\ell}(\eta') - \hat{\ell}(k'(k,\eta),\eta'),\eta')}{\dfrac{\partial u}{\partial c}(c(k,\eta), \overline{\ell}(\eta) - \hat{\ell}(k,\eta),\eta)}$$

- $c(k,\eta) + k'(k,\eta) - (1-\delta)k = f(k, \, \ell(\eta), \, \eta)$, all k, η.

Notice that equilibrium prices and quantities are Markov. That is, they depend only on the current dynamic programming state (k,η) and not on the Arrow-Debreu state, which is the entire history of events.

Be careful here about the use of the word “state”! The dynamic programming state (k,η) is a very different concept from the Arrow-Debreu state $s=(\eta_0,\eta_1,...,\eta_s)$.