A Version of Kei-Mu Yi's Model

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Environment:

Goods are produced using labor and intermediate goods:

$$\begin{aligned} y_j(z) &= \left[\theta v_j(z)^\eta + (1-\theta)\ell_j(z)^\eta\right]^{\frac{1}{\eta}} / a_j(z) \\ &= v_j(z)^\theta \ell_j(z)^{1-\theta} / a_j(z) \quad \text{if } \eta = 0 \,. \end{aligned}$$

where

$$\begin{aligned} a_1(z) &= e^{\alpha z} \\ a_2(z) &= e^{\alpha(1-z)} \,. \end{aligned}$$

Here $y_j(z)$ is the production of good z in country j; $v_j(z)$ is the input of the aggregate intermediate good; and $\ell_j(z)$ is the input of labor.

The utility function is

$$\left(\int_0^1 c_j(z)^\rho dz - 1\right) / \rho \qquad = \int_0^1 \log c_j(z)\, dz \quad \text{if } \rho = 0 \,.$$

The budget constraint is

$$\int_0^1 p_j(z) c_j(z)\, dz = w_j \bar{\ell}_j + T_j \,,$$

where T_j is the lump-sum rebate of tariff revenue.

The technology for producing intermediate goods is

$$\begin{aligned} v_j &= \left(\int_0^1 x_j(z)^\rho dz\right)^{\frac{1}{\rho}} \\ &= \exp \int_0^1 \log\, x_j(z)\, dz \quad \text{if } \rho = 0 \,, \end{aligned}$$

where v_j is the total production of intermediate goods in country j.

Cost minimization:

We calculate some price indices by solving cost minimization problems:

For the aggregate consumption/intermediate good in country j:

$$\min \int_0^1 p_j(z)x_j(z)\, dz$$

$$\text{s.t.} \left(\int_0^1 x_j(z)^\rho dz\right)^{\frac{1}{\rho}} = \overline{x}_j$$

$$x_j(z) = \frac{\overline{x}_j \left(\int_0^1 p_j(\zeta)^{-\frac{\rho}{1-\rho}} d\zeta\right)^{-\frac{1}{\rho}}}{p_j(z)^{\frac{1}{1-\rho}}} \qquad = \frac{\overline{x}_j \left(\exp \int_0^1 \log\, p_j(z)\, dz\right)}{p_j(z)} \quad \text{if } \rho = 0$$

$$\int_0^1 p_j x_j(z)\, dz = \left(\int_0^1 p_j(z)^{-\frac{\rho}{1-\rho}} dz\right)^{-\frac{1-\rho}{\rho}} \overline{x}_j \qquad = \left(\exp \int_0^1 \log\, p_j(z)\, dz\right)\overline{x}_j \quad \text{if } \rho = 0$$

$$\int_0^1 p_j x_j(z)\, dz = q_j \overline{x}_j$$

$$x_j(z) = \frac{q_j^{\frac{1}{1-\rho}}\, \overline{x}_j}{p_j(z)^{\frac{1}{1-\rho}}} \qquad = \frac{q_j \overline{x}_j}{p_j(z)} \quad \text{if } \rho = 0.$$

Here q_j is the price of the aggregate consumption/intermediate good in country j.

For the good z produced in country j:

$$\min\ q_j v_j(z) + w_j \ell_j(z)$$

$$\text{s.t.} \left[\theta v_j(z)^\eta + (1-\theta)\ell_j(z)^\eta\right]^{\frac{1}{\eta}} = a_j(z) y_j(z)$$

$$q_j(z)y_j(z) = q_j v_j(z) + w_j \ell_j(z) = a_j(z)y_j(z)\left[\theta^{\frac{1}{1-\eta}} q_j^{-\frac{\eta}{1-\eta}} + (1-\theta)^{\frac{1}{1-\eta}} w_j^{-\frac{\eta}{1-\eta}}\right]^{-\frac{1-\eta}{\eta}}$$

$$= \frac{a_j(z)y_j(z)q_j^\theta w_j^{1-\theta}}{\theta^\theta (1-\theta)^{1-\theta}} \quad \text{if } \eta = 0$$

$$q_j(z)y_j(z) = q_j v_j(z) + w_j \ell_j(z) = a_j(z)\overline{p}_j y_j(z)$$

$$q_j(z) = a_j(z)\overline{p}_j$$

where

$$q_j(z) = a_j(z)\left[\theta^{\frac{1}{1-\eta}} q_j^{-\frac{\eta}{1-\eta}} + (1-\theta)^{\frac{1}{1-\eta}} w_j^{-\frac{\eta}{1-\eta}}\right]^{-\frac{1-\eta}{\eta}}$$

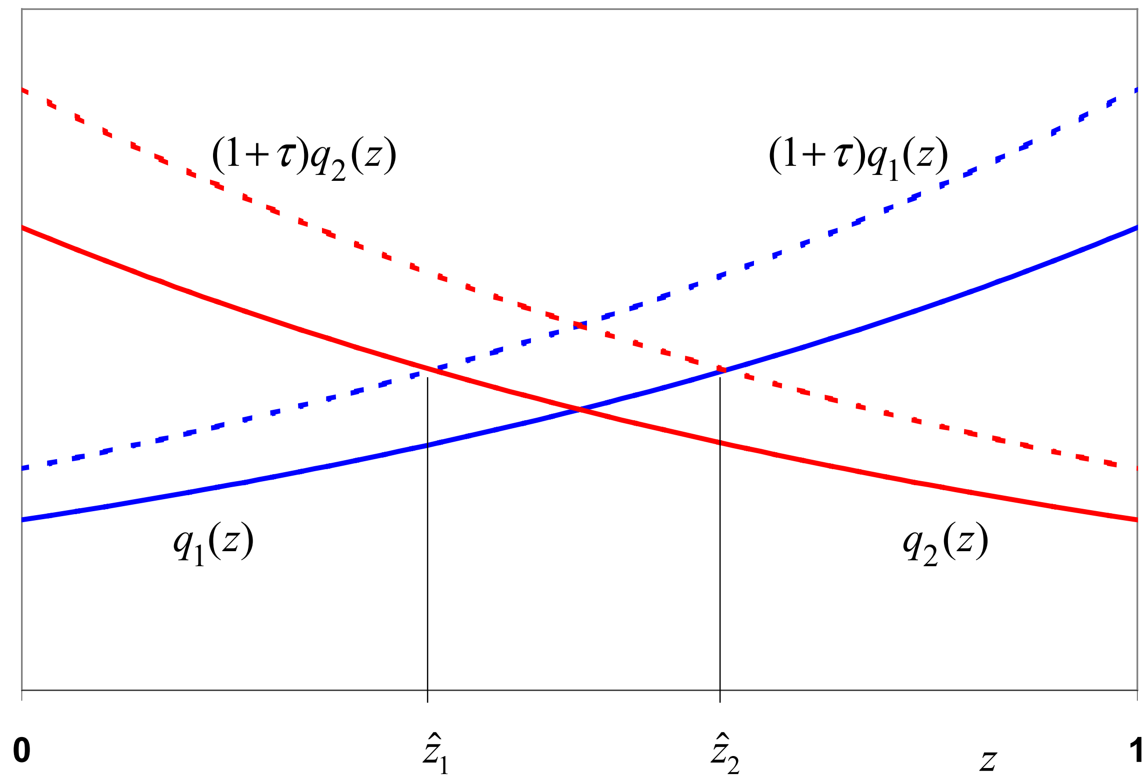
$$= \frac{a_j(z) q_j^{\theta} w_j^{1-\theta}}{\theta^{\theta}(1-\theta)^{1-\theta}} \quad \text{if } \eta = 0 \ .$$

$$v_j(z) = \frac{\theta^{\frac{1}{1-\eta}} \overline{p}_j^{\frac{1}{1-\eta}} a_j(z) y_j(z)}{q_j^{\frac{1}{1-\eta}}} \qquad = \frac{\theta \overline{p}_j a_j(z) y_j(z)}{q_j} \quad \text{if } \eta = 0.$$

$$\ell_j(z) = \frac{(1-\theta)^{\frac{1}{1-\eta}} \overline{p}_j^{\frac{1}{1-\eta}} a_j(z) y_j(z)}{w_j^{\frac{1}{1-\eta}}} \qquad = \frac{(1-\theta)\, \overline{p}_j\, a_j(z) y_j(z)}{w_j} \quad \text{if } \eta = 0.$$

Here $q_j(z)$ is the price of good z produced in country j and $\overline{p}_j$ is the price of the aggregate of the intermediate good and labor. Notice that, if $\theta = 0$, $\overline{p}_j = w_j$.

Patterns of production and trade



Symmetric Cobb-Douglas case:

Suppose that $\eta=\rho=0$. Further suppose that labor endowments and tariffs are equal across countries. Then

$$p_1(z)=\min\left[q_1(z),\ (1+\tau)q_2(z)\right]=\min\left[\frac{a_1(z)q_1^{\theta}w_1^{1-\theta}}{\theta^{\theta}(1-\theta)^{1-\theta}},\ (1+\tau)\frac{a_2(z)q_2^{\theta}w_2^{1-\theta}}{\theta^{\theta}(1-\theta)^{1-\theta}}\right].$$

Here $p_1(z)$ is the price paid for good z in country j.

Symmetry implies that

$$\begin{aligned}
w_1&=w_2=1\\
q_1&=q_2=q\ \ \hat{z}_1=1-\hat{z}_2\\
(1+\tau)e^{\alpha\hat{z}_1}&=e^{\alpha(1-\hat{z}_1)}\\
\hat{z}_1&=\frac{\alpha-\log(1+\tau)}{2\alpha}\\
\hat{z}_2&=\frac{\alpha+\log(1+\tau)}{2\alpha}.
\end{aligned}$$

We can calculate equilibrium prices:

$$q=q_1=\exp\ \int_0^1 \log\ p_1(z)\ dz=\exp\left[\int_0^{\hat{z}_2}\ \log\ \overline{p}e^{\alpha z}dz+\int_{\hat{z}_2}^1\ \log\ (1+\tau)\overline{p}e^{\alpha(1-z)}dz\right]$$
$$=\overline{p}(1+\tau)^{1-\hat{z}_2}e^{\alpha(2\hat{z}_2-1)}=\overline{p}(1+\tau)^{2-\hat{z}_2}$$

$$q=\overline{p}(1+\tau)^{2-\hat{z}_2}=\frac{q^{\theta}}{\theta^{\theta}(1-\theta)^{1-\theta}}(1+\tau)^{2-\hat{z}_2}=\frac{(1+\tau)^{\frac{2-\hat{z}_2}{1-\theta}}}{\theta^{\frac{\theta}{1-\theta}}(1-\theta)}.$$

$$\overline{p}=\frac{(1+\tau)^{\frac{(2-\hat{z}_2)\theta}{1-\theta}}}{\theta^{\frac{\theta}{1-\theta}}(1-\theta)}$$

$$q_j(z)=a_j(z)\frac{(1+\tau)^{\frac{(2-\hat{z}_2)\theta}{1-\theta}}}{\theta^{\frac{\theta}{1-\theta}}(1-\theta)}$$

$$p_1(z) = \begin{cases} e^{a z} \dfrac{(1+\tau)^{\frac{(2-\hat{z}_2)\theta}{1-\theta}}}{\theta^{\frac{\theta}{1-\theta}}(1-\theta)} & \text{if } z \le \hat{z}_2 \\ (1+\tau)e^{\alpha(1-z)} \dfrac{(1+\tau)^{\frac{(2-\hat{z}_2)\theta}{1-\theta}}}{\theta^{\frac{\theta}{1-\theta}}(1-\theta)} & \text{if } z \ge \hat{z}_2 \end{cases}$$

$$p_2(z) = \begin{cases} (1+\tau)e^{a z} \dfrac{(1+\tau)^{\frac{(1+\hat{z}_1)\theta}{1-\theta}}}{\theta^{\frac{\theta}{1-\theta}}(1-\theta)} & \text{if } z \le \hat{z}_1 \\ e^{\alpha(1-z)} \dfrac{(1+\tau)^{\frac{(1+\hat{z}_1)\theta}{1-\theta}}}{\theta^{\frac{\theta}{1-\theta}}(1-\theta)} & \text{if } z \ge \hat{z}_1 \end{cases}$$

Let $u_1(z) = a_1(z)y_1(z) = v_1(z)^{\theta}\ell_1(z)^{1-\theta}$ be the aggregate input of intermediate goods and labor in the production of good z in country 1. We know that

$$u_1 = \int_0^{\hat{z}_1} u_1(z)dz + \int_0^{\hat{z}_2} u_1(z)dz = v_1^{\theta}\ell_1^{1-\theta}$$

$$v_1 = \frac{\theta \bar{p} u_1}{q} = \theta(1+\tau)^{-(2-\hat{z}_2)}u_1$$

$$\ell_1 = (1-\theta)\,\bar{p}\,u_1 = \frac{(1+\tau)^{\frac{(2-\hat{z}_2)\theta}{1-\theta}}}{\theta^{\frac{\theta}{1-\theta}}}u_1.$$

Therefore

$$u_1 = \frac{\theta^{\frac{\theta}{1-\theta}}}{(1+\tau)^{\frac{(2-\hat{z}_2)\theta}{1-\theta}}}\bar{\ell}$$

$$v_1 = \frac{\theta^{\frac{1}{1-\theta}}}{(1+\tau)^{\frac{2-\hat{z}_2}{1-\theta}}}\bar{\ell}.$$

The value of exports is

$$\begin{aligned} X &= \int_0^{\hat{z}_1} q_1(z)\left[c_1(z)+x_1(z)\right]dz \\ &= \int_0^{\hat{z}_1} q_1(z)\left[\frac{\overline{\ell}+T}{(1+\tau)q_1(z)}+\frac{qv_1}{(1+\tau)q_1(z)}\right]dz \\ &= \int_0^{\hat{z}_1} q_1(z)\left[\frac{\overline{\ell}+T}{(1+\tau)q_1(z)}+\frac{\theta\overline{\ell}}{(1-\theta)(1+\tau)q_1(z)}\right]dz \\ &= \frac{\hat{z}_1}{(1+\tau)}\left[\frac{\overline{\ell}}{1-\theta}+T\right] \ . \end{aligned}$$

Tariff revenues are

$$T = \tau X = \frac{\tau\hat{z}_1}{(1+\tau)}\left[\frac{\overline{\ell}}{1-\theta}+T\right]$$

$$T = \frac{\tau\hat{z}_1\overline{\ell}}{(1-\theta)(1+\tau-\tau\hat{z}_1)} = \frac{\tau\left[\alpha-\log(1+\tau)\right]\overline{\ell}}{(1-\theta)\left[2\alpha+\alpha\tau+\tau\log(1+\tau)\right]} \ .$$

The value of exports is

$$X = \frac{\left[\alpha-\log(1+\tau)\right]\overline{\ell}}{(1-\theta)\left[2\alpha+\alpha\tau+\tau\log(1+\tau)\right]} \ .$$

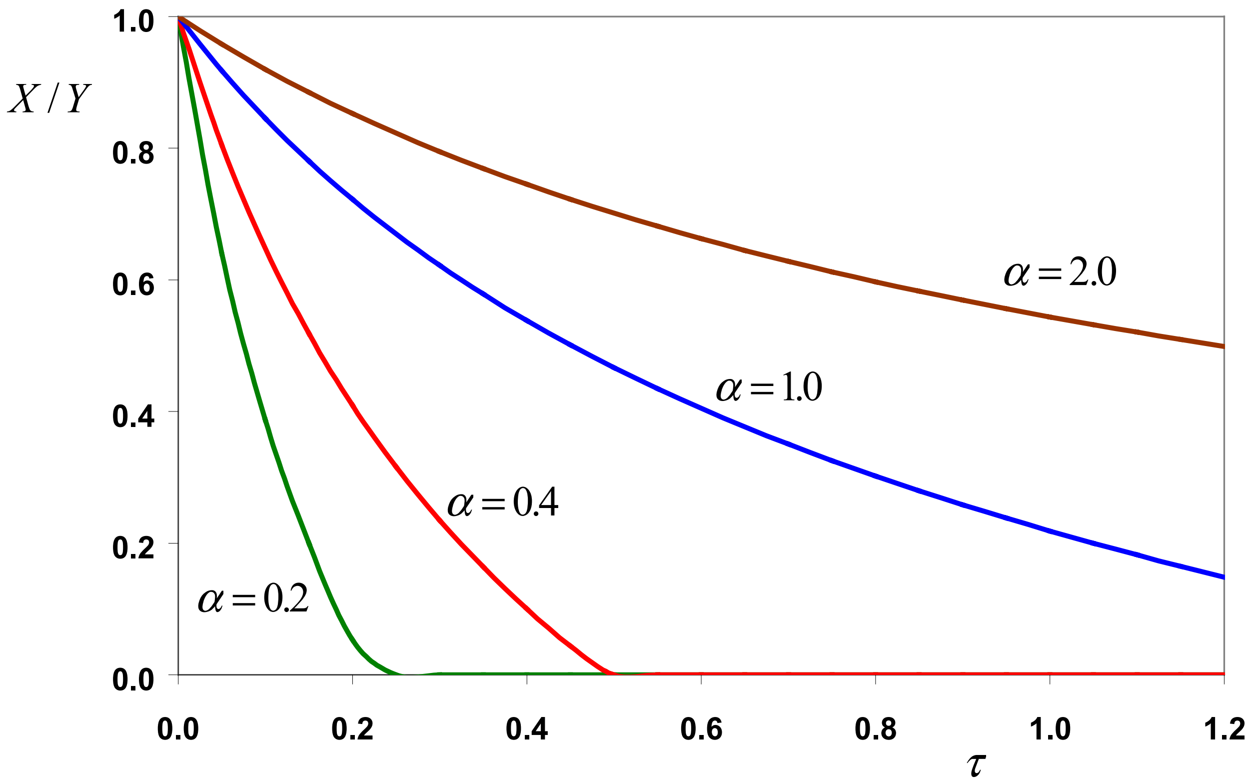
GDP is

$$\begin{aligned} Y = \ell + T &= \frac{(1-\theta)\left[2\alpha+\alpha\tau+\tau\log(1+\tau)\right]\overline{\ell}+\tau\left[\alpha-\log(1+\tau)\right]\overline{\ell}}{(1-\theta)\left[2\alpha+\alpha\tau+\tau\log(1+\tau)\right]} \\ &= \frac{\left[2(1-\theta)\alpha+(2-\theta)\alpha\tau-\theta\tau\log(1+\tau)\right]\overline{\ell}}{(1-\theta)\left[2\alpha+\alpha\tau+\tau\log(1+\tau)\right]} \ . \end{aligned}$$

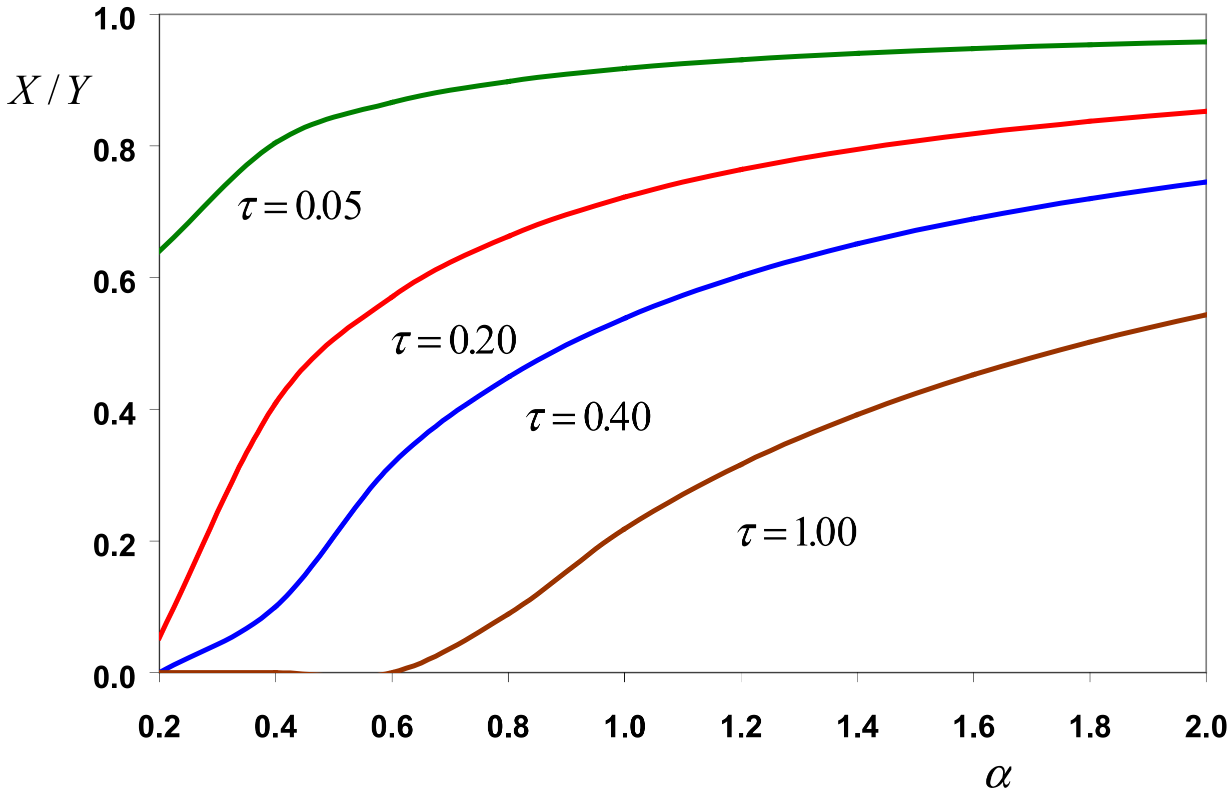
The ratio Exports/GDP is

$$X = \frac{\left[\alpha-\log(1+\tau)\right]}{\left[2(1-\theta)\alpha+(2-\theta)\alpha\tau-\theta\tau\log(1+\tau)\right]} .$$

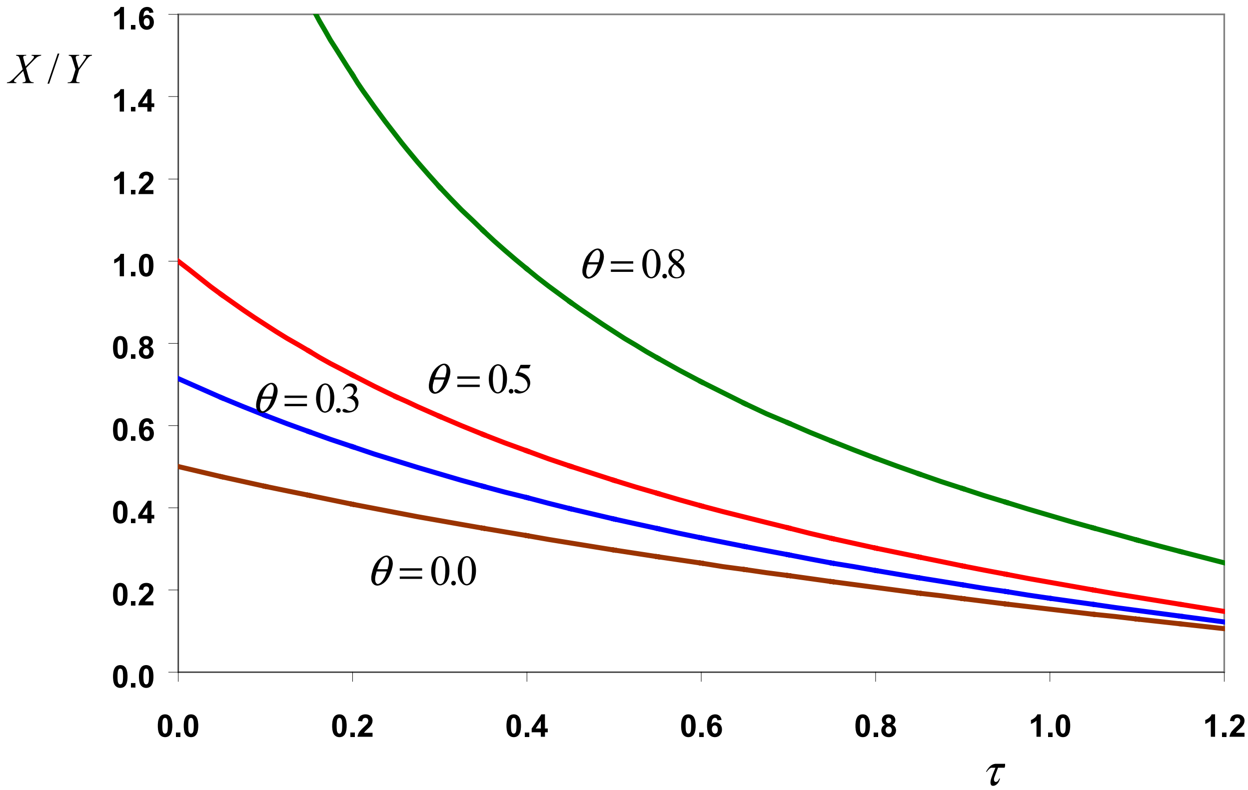
Fraction of world trade ($\theta = 0.5$)



Fraction of world trade ($\theta = 0.5$)



Fraction of world trade ($\alpha = 1.0$)



Fraction of world trade ($\alpha = 1.0$)