Calibrating the Growth Model

Kaldor's "stylized facts"

1. Y_t / N_t (output per worker) exhibits continual growth.

2. K_t / N_t (capital per worker) exhibits continual growth.

3. $r_t - \delta$ (real interest rate) is roughly constant.

4. K_t / Y_t (capital-output ratio) is roughly constant.

5. $r_t K_t / Y_t$, $w_t N_t / Y_t$ (factor shares) are roughly constant.

6. There are wide differences in the rate of growth of productivity across countries.

N. Kaldor (1961), "Capital Accumulation and Economic Growth," in F. A. Lutz and D. C. Hague, editors, *The Theory of Capital*. New York: St. Martin's Press.

The growth model

$$
\begin{aligned}
\max \ & \sum\nolimits_{t=0}^{\infty} \beta^t \log \ C_t / N_t \\
\text{s. t.} \ & C_t + K_{t+1} - (1-\delta)K_t \leq (\gamma^{1-\alpha})^t \theta K_t^{\alpha} N_t^{1-\alpha} \\
& C_t,\ K_t \geq 0 \\
& K_0 = \overline{K}_0 \\
& N_t = \eta^t N_0
\end{aligned}
$$

First-order conditions:

$$
\begin{aligned}
\beta^t / C_t &= P_t \\
P_{t-1} &= P_t(\alpha(\gamma^{1-\alpha})^t \theta K_t^{\alpha-1} N_t^{1-\alpha} + 1 - \delta) \\
C_t + K_{t+1} - (1-\delta)K_t &= (\gamma^{1-\alpha})^t \theta K_t^{\alpha} N_t^{1-\alpha}.
\end{aligned}
$$

Impose balanced growth conditions

$$
\frac{C_{t+1} / N_{t+1}}{C_t / N_t} = \gamma_c, \quad \frac{K_{t+1} / N_{t+1}}{K_t / N_t} = \gamma_k .
$$

Simple algebra shows that

$$\frac{C_{t+1}/N_{t+1}}{C_t/N_t}=\frac{K_{t+1}/N_{t+1}}{K_t/N_t}=\frac{Y_{t+1}/N_{t+1}}{Y_t/N_t}=\gamma\ .$$

Redefine variables in terms of effective labor units $\widetilde{N}_t=\gamma^t N_t=(\gamma\eta)^t N_0$:

$$\begin{aligned}
&\widetilde{c}_t=C_t/\widetilde{N}_t=\gamma^{-t}(C_t/N_t)\\
&\widetilde{k}_t=K_t/\widetilde{N}_t=\gamma^{-t}(K_t/N_t)\\
&\log\ C_t/N_t=\log\ \gamma^{t}\widetilde{c}_t=\log\ \widetilde{c}_t+t\log\ \gamma.
\end{aligned}$$

Notice that the balanced growth path is the steady state $\widetilde{c}_t=\widetilde{c}$, $\widetilde{k}_t=\widetilde{k}$ of the redefined model

$$\begin{aligned}
\max\ &\textstyle\sum_{t=0}^{\infty}\beta^t\log\ \widetilde{c}_t\\
\text{s. t. }\ &\widetilde{c}_t+\gamma\eta\widetilde{k}_{t+1}-(1-\delta)\widetilde{k}_t\le\theta\widetilde{k}_t^{\alpha}\\
&\widetilde{c}_t,\ \widetilde{k}_t\ge 0\\
&\widetilde{k}_0=\overline{K}_0.
\end{aligned}$$

The balanced growth path also matches Kaldor's stylized facts (although the explanation for fact 6 is not very interesting):

1. $Y_t/N_t=(\gamma^{1-\alpha})^t\theta(K_t/N_t)^{\alpha}=\gamma^t\theta\widetilde{k}^{\alpha}$ grows at rate $\gamma-1$.

2. $K_t/N_t=\gamma^t\widetilde{k}$ grows at rate $\gamma-1$.

3. $r_t-\delta=\alpha(\gamma^{1-\alpha})^t\theta K_t^{\alpha-1}N_t^{1-\alpha}-\delta=\alpha\theta\widetilde{k}^{\alpha-1}-\delta=\gamma\eta/\beta-1$ is constant.

4. $K_t/Y_t=\widetilde{k}^{1-\alpha}/\theta$ is constant.

5. $r_tK_t/Y_t=\alpha$, $w_tN_t/Y_t=1-\alpha$ are constant.

6. rate of growth of Y_t/N_t is determined solely by γ.

Calibration to the U.S. data

Y_t = Real gross domestic product (*Economic Report of the President*, 2002, Table B-2) (Billions of 1996 dollars)
N_t = Civilian Employment, Table (B-32) (Thousands of Persons)

	Y_t	N_t	Y_t / N_t	γ	η
1960	2,376.7	65,778	36,132		
1970	3,578.0	78,678	45,476	1.0233	1.0181
1980	4,900.9	99,303	49,353	1.0082	1.0236
1990	6,707.9	118,793	56,467	1.0136	1.0181
2000	9,224.0	140,863	65,482	1.0149	1.0172

$$\log\ Y_{t'} / N_{t'} - \log\ Y_t / N_t = \log\ \gamma^{t'}\tilde{k}^{\alpha} - \log\ \gamma^{t}\tilde{k}^{\alpha} = (t'-t)\log\ \gamma .$$

$$\gamma = 1.0150, \qquad \eta = 1.0192$$

Y_t = Gross domestic product (B-26) - proprietors' income (B-28) - indirect business tax (B-26) (Billions of current dollars)
$w_t N_t$ = Compensation of employees (B-28) (Billions of current dollars)

(We distribute proprietors' income and indirect business taxes proportionally between labor income and capital income.)

	Y_t	$w_t N_t$	$1-\alpha$
1960	430.0	296.4	0.6893
1970	865.6	617.2	0.7130
1980	2,406.0	1,651.7	0.6865
1990	4,974.9	3,351.0	0.6736
2000	8,395.2	5,715.2	0.6808

$$1-\alpha = 0.6886, \qquad \alpha = 0.3114$$

How good is the assumption that we are in a balanced growth path? Let 1960 be $t = 0$:
$Y_t / (\gamma^t N_t) = (Y_t / N_t) / \gamma^t = \theta\tilde{k}^{\alpha}$ should be constant.

	$\frac{Y_t}{\gamma^t N_t}$
1960	36,132
1970	39,195
1980	36,661
1990	36,151
2000	36,132

$$\theta\tilde{k}^{\alpha} = 36,854$$

$K_{t+1} - (1-\delta)K_t =$ Gross private domestic investment (B-1) + government gross investment (B-20) (Billions of current dollars)
$\delta K_t =$ Capital consumption allowances (B-26) (Billions of current dollars)
$Y_t =$ Gross domestic product (B-1) (Billions of current dollars)

	$K_{t+1} - (1-\delta)K_t$	δK_t	$K_{t+1} - K_t$	Y_t	$\frac{K_{t+1} - K_t}{Y_t}$	$\frac{\delta K_t}{Y_t}$
1960	107.2	56.9	50.3	527.4	0.0954	0.1079
1970	197.1	152.4	88.0	1,039.7	0.0846	0.1049
1980	578.2	345.2	233.0	2,795.6	0.0833	0.1235
1990	1,077.4	711.3	366.1	5,803.2	0.0631	0.1226
2000	2,085.8	1,241.3	844.5	9,872.9	0.0855	0.1257

Calculation of parameters

$(K_{t+1} - K_t)/Y_t = (\gamma\eta - 1)\tilde{k}/(\theta\tilde{k}^{\alpha}) = 0.0824$
$\theta\tilde{k}^{\alpha} = 36,854$
$\tilde{k} = 0.0979\left(\theta\tilde{k}^{\alpha}/(\gamma\eta - 1)\right) = 0.0824(36,854/0.0345) = 86,335$
$K_t/Y_t = \tilde{k}/(\theta\tilde{k}^{\alpha}) = 86,335/36,854 = 2.3426$
$\delta K_t/Y_t = 0.1169$
$\delta = 0.0499$
$r_t K_t/Y_t = 0.3114$
$r = 0.1329$
$\theta = 36,854/(86,335^{0.3114}) = 1,070.37$
$r - \delta = 0.1329 - 0.0499 = 0.0830 = \gamma\eta/\beta - 1$
$\beta = \gamma\eta/(1 + r - \delta) = 1.0345/1.0830 = 0.9552$

Summary

$\beta = 0.9552$
$\eta = 1.0192$
$\gamma = 1.0150$
$\delta = 0.0499$
$\theta = 1,070.37$
$\alpha = 0.3114$

Kaldor's stylized facts (again)

1. $Y_t / N_t = (1.0150)^t 36{,}854$ (t = 0 in 1960)

2. $K_t / N_t = (1.0150)^t 86{,}335$

3. $r_t - \delta = 0.0830$

4. $K_t / Y_t = 2.3426$

5. $r_t K_t / Y_t = 0.3114,\ w_t N_t / Y_t = 0.6886$

6. $\gamma = 1.0150$

A puzzle: Interest rates on bonds

i_t = Corporate bond rate (Moody's Aaa) (B-73)
π_t = Percentage change in GNP deflator (B-3)

	i_t (%)	π_t (%)	$i_t - \pi_t$
1960-1969	5.01	2.36	2.65
1970-1979	8.62	6.99	1.63
1980-1989	11.34	4.80	6.54
1990-1999	7.72	2.30	5.42
2000	7.62	2.30	5.32

Arbitrage implies that

$$r_t - \delta = 0.0830 \approx i_t - \pi_t = 0.0409$$

There was a large equity premium up until the 1980s. See

R. Mehra and E. C. Prescott (1985), "The Equity Premium: A Puzzle," *Journal of Monetary Economics*, 15, 145-161.

E. R. McGrattan and E. C. Prescott (2000), "Is the Stock Market Overvalued? " *Federal Reserve Bank of Minneapolis Quarterly Review*, 24(4), 20–40.