## ANSWERS TO EXAMINATION

## 1. [Ricardian model with a continuum of goods]

a) Definition of equilibrium: An equilibrium is a price function $\hat{p}(z)$, wage rates $\hat{w}_{1}, \hat{w}_{2}$, consumption functions $\hat{c}_{1}(z), \hat{c}_{2}(z)$, and production plans $\hat{y}_{1}(z), \hat{\ell}_{1}(z), \hat{y}_{2}(z), \hat{\ell}_{2}(z)$ such that

- Given $\hat{p}(z), \hat{w}_{j}$, the consumer in country $j, j=1,2$, chooses $\hat{c}_{1}(z)$ to solve

$$
\begin{array}{ll}
\max & \int_{0}^{1} \log c_{j}(z) d z \\
\text { s. t. } & \int_{0}^{1} \hat{p}(z) c_{j}(z) d z \leq \hat{w}_{j} \bar{\ell}_{j} \\
& c_{j}(z) \geq 0 .
\end{array}
$$

- $\hat{p}(z)-a_{j}(z) \hat{w}_{j} \leq 0,=0$ if $\hat{y}_{j}(z)>0, j=1,2, z \in[0,1]$
- $\hat{c}_{1}(z)+\hat{c}_{2}(z)=\hat{y}_{1}(z)+\hat{y}_{2}(z), z \in[0,1]$.
- $\int_{0}^{1} \hat{\ell}_{j}(z) d z=\bar{\ell}, j=1,2$.

Because of symmetry, we know that there is an equilibrium in which $\hat{w}_{1}=\hat{w}_{2}=1$. This implies that the pattern of production, trade, and specialization is


Country 1 produces and exports the goods in the interval $[0, \bar{z}]$ while country 2 produces and exports the goods in the interval $(\bar{z}, 1]$.

The prices of the goods are

$$
\hat{p}(z)=\left\{\begin{array}{ll}
e^{\alpha z} & z \in[0, \bar{z}] \\
e^{\alpha(1-z)} & z \in(\bar{z}, 1]
\end{array} .\right.
$$

The consumption levels are

$$
\hat{c}_{1}(z)=\hat{c}_{2}(z)=\frac{\bar{\ell}}{\hat{p}(z)} .
$$

The production plans are

$$
\begin{aligned}
& \hat{y}_{1}(z)=\frac{2 \bar{\ell}}{\hat{p}(z)}, \hat{\ell}_{1}(z)=2 \bar{\ell}, \hat{y}_{2}(z)=\hat{\ell}_{2}(z)=0, z \in[0, \bar{z}] \\
& \hat{y}_{1}(z)=\hat{\ell}_{1}(z)=0, \hat{y}_{2}(z)=\frac{2 \bar{\ell}}{\hat{p}(z)}, \hat{\ell}_{2}(z)=2 \bar{\ell}, z \in(\bar{z}, 1]
\end{aligned}
$$

b) Definition of equilibrium: An equilibrium is
producer price functions $\hat{p}_{1}(z), \hat{p}_{2}(z)$
consumer price functions $\hat{q}_{1}(z), \hat{q}_{2}(z)$
wage rates $\hat{w}_{1}, \hat{w}_{2}$,
consumption functions $\hat{c}_{1}(z), \hat{c}_{2}(z)$,
production plans $\hat{y}_{1}(z), \hat{\ell}_{1}(z), \hat{y}_{2}(z), \hat{\ell}_{2}(z)$,
and tariff revenues $\hat{T}_{1}, \hat{T}_{2}$
such that

- $\hat{q}_{1}(z)=\min \left[a_{1}(z) \hat{w}_{1},(1+\tau) a_{2}(z) \hat{w}_{2}\right]$

$$
\hat{q}_{2}(z)=\min \left[(1+\tau) a_{1}(z) \hat{w}_{1}, a_{2}(z) \hat{w}_{2}\right] .
$$

- Given $\hat{p}(z), \hat{w}_{j}$, the consumer in country $j, j=1,2$, chooses $\hat{c}_{1}(z)$ to solve

$$
\max \int_{0}^{1} \log c_{j}(z) d z
$$

s. t. $\int_{0}^{1} \hat{q}_{1}(z) c_{j}(z) d z \leq \hat{w}_{j} \bar{\ell}_{j}+\widehat{T}$

$$
c_{j}(z) \geq 0 .
$$

- $\hat{p}_{1}(z)-a_{j}(z) \hat{w}_{j} \leq 0,=0$ if $\hat{y}_{j}(z)>0, j=1,2, z \in[0,1]$
- $\hat{c}_{1}(z)+\hat{c}_{2}(z)=\hat{y}_{1}(z)+\hat{y}_{2}(z), z \in[0,1]$.
- $\hat{T}_{1}=\int_{a_{1}(z) \hat{w}_{1}>(1+\tau) a_{2}(z) \hat{w}_{2}} \tau \hat{p}_{2}(z) \hat{c}_{1}(z) d z$

$$
\hat{T}_{2}=\int_{a_{2}(z) \hat{w}_{2}>(1+\tau) a_{1}(z) \hat{w}_{1}} \tau \hat{p}_{1}(z) \hat{c}_{2}(z) d z \text {. }
$$

- $\int_{0}^{1} \hat{\ell}_{j}(z) d z=\bar{\ell}, j=1,2$.

Once again, because of symmetry, we know that there is an equilibrium in which $\hat{w}_{1}=\hat{w}_{2}=1$.

There are two possibilities: either there is no trade in equilibrium or there is trade in trade in equilibrium.

First, $\tau$ is so large and/or $\alpha$ is so small that there is no trade in equilibrium because $(1+\tau)>e^{\alpha}$, which implies that $a_{1}(z) \hat{w}_{1}<(1+\tau) a_{2}(z) \hat{w}_{2}$ and $(1+\tau) a_{1}(z) \hat{w}_{1}>a_{2}(z) \hat{w}_{2}$ for all $z \in[0,1]$.

Second, if $(1+\tau)<e^{\alpha}$, then the pattern of production, trade, and specialization looks something like this:


Country 1 produces the goods in the interval $[0,1-\bar{z}]$ and exports the goods in the interval $[0, \bar{z}]$. Country 2 produces the goods in the interval $[0, \bar{z}]$ and exports the goods in the interval $[0,1-\bar{z}]$. The goods in the interval $[\bar{z}, 1-\bar{z}]$ are not traded.

$$
\begin{gathered}
(1+\tau) e^{\alpha \bar{z}}=e^{\alpha(1-\bar{z})} \\
\log (1+\tau)+\alpha \bar{z}=\alpha(1-\bar{z}) \\
\bar{z}=\frac{1}{2}-\frac{\log (1+\tau)}{2 \alpha}
\end{gathered}
$$

To find $\hat{T}_{1}=\hat{T}_{2}=T$, we use the solution to the consumer's problem

$$
c_{j}(z)=\frac{w_{j} \bar{\ell}+T_{j}}{p_{j}(z)}
$$

to obtain

$$
\begin{gathered}
T=\int_{0}^{\bar{z}} \tau p(z) \frac{\bar{\ell}+T}{(1+\tau) p(z)} d z \\
\hat{T}_{1}=\hat{T}_{2}=\hat{T}=\frac{\tau \bar{z} \bar{\ell}}{1+\tau(1-\bar{z})} .
\end{gathered}
$$

The producer prices goods are

$$
\begin{aligned}
& \hat{p}_{1}(z)=e^{\alpha z}, \quad z \in[0,1-\bar{z}] \\
& \quad \hat{p}_{2}(z)=e^{\alpha(1-z)}, \quad z \in(\bar{z}, 1] .
\end{aligned}
$$

The consumer prices are

$$
\begin{gathered}
\hat{q}_{1}(z)= \begin{cases}e^{\alpha z} & z \in[0,1-\bar{z}] \\
(1+\tau) e^{\alpha(1-z)} & z \in(1-\bar{z}, 1]\end{cases} \\
\hat{q}_{2}(z)= \begin{cases}(1+\tau) e^{\alpha z} & z \in[0, \bar{z}] \\
e^{\alpha(1-z)} & z \in(\bar{z}, 1]\end{cases}
\end{gathered}
$$

The consumption levels are

$$
\begin{aligned}
& \hat{c}_{1}(z)=\frac{\bar{\ell}+\hat{T}}{\hat{q}_{1}(z)} \\
& \hat{c}_{2}(z)=\frac{\bar{\ell}+\hat{T}}{\hat{q}_{2}(z)}
\end{aligned}
$$

The production plans are

$$
\begin{gathered}
\hat{y}_{1}(z)=\frac{(2+\tau)(\bar{\ell}+\hat{T})}{(1+\tau) \hat{p}_{1}(z)}, \hat{\ell}_{1}(z)=\frac{(2+\tau)(\bar{\ell}+\hat{T})}{(1+\tau)}, \hat{y}_{2}(z)=\hat{\ell}_{2}(z)=0, z \in[0, \bar{z}] \\
\hat{y}_{1}(z)=\frac{\bar{\ell}+\hat{T}}{\hat{p}_{1}(z)}, \hat{\ell}_{1}(z)=\bar{\ell}+\hat{T}, \hat{y}_{2}(z)=\frac{\bar{\ell}+\hat{T}}{\hat{p}_{2}(z)}, \hat{\ell}_{2}(z)=\bar{\ell}+\hat{T}, \quad z \in(\bar{z}, 1-\bar{z}] \\
\hat{y}_{1}(z)=\hat{\ell}_{1}(z)=0, \hat{y}_{2}(z)=\frac{(2+\tau)(\bar{\ell}+\hat{T})}{(1+\tau) \hat{p}_{2}(z)}, \hat{\ell}_{2}(z)=\frac{(2+\tau)(\bar{\ell}+\hat{T})}{(1+\tau)}, z \in(1-\bar{z}, 1] .
\end{gathered}
$$

c) The value of exports $\hat{X}_{1}=\hat{X}_{2}=X$ is

$$
X= \begin{cases}0 & \text { if }(1+\tau) \geq e^{\alpha} \\ \int_{0}^{\bar{z}} \hat{p}_{1}(z) \frac{\bar{\ell}+\hat{T}}{(1+\tau) \hat{p}_{1}(z)} d z & \text { if }(1+\tau) \leq e^{\alpha}\end{cases}
$$

and the value of GDP $\hat{Y}_{1}=\hat{Y}_{2}=Y$ is

$$
Y=\left\{\begin{array}{ll}
\bar{\ell} & \text { if }(1+\tau) \geq e^{\alpha} \\
\bar{\ell}+\hat{T} & \text { if }(1+\tau) \leq e^{\alpha}
\end{array} .\right.
$$

In the case where $(1+\tau) \leq e^{\alpha}$

$$
\begin{gathered}
X=\frac{\bar{\ell}+\hat{T}}{(1+\tau)} \bar{z}=\frac{\bar{z} \bar{\ell}}{1+\tau(1-\bar{z})}=\frac{(\alpha-\log (1+\tau)) \bar{\ell}}{2 \alpha+\tau \alpha+\tau \log (1+\tau)} \\
Y=\bar{\ell}+\hat{T}=\frac{(1+\tau) \bar{\ell}}{1+\tau(1-\bar{z})}=\frac{2 \alpha(1+\tau) \bar{\ell}}{2 \alpha+\tau \alpha+\tau \log (1+\tau)} \\
\frac{X}{Y}=\frac{(\alpha-\log (1+\tau)) \bar{\ell}}{2 \alpha(1+\tau) \bar{\ell}}=\frac{\alpha-\log (1+\tau)}{2 \alpha(1+\tau)} .
\end{gathered}
$$

In the case where $(1+\tau) \geq e^{\alpha}$

$$
\begin{aligned}
X & =0 \\
Y & =\bar{\ell} \\
\frac{X}{Y} & =0 .
\end{aligned}
$$

d) Definition of equilibrium: An equilibrium is
a price function $\hat{p}(z)$,
factor prices $\hat{r}_{1}, \hat{w}_{1}, \hat{r}_{2}, \hat{w}_{2}$,
consumption functions $\hat{c}_{1}(z), \hat{c}_{2}(z)$,
and production plans $\hat{y}_{1}(z), \hat{k}_{1}(z), \hat{\ell}_{1}(z), \hat{y}_{2}(z), \hat{k}_{2}(z), \hat{\ell}_{2}(z)$
such that

- Given $\hat{p}(z), \hat{w}_{j}$, the consumer in country $j, j=1,2$, chooses $\hat{c}_{1}(z)$ to solve

$$
\begin{array}{ll}
\max & \int_{0}^{1} \log c_{j}(z) d z \\
\text { s. t. } & \int_{0}^{1} \hat{p}(z) c_{j}(z) d z \leq \hat{r}_{j} \bar{k}_{j}+\hat{w}_{j} \bar{\ell}_{j} \\
& c_{j}(z) \geq 0 .
\end{array}
$$

- $\hat{p}(z) \alpha(z) \hat{k}_{j}^{\alpha(z)-1} \hat{\ell}_{j}^{1-\alpha(z)}-\hat{r}_{j} \leq 0,=0$ if $\hat{y}_{j}(z)>0, j=1,2, z \in[0,1]$

$$
\hat{p}(z)(1-\alpha(z)) \hat{k}_{j}^{\alpha(z)} \hat{\ell}_{j}^{-\alpha(z)}-\hat{w}_{j} \leq 0,=0 \text { if } \hat{y}_{j}(z)>0, j=1,2, z \in[0,1]
$$

- $\hat{c}_{1}(z)+\hat{c}_{2}(z)=\hat{y}_{1}(z)+\hat{y}_{2}(z), z \in[0,1]$.
- $\int_{0}^{1} \hat{\ell}_{j}(z) d z=\bar{\ell}, j=1,2$.

Because of symmetry, we know that there is an equilibrium on which $\hat{r}_{1}=\hat{w}_{2}$ and $\hat{w}_{1}=\hat{r}_{2}$. There are two possibilities: either $\hat{r}_{1}=\hat{w}_{2}>\hat{w}_{1}=\hat{r}_{2}=1$ or $\hat{r}_{1}=\hat{w}_{2}=\hat{w}_{1}=\hat{r}_{2}=1$. If $\hat{r}_{1}=\hat{w}_{2}>\hat{w}_{1}=\hat{r}_{2}=1$, then country 1 specializes in all of the goods less capital intensive then a specific level $\bar{z}$, that is all $z \leq \bar{z}$, and country 2 specializes in all goods more capital intensive than the same $\bar{z}$, that is all $z>\bar{z}$. Because of symmetry, $\bar{z}=1 / 2$. The graph is the same as for part (a). On the other hand, if $\hat{r}_{1}=\hat{w}_{2}=\hat{w}_{1}=\hat{r}_{2}=1$, then the structure of production and trade is indeterminate.

We use the first-order conditions for firm $z$ in country $j$ to obtain

$$
\begin{gathered}
\ell_{j}(z)=\left(\frac{r_{j}(1-z)}{w_{j} z}\right)^{z} y_{j}(z) \\
k_{j}(z)=\left(\frac{w_{j} z}{r_{j}(1-z)}\right)^{1-z} y_{j}(z) \\
p(z)=\frac{r_{j}^{z} w_{j}^{1-z}}{z^{z}(1-z)^{1-z}} .
\end{gathered}
$$

To see which of the two cases that we are in, we suppose that $\hat{r}_{1}=\hat{w}_{2}=\hat{w}_{1}=\hat{r}_{2}=1$. Let us calculate the demand for labor in country 1 under the assumption that that country 1 produces all of the goods $z \leq 1 / 2$. If this amount of labor is less than $\bar{\ell}$, then we know that we are in the other case, where $\hat{r}_{1}=\hat{w}_{2}>\hat{w}_{1}=\hat{r}_{2}=1$.

$$
\begin{gathered}
\ell_{1}(z)=\left(\frac{(1-z)}{z}\right)^{z} y_{1}(z) \\
p(z)=\frac{1}{z^{z}(1-z)^{1-z}} \\
c_{1}(z)=\frac{\bar{k}_{1}+\bar{\ell}_{1}}{p(z)} \\
c_{2}(z)=\frac{\bar{k}_{2}+\bar{\ell}_{2}}{p(z)}
\end{gathered}
$$

which imply that

$$
\begin{gathered}
y_{1}(z)=c_{1}(z)+c_{2}(z)=\frac{\bar{k}_{1}+\bar{\ell}_{1}+\bar{k}_{2}+\bar{\ell}_{2}}{p(z)}=z^{z}(1-z)^{1-z}\left(\bar{k}_{1}+\bar{\ell}_{1}+\bar{k}_{2}+\bar{\ell}_{2}\right) \\
\ell_{1}(z)=\left(\frac{(1-z)}{z}\right)^{z} z^{z}(1-z)^{1-z}\left(\bar{k}_{1}+\bar{\ell}_{1}+\bar{k}_{2}+\bar{\ell}_{2}\right)=(1-z)\left(\bar{k}_{1}+\bar{\ell}_{1}+\bar{k}_{2}+\bar{\ell}_{2}\right) .
\end{gathered}
$$

The total demand for labor in country 1 is

$$
\int_{0}^{1 / 2}(1-z)\left(\bar{k}_{1}+\bar{\ell}_{1}+\bar{k}_{2}+\bar{\ell}_{2}\right) d z=\left.\left(\bar{k}_{1}+\bar{\ell}_{1}+\bar{k}_{2}+\bar{\ell}_{2}\right)\left(z-\frac{z^{2}}{2}\right)\right|_{0} ^{1 / 2}=\frac{3}{8}\left(\bar{k}_{1}+\bar{\ell}_{1}+\bar{k}_{2}+\bar{\ell}_{2}\right) .
$$

If

$$
\bar{\ell}_{1}<\frac{3}{8}\left(\bar{k}_{1}+\bar{\ell}_{1}+\bar{k}_{2}+\bar{\ell}_{2}\right)
$$

then we know that we are in the case where $\hat{r}_{1}=\hat{w}_{2}>\hat{w}_{1}=\hat{r}_{2}=1$. Since $\bar{k}_{2}=\bar{\ell}_{1}$ and $\bar{\ell}_{2}=\bar{k}_{1}$, this condition is

$$
\begin{gathered}
\bar{\ell}_{1}<\frac{3}{4}\left(\bar{k}_{1}+\bar{\ell}_{1}\right) \\
\bar{\ell}_{1}<\frac{1}{3} \bar{k}_{1} .
\end{gathered}
$$

Consequently, the crucial condition for factor prices not being equal and the pattern of production being uniquely determined is

$$
\bar{\ell}_{1}<\frac{1}{3} \bar{k}_{1},
$$

or, equivalently,

$$
\bar{\ell}_{2}>3 \bar{k}_{2}
$$

If $\bar{\ell}_{1}<\frac{1}{3} \bar{k}_{1}$, let us solve for $\hat{w}_{2}=\hat{r}_{1}=r$

$$
\begin{gathered}
p(z)=\frac{r^{z}}{z^{z}(1-z)^{1-z}} \\
y_{1}(z)=c_{1}(z)+c_{2}(z)=\frac{2\left(r \bar{k}_{1}+\bar{\ell}_{1}\right)}{p(z)}=\frac{2 z^{z}(1-z)^{1-z}\left(r \bar{k}_{1}+\bar{\ell}_{1}\right)}{r^{z}} \\
\ell_{1}(z)=\left(\frac{r(1-z)}{z}\right)^{z} \frac{2 z^{z}(1-z)^{1-z}\left(r \bar{k}_{1}+\bar{\ell}_{1}\right)}{r^{z}} \\
\ell_{1}(z)=2(1-z)\left(r \bar{k}_{1}+\bar{\ell}_{1}\right) \\
\int_{0}^{1 / 2} 2(1-z)\left(r \bar{k}_{1}+\bar{\ell}_{1}\right) d z=\left.2\left(r \bar{k}_{1}+\bar{\ell}_{1}\right)\left(z-\frac{z^{2}}{2}\right)\right|_{0} ^{1 / 2}=\frac{3}{4}\left(r \bar{k}_{1}+\bar{\ell}_{1}\right) .
\end{gathered}
$$

Solving for $\hat{w}_{2}=\hat{r}_{1}=r$, we obtain

$$
\begin{aligned}
& \frac{3}{4}\left(r \bar{k}_{1}+\bar{\ell}_{1}\right)=\bar{\ell}_{1} \\
& \hat{w}_{2}=\hat{r}_{1}=r=\frac{\bar{\ell}_{1}}{3 \bar{k}_{1}}
\end{aligned}
$$

## 2. [Monopolistic competition and trade]

a) Definition of equilibrium: An equilibrium is a number of manufacturing firm $\hat{n}$, a price $\hat{p}_{0}$ for the agricultural good,
a price $\hat{p}_{j}$ for each manufacturing firm that operates at a positive level, a wage rate $\hat{w}$, a consumption plan $\hat{c}_{0}, \hat{c}_{1}, \hat{c}_{2}, \ldots, \hat{c}_{\hat{n}}$,
and production plans, $\hat{y}_{0}, \hat{\ell}_{0}$ for the primary good and $\hat{y}_{j}, \hat{\ell}_{j}$ for each manufacturing firm that operates at a positive level,
such that

- Given $\hat{p}_{0}, \hat{p}_{1}, \hat{p}_{2}, \ldots, \hat{p}_{\hat{n}}, \hat{r}, \hat{w}$, the consumer chooses $\hat{c}_{0}, \hat{c}_{1}, \hat{c}_{2}, \ldots, \hat{c}_{\hat{n}}$ to solve

$$
\begin{array}{ll}
\max & \log c_{0}+(1 / \rho) \log \left(\sum_{j=1}^{n} c_{j}^{\rho}\right) \\
\text { s. t. } & \hat{p}_{0} c_{0}+\sum_{j=1}^{\hat{n}} \hat{p}_{j} c_{j} \leq \hat{w} \bar{\ell} \\
& c_{j} \geq 0 .
\end{array}
$$

- $\hat{p}_{0}-\hat{w} \leq 0,=0$ if $\hat{y}_{0}>0$.
- Given the indirect demand function $p_{i}\left(c_{1}, \ldots, c_{i}, \ldots ., c_{n}\right)$ that comes from solving the representative consumer's utility maximization problem, firm $i, i=1,2, \ldots, \hat{n}$, chooses $\hat{y}_{i}$ to solve

$$
\max p_{i}\left(\hat{y}_{1}, \ldots, y_{i}, \ldots ., \hat{y}_{\hat{n}}\right) y_{i}-\hat{w} b y_{i}-\hat{w} f .
$$

- $\hat{p}_{j} \hat{y}_{j}-\hat{w} \hat{\ell}_{j}=0, j=1,2, \ldots, \hat{n}$.
- $\hat{c}_{j}=\hat{y}_{j}, j=0,1,2, \ldots, \hat{n}$.
- $\hat{\ell}_{0}+\sum_{j=1}^{\hat{n}} \hat{\ell}_{j}=\bar{\ell}$.
b) Solving the representative consumer's problem, we obtain the indirect demand function for manufactured good $i$

$$
p_{i}=\frac{w \bar{\ell}}{2} \frac{c_{i}^{\rho-1}}{\sum_{j=1}^{n} c_{j}^{\rho}} .
$$

The (ordinary) demand function for agricultural goods is

$$
c_{0}=\frac{w \bar{\ell}}{2 p_{0}} .
$$

As a Cournot competitor, manufacturing firm $i$ choose $y_{i}$ to maximize

$$
\frac{w \bar{\ell}}{2} \frac{y_{i}^{\rho-1}}{\sum_{j=1}^{n} y_{j}^{\rho}} y_{i}-b w y_{i}-f
$$

assuming that the output levels $y_{j}$ of all other firms are not influenced by this decision. The solution to this problem is

$$
\frac{w \bar{\ell}}{2}\left(\frac{\left(\sum_{j=1}^{n} x_{j}^{\rho}\right) \rho y_{i}^{\rho-1}-y_{i}^{\rho} \rho y_{i}^{\rho-1}}{\left(\sum_{j=1}^{n} y_{j}^{\rho}\right)^{2}}\right)=b .
$$

Since all firms that produce are in the same position, we impose the symmetry condition $y_{i}=\bar{y}$. In addition, we impose the price normalization $w=1$.

$$
\begin{gathered}
\frac{\bar{\ell}}{2}\left(\frac{\left(n \bar{y}^{\rho}\right) \rho \bar{y}^{\rho-1}-\bar{y}^{\rho} \rho \bar{y}^{\rho-1}}{\left(n \bar{y}^{\rho}\right)^{2}}\right)=b \\
\frac{\bar{\ell}}{2}\left(\frac{\rho(n-1)}{n^{2} \bar{y}}\right)=b \\
\bar{y}=\frac{\rho(n-1) \bar{\ell}}{2 n^{2} b} \\
p_{i}=\bar{p}=\frac{\bar{\ell} \bar{y}^{\rho-1}}{2 n \bar{y}^{\rho}}=\frac{\bar{\ell}}{2 n \bar{y}}=\frac{b n}{\rho(n-1)} .
\end{gathered}
$$

This implies that the profits of any firm that produces are

$$
\overline{p y}-b \bar{y}-f=\frac{\bar{\ell}}{2 n}-\frac{\rho(n-1) \bar{\ell}}{2 n^{2}}-f
$$

Since there is free entry and exit until profits equal zero,

$$
\begin{gathered}
2 f n^{2}-(1-\rho) \bar{\ell}-\rho \bar{\ell}=0 \\
n=\frac{(1-\rho) \bar{\ell}+\sqrt{(1-\rho)^{2} \bar{\ell}^{2}+4(\rho \bar{\ell})(2 f)}}{4 f}
\end{gathered}
$$

Putting in the parameters $b=1, f=3, \rho=1 / 2$, and $\bar{\ell}=50$ into these formulas, we obtain

| $\hat{n}$ | $\hat{w}$ | $\hat{p}_{0}$ | $\hat{p}_{j}$ | $\hat{c}_{0}$ | $\hat{c}_{j}$ | $\hat{y}_{0}$ | $\hat{\ell}_{0}$ | $\hat{y}_{j}$ | $\hat{\ell}_{j}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 5 | 1 | 1 | 2.5 | 25 | 2 | 25 | 25 | 2 | 5 |

c) If $\bar{\ell}=200$, we can calculate

| $\hat{n}$ | $\hat{w}$ | $\hat{p}_{0}$ | $\hat{p}_{j}$ | $\hat{c}_{0}$ | $\hat{c}_{j}$ | $\hat{y}_{0}$ | $\hat{\ell}_{0}$ | $\hat{y}_{j}$ | $\hat{\ell}_{j}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 17.6129 | 1 | 1 | 2.1204 | 100 | 2.6776 | 100 | 100 | 2.6776 | 5.6776 |

d) If there are two countries, one with $\bar{\ell}^{1}=50$ and the other with $\bar{\ell}^{2}=150$, we use the results from part (c) to calculate

| $\hat{n}^{1}$ | $\hat{n}^{2}$ | $\hat{w}$ | $\hat{p}_{0}$ | $\hat{p}_{j}$ | $\hat{c}_{0}^{1}$ | $\hat{c}_{0}^{2}$ | $\hat{c}_{j}^{1}$ | $\hat{c}_{j}^{2}$ | $\hat{y}_{0}$ | $\hat{\ell}_{0}$ | $\hat{y}_{j}$ | $\hat{\ell}_{j}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 13.2097 | 4.4032 | 1 | 1 | 2.1204 | 25 | 75 | 0.6694 | 2.0082 | 100 | 100 | 2.6776 | 5.6776 |

Even without a calculator, we can reason the number of varieties available will go up, but the number of domestic firms will go down. The output of each manufacturing firm will go up and the price will go down. This process is called rationalization.

For country 1, we can calculate the gains from trade. Utility in autarky is

$$
\log (25)+2 \log \left(5 \times 2^{1 / 2}\right)=7.1309
$$

Utility in trade is

$$
\log (25)+2 \log \left(17.6129 \times 0.6694^{1 / 2}\right)=8.5548
$$

To calculate increases in real income, we calculate real income indices using the homogenous-of-degree-one representation of utility:

$$
I=e^{u / 2}
$$

Real income increases by the factor

$$
\frac{e^{8.5548 / 2}}{e^{7.1309 / 2}}=\frac{72.0529}{35.3554}=2.0380
$$

that is, real income increases by 103.8 percent.

e) If firms are Bertrand competitors, the only part of the definition of equilibrium that changes is that

- Given the demand function $c_{i}\left(p_{1}, \ldots, p_{i}, \ldots ., p_{n}\right)$ that comes from solving the representative consumer's utility maximization problem, firm $i, i=1,2, \ldots, \hat{n}$, chooses $\hat{p}_{i}$ to solve

$$
\max p_{i} c_{i}\left(\hat{p}_{1}, \ldots, p_{i}, \ldots, \hat{p}_{\hat{n}}\right)-\hat{w} b c_{i}\left(\hat{p}_{1}, \ldots, p_{i}, \ldots, \hat{p}_{\hat{n}}\right)-\hat{w} f .
$$

f) The benefits of trade that come about because of the rationalization effect will be smaller under Bertrand competition because there will be less inefficiency due to monopoly power before trade liberalization.

The demand function for manufacturing good $j$ is

$$
c_{i}\left(p_{1}, \ldots, p_{i}, \ldots, p_{n}\right)=\frac{w \bar{\ell}}{2 p_{i}^{\frac{1}{1-\rho}} \sum_{j=1}^{n} p_{i}^{\frac{-\rho}{1-\rho}}} .
$$

Solving the profit maximization condition of firm $i$, we obtain

$$
\begin{gathered}
\frac{w \bar{\ell}}{2 p_{i}^{\frac{1}{1-\rho}} \sum_{j=1}^{n} p_{j}^{\frac{-\rho}{1-\rho}}}-\left(p_{i}-b\right) \frac{w \bar{\ell}\left(\frac{1}{1-\rho} p_{i}^{\frac{\rho}{1-\rho}} \sum_{j=1}^{n} p_{j}^{\frac{-\rho}{1-\rho}}-\frac{\rho}{1-\rho} p_{i}^{\frac{1}{1-\rho}} p_{i}^{\frac{-1}{1-\rho}}\right)}{2\left(p_{i}^{\frac{1}{1-\rho}} \sum_{j=1}^{n} p_{j}^{\frac{-\rho}{1-\rho}}\right)^{2}}=0 \\
\left(p_{i}^{\frac{1}{1-\rho}} \sum_{j=1}^{n} p_{j}^{\frac{-\rho}{1-\rho}}\right)-\left(p_{i}-b\right)\left(\frac{1}{1-\rho} p_{i}^{\frac{\rho}{1-\rho}} \sum_{j=1}^{n} p_{j}^{\frac{-\rho}{1-\rho}}-\frac{\rho}{1-\rho} p_{i}^{\frac{1}{1-\rho}} p_{i}^{\frac{-1}{1-\rho}}\right)=0 .
\end{gathered}
$$

Imposing symmetry, $p_{i}=\bar{p}$, we obtain

$$
\begin{gathered}
n \bar{p}-\frac{(\bar{p}-b)(n-\rho)}{1-\rho}=0 \\
\bar{p}=\frac{n-\rho}{n \rho-\rho} b \\
\bar{y}=\frac{w \bar{\ell}(n \rho-\rho)}{2 n(n-\rho) b} .
\end{gathered}
$$

Imposing the price normalization condition $w=1$ and the zero profit condition, we obtain

$$
\begin{gathered}
\overline{p y}-b \bar{y}-f=\frac{\bar{\ell}}{2 n}-\frac{\bar{\ell}(n \rho-\rho)}{2 n(n-\rho)}-f \\
n=\frac{(1-\rho) \bar{\ell}+2 \rho f}{2 f}
\end{gathered}
$$

The autarky equilibrium is

| $\hat{n}$ | $\hat{w}$ | $\hat{p}_{0}$ | $\hat{p}_{j}$ | $\hat{c}_{0}$ | $\hat{c}_{j}$ | $\hat{y}_{0}$ | $\hat{\ell}_{0}$ | $\hat{y}_{j}$ | $\hat{\ell}_{j}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 4.6667 | 1 | 1 | 2.2727 | 25 | 2.3572 | 25 | 25 | 2.3572 | 5.3572 |

The trade equilibrium is

| $\hat{n}^{1}$ | $\hat{n}^{2}$ | $\hat{w}$ | $\hat{p}_{0}$ | $\hat{p}_{j}$ | $\hat{c}_{0}^{1}$ | $\hat{c}_{0}^{2}$ | $\hat{c}_{j}^{1}$ | $\hat{c}_{j}^{2}$ | $\hat{y}_{0}$ | $\hat{\ell}_{0}$ | $\hat{y}_{j}$ | $\hat{\ell}_{j}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 12.8750 | 4.2917 | 1 | 1 | 2.0619 | 25 | 75 | 0.7063 | 2.1189 | 100 | 100 | 2.8252 | 5.8252 |

Utility goes from

$$
\log (25)+2 \log \left(4.6667 \times 2.3572^{1 / 2}\right)=7.1572
$$

to

$$
\log (25)+2 \log \left(17.1667 \times 0.7063^{1 / 2}\right)=8.5570
$$

Real income increases by a factor of

$$
\frac{e^{8.5571 / 2}}{e^{7.1572 / 2}}=\frac{72.1358}{35.8241}=2.0136
$$

that is, real income increases by 101.4 percent.

## 3. [A dynamic Heckscher-Ohlin model]

a) Definition of equilibrium: An equilibrium is prices for the produced goods $p_{1}(t), p_{2}(t)$, a rental rate $r(t)$, a wage rate $w(t)$, a consumption plan $c_{1}(t), c_{2}(t)$, an investment plan $x_{1}(t), x_{2}(t)$, and a capital stock $k(t)$ such that

1. Given $p_{1}(t), p_{2}(t), r(t), w(t)$, the consumer chooses $c_{1}(t), c_{2}(t), k(t)$ to solve

$$
\begin{array}{ll}
\max & \int_{0}^{\infty} e^{-\rho t} \log \left(c_{1}^{b}+c_{2}^{b}\right)^{1 / b} d t \\
\text { s.t. } & p_{1} c_{1}+p_{2} c_{2}+\dot{k}-\delta k=w+r k \\
& k(0)=\bar{k}(0) \\
& c_{1}, c_{2}, k \geq 0 .
\end{array}
$$

Notice that we are using the investment good as numeraire.
2. $p_{1}=r$.
3. $p_{2}=w$.
4. Given $p_{1}(t), p_{2}(t), k(t)$, and $\dot{k}(t)$, producers of the investment good choose $x_{1}(t), x_{2}(t)$ to solve

$$
\begin{aligned}
& \min p_{1} x_{1}+p_{2} x_{2} \\
& \text { s.t. }\left(x_{1}^{b}+x_{2}^{b}\right)^{1 / b}=\dot{k}+\delta k \\
& \quad x_{1}, x_{2} \geq 0
\end{aligned}
$$

5. $\dot{k}+\delta k-p_{1} x_{1}-p_{2} x_{2}=0$.
6. $c_{1}+x_{1}=k$.
7. $c_{2}+x_{2}=1$.
b) Necessary and sufficient conditions for $\left(p_{1}(t), p_{2}(t), r(t), w(t), c_{1}(t), c_{2}(t), x_{1}(t)\right.$, $\left.x_{2}(t), k(t)\right)$ to be an equilibrium is that there exist a Lagrange multiplier $\lambda(t)$ such that these variables satisfy the first order conditions to the consumer's problem

$$
\begin{gathered}
e^{-\rho t} c_{1}^{b-1}\left(c_{1}^{b}+c_{1}^{b}\right)^{-1}-\lambda p_{1}=0 \\
e^{-\rho t} c_{2}^{b-1}\left(c_{1}^{b}+c_{1}^{b}\right)^{-1}-\lambda p_{2}=0 \\
\lambda(r-\delta)=-\dot{\lambda},
\end{gathered}
$$

the transversality condition

$$
\lim _{t \rightarrow \infty} \lambda(t) k(t)=0
$$

the first order conditions to the investment goods producers' problem

$$
\begin{aligned}
& -p_{1}+x_{1}^{b-1}\left(x_{1}^{b}+x_{1}^{b}\right)^{1 / b-1}=0 \\
& -p_{2}+x_{2}^{b-1}\left(x_{1}^{b}+x_{1}^{b}\right)^{1 / b-1}=0,
\end{aligned}
$$

the profit maximization conditions

$$
\begin{gathered}
p_{1}=r \\
p_{2}=w,
\end{gathered}
$$

the budget constraint, and the feasibility constraints.
The necessary and sufficient conditions for $(c(t), k(t))$ to solve the problem

$$
\begin{array}{ll}
\max & \int_{0}^{\infty} e^{-\rho t} \log c d t \\
\text { s.t. } & c+\dot{k}+\delta k=\left(k^{b}+1\right)^{1 / b} \\
& k(0)=\bar{k}(0) \\
& c, k \geq 0 .
\end{array}
$$

is that there exist a Lagrange multiplier $\pi(t)$ such that

$$
\begin{gathered}
e^{-\rho t} c^{-1}-\pi=0 \\
\pi\left(k^{b-1}\left(k^{b}+1\right)^{1 / b-1}-\delta\right)=-\dot{\pi} \\
\lim _{t \rightarrow \infty} \pi(t) k(t)=0
\end{gathered}
$$

and the feasibility condition.
Suppose that $(c(t), k(t))$ solves the one-sector optimal growth problem. Let

$$
\begin{gathered}
c_{1}=\left[1 /\left(k^{b}+1\right)^{1 / b}\right] c \\
c_{2}=\left[k /\left(k^{b}+1\right)^{1 / b}\right] c \\
x_{1}=\left[1 /\left(k^{b}+1\right)^{1 / b}\right](\dot{k}+\delta k) \\
x_{2}=\left[k /\left(k^{b}+1\right)^{1 / b}\right](\dot{k}+\delta k)
\end{gathered}
$$

and

$$
\begin{aligned}
& r=p_{1}=k^{b}\left(k^{b}+1\right)^{1 / b-1}=0 \\
& w=p_{2}=\left(k^{b}+1\right)^{1 / b-1}=0 .
\end{aligned}
$$

Then $\left(p_{1}(t), p_{2}(t), r(t), w(t), c_{1}(t), c_{2}(t), x_{1}(t), x_{2}(t), k(t)\right)$ is an equilibrium of the two-sector economy.

The crucial step is to observe that the first order conditions from the consumer problem and the definitions of the functions $c_{1}(t), c_{2}(t)$ imply that

$$
\lambda=e^{-\rho t}\left(c_{1}^{b}+c_{1}^{b}\right)^{-1 / b}=e^{-\rho t} c^{-1}=\pi .
$$

The differential equations are

$$
\begin{gathered}
e^{-\rho t} c^{-1}-\pi=0 \\
\pi\left(k^{b-1}\left(k^{b}+1\right)^{1 / b-1}-\delta\right)=-\dot{\pi} \\
c+\dot{k}+\delta k=\left(k^{b}+1\right)^{1 / b}
\end{gathered}
$$

Substituting out $\pi$, we obtain a system of two differential equations in $\dot{c}$ and $\dot{k}$ :

$$
\begin{gathered}
\frac{\dot{c}}{c}=k^{b-1}\left(k^{b}+1\right)^{1 / b-1}-\delta-\rho \\
\frac{\dot{k}}{k}=\frac{\left(k^{b}+1\right)^{1 / b}}{k}-\delta-\frac{c}{k}
\end{gathered}
$$

Jaume Venura chooses to work instead with $\dot{z}$ and $\dot{k}$ where $z=c / k$ :

$$
\frac{\dot{z}}{z}=\frac{\dot{c}}{c}-\frac{\dot{k}}{k}=k^{b-1}\left(k^{b}+1\right)^{1 / b-1}-\delta-\rho-\frac{\left(k^{b}+1\right)^{1 / b}}{k}+\delta+z=-\frac{\left(k^{b}+1\right)^{1 / b-1}}{k}+z-\rho .
$$

The differential equations are

$$
\begin{gathered}
\frac{\dot{k}}{k}=\frac{\left(k^{b}+1\right)^{1 / b}}{k}-\delta-\frac{c}{k} \\
\frac{\dot{z}}{z}=-\frac{\left(k^{b}+1\right)^{1 / b-1}}{k}+z-\rho,
\end{gathered}
$$

and the transversality condition is

$$
\lim _{t \rightarrow \infty} e^{-\rho t} \frac{1}{z(t) k(t)}=0 .
$$

c)

d) To define a trade equilibrium we put superscripts on the variables $r^{j}(t), w^{j}(t)$, $c_{1}^{j}(t), c_{2}^{j}(t), x_{1}^{j}(t), x_{2}^{j}(t), k^{j}(t)$ to denote country, but not on $p_{1}(t), p_{2}(t)$. (Since factor price equalization holds, we can quickly show that $r^{j}(t), w^{j}(t)$ do not vary across countries.) To keep things as much like the previous definition of equilibrium, we let all of the quantities denote per capita values so that, for example, $\bar{\ell}_{j} c_{1}^{j}(t)$ is the total consumption of good 1 by country $j$ at time $t$. Each country $j$ satisfies a balance of trade condition

$$
p_{1}\left(c_{1}^{j}+x_{1}^{j}-k^{j}\right)+p_{2}\left(c_{2}^{j}+x_{2}^{j}-1\right)=0 .
$$

The aggregate resource constraints are

$$
\begin{aligned}
& \sum_{j=1}^{n} \bar{\ell}_{j}\left(c_{1}^{j}+x_{1}^{j}\right)=\sum_{j=1}^{n} \bar{\ell}_{j} k^{j} \\
& \sum_{j=1}^{n} \bar{\ell}_{j}\left(c_{2}^{j}+x_{2}^{j}\right)=\sum_{j=1}^{n} \bar{\ell}_{j} .
\end{aligned}
$$

e) To calculate the world equilibrium, we use an integrated equilibrium approach. We first solve for the equilibrium of the world economy in which the endowment of labor each period is $\sum_{j=1}^{n} \bar{\ell}_{j}$ and the initial endowment of capital is $\sum_{j=1}^{n} \bar{\ell}_{j} \bar{k}^{j}(0)$, which in per capita terms are 1 and $\bar{k}(0)=\sum_{j=1}^{n} \bar{\ell}_{j} \bar{k}^{j}(0) / \sum_{j=1}^{n} \bar{\ell}_{j}$. Since factor prices are always equal, we can assign consumption to country $j$ based on its share of the world's present discounted value of income:

$$
c_{i}^{j}(t)=\frac{\int_{0}^{\infty} e^{-\int_{0}^{u}(r(v)-\delta) d v} w(u) d u+p_{1}(0) \bar{k}^{j}(0)}{\int_{0}^{\infty} e^{-\int_{0}^{u}(r(v)-\delta) d v} w(u) d u+p_{1}(0) \bar{k}(0)} c_{i}(t), i=1,2
$$

To assign capital to each country takes a more work, but we can show that

$$
\frac{k^{j}(t)}{k(t)}-1=\frac{z(t)}{z(0)}\left(\frac{\bar{k}^{j}(0)}{\bar{k}(0)}-1\right)
$$

(This is why the transformation $z=c / k$ is useful.) If $z(t)$ is an increasing function of time, then a country whose capital per worker is lower than the world average will find itself with a falling share of the world capital stock. If, however, $z(t)$ is a decreasing function of time, then a country whose capital per worker is lower than the world average will find itself with rising share of the world capital stock.
f) Notice that convergence/divergence of $k^{i}$ and $k$ is not the same as convergence/divergence of $k^{i}$ and $k$, where

$$
y^{i}=w+r k^{i} .
$$

Instead, we could study the behavior of

$$
\frac{y^{i}(t)-y(t)}{y(t)}=\frac{z(t)}{z(0)}\left(\frac{y^{i}(0)-y(0)}{y(0)}\right)
$$

where

$$
z(t)=\frac{r(t) c(t)}{y(t)},
$$

by analyzing phase diagrams in $(k, z)$ space.
In the case where $\delta=0$, this becomes

$$
z(t)=\frac{r(t) c(t)}{y(t)}=\frac{f^{\prime}(k) c(t)}{f(k)}
$$

where $f(k)=\left(1+k^{b}\right)^{1 / b}$.
We use the first-order conditions

$$
\begin{aligned}
& \frac{\dot{c}}{c}=f(k)-\rho \\
& \frac{\dot{k}}{k}=\frac{f(k)}{k}-\frac{c}{k}
\end{aligned}
$$

to obtain

$$
\begin{gathered}
\frac{\dot{z}}{z}=f^{\prime}(k)-\rho-\left(\frac{f^{\prime}(k)^{2}-f(k) f^{\prime \prime}(k)}{f^{\prime}(k)^{2}}\right)\left(f^{\prime}(k)-z\right) \\
\frac{\dot{k}}{k}=\frac{f(k)}{f^{\prime}(k) k}\left(f^{\prime}(k)-z\right) .
\end{gathered}
$$

