MACROECONOMIC THEORY **T. J. KEHOE**
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ANSWERS TO MIDTERM EXAMINATION

1. (a) With an Arrow-Debreu markets structure futures markets for goods are open in period 0. Consumers trade futures contracts among themselves.

An **Arrow-Debreu equilibrium** is sequence of prices $\hat{p}_0, \hat{p}_1, \hat{p}_2, \ldots$ and consumption levels $\hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots; \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots; \hat{c}_0^3, \hat{c}_1^3, \hat{c}_2^3, \ldots$ such that

- Given $\hat{p}_0, \hat{p}_1, \hat{p}_2, \ldots$, consumer i, $i = 1,2,3$, chooses $\hat{c}_0^i, \hat{c}_1^i, \hat{c}_2^i, \ldots$ to solve

$$\max \ \sum_{t=0}^{\infty} \beta^t \log c_t^i$$
$$\text{s.t. } \sum_{t=0}^{\infty} \hat{p}_t c_t^i \leq \sum_{t=0}^{\infty} \hat{p}_t w_t^i$$
$$c_t^i \geq 0 .$$

- $\hat{c}_t^1 + \hat{c}_t^2 + \hat{c}_t^3 = w_t^1 + w_t^2 + w_t^3$, $t = 0,1,\ldots$.

(b) With sequential market markets structure, there are markets for goods and bonds open every period. Consumers trade goods and bonds among themselves.

A **sequential markets equilibrium** is sequences of interest rates $\hat{r}_1, \hat{r}_2, \hat{r}_3, \ldots$, consumption levels $\hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots; \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots; \hat{c}_0^3, \hat{c}_1^3, \hat{c}_2^3, \ldots$, and asset holdings $\hat{b}_1^1, \hat{b}_2^1, \hat{b}_3^1, \ldots; \hat{b}_1^2, \hat{b}_2^2, \hat{b}_3^2, \ldots; \hat{b}_1^3, \hat{b}_2^3, \hat{b}_3^3, \ldots$ such that

- Given $\hat{r}_1, \hat{r}_2, \hat{r}_3, \ldots$, the consumer i chooses $\hat{c}_0^i, \hat{c}_1^i, \hat{c}_2^i, \ldots; \hat{b}_1^i, \hat{b}_2^i, \hat{b}_3^i, \ldots$ to solve

$$\max \sum_{t=0}^{\infty} \beta^t \log c_t^i$$
$$\text{s.t. } c_0^i + b_1^i \leq w_0^i$$
$$c_t^i + b_{t+1}^i \leq w_t^i + (1 + \hat{r}_t) b_t^i, \ t = 1,2,\ldots$$
$$b_t^i \geq -B$$
$$c_t^i \geq 0 .$$

Here $b_t^i \geq -B$, where $B > 0$ is chosen large enough, rules out Ponzi schemes but does not otherwise bind in equilibrium.

- $\hat{c}_t^1 + \hat{c}_t^2 + \hat{c}_t^3 = w_t^1 + w_t^2 + w_t^3$, $t = 0,1,\ldots$

- $\hat{b}_t^1 + \hat{b}_t^2 + \hat{b}_t^3 = 0$, $t = 0,1,\ldots$

(c) **Proposition 1:** Suppose that $\hat{p}_0, \hat{p}_1, \hat{p}_2, \ldots;\ \hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots;\ \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots;\ \hat{c}_0^3, \hat{c}_1^3, \hat{c}_2^3, \ldots$ is an Arrow-Debreu equilibrium. Then $\hat{r}_1, \hat{r}_2, \hat{r}_3, \ldots;\ \hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots;\ \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots;\ \hat{c}_0^3, \hat{c}_1^3, \hat{c}_2^3, \ldots;$ $\hat{b}_1^1, \hat{b}_2^1, \hat{b}_3^1, \ldots;\ \hat{b}_1^2, \hat{b}_2^2, \hat{b}_3^2, \ldots;\ \hat{b}_1^3, \hat{b}_2^3, \hat{b}_3^3, \ldots$ is a sequential markets equilibrium where

$$\hat{r}_t = \frac{\hat{p}_{t-1}}{\hat{p}_t} - 1$$

$$\hat{b}_1^i = w_0^i - \hat{c}_0^i$$

$$\hat{b}_{t+1}^i = w_t^i + (1+\hat{r}_t)\hat{b}_t^i - \hat{c}_t^i,\ t = 1, 2, \ldots.$$

Proposition 2: Suppose that $\hat{r}_1, \hat{r}_2, \hat{r}_3, \ldots;\ \hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots;\ \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots;\ \hat{c}_0^3, \hat{c}_1^3, \hat{c}_2^3, \ldots;$ $\hat{b}_1^1, \hat{b}_2^1, \hat{b}_3^1, \ldots;\ \hat{b}_1^2, \hat{b}_2^2, \hat{b}_3^2, \ldots;\ \hat{b}_1^3, \hat{b}_2^3, \hat{b}_3^3, \ldots$ is a sequential markets equilibrium. Then $\hat{p}_0, \hat{p}_1, \hat{p}_2, \ldots; \hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots;\ \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots;\ \hat{c}_0^3, \hat{c}_1^3, \hat{c}_2^3, \ldots$ is an Arrow-Debreu equilibrium where

$$\hat{p}_0 = 1$$

$$\hat{p}_t = \prod_{s=1}^{t} \frac{1}{(1+\hat{r}_s)},\ t = 1, 2, \ldots.$$

(d) We can multiply the three consumers' first order conditions

$$\frac{\beta^t}{c_t^i} = \lambda^i p_t,$$

by c_t^i, divide by λ_i, and sum across consumers to obtain

$$\beta^t \left(\frac{1}{\lambda^1} + \frac{1}{\lambda^2} + \frac{1}{\lambda^3} \right) = p_t (c_t^1 + c_t^2 + c_t^3)$$

$$p_t = \beta^t \frac{\frac{1}{\lambda^1} + \frac{1}{\lambda^2} + \frac{1}{\lambda^3}}{c_t^1 + c_t^2 + c_t^3} = \frac{\frac{1}{\lambda^1} + \frac{1}{\lambda^2} + \frac{1}{\lambda^3}}{4}.$$

Normalizing $p_0 = 1$, we obtain

$$p_t = \beta^t,\ t = 0, 1, 2, \ldots$$

Consequently, for each consumer i, $i = 1, 2, 3$,

$$\frac{\beta^t}{c_t^i} = \lambda^i p_t$$

$$c_t^i = \frac{1}{\lambda^i} = c_0^i$$

The budget constraint of consumer i is

$$c_0^i \sum_{t=0}^{\infty} p_t = \sum_{t=0}^{\infty} p_t w_t^i$$

$$c_0^i = \frac{\sum_{t=0}^{\infty} p_t w_t^i}{\sum_{t=0}^{\infty} p_t}$$

Notice that, for consumer 3,

$$\sum_{t=0}^{\infty} p_t w_t^3 = \sum_{t=0}^{\infty} p_t = \sum_{t=0}^{\infty} \beta^t = \frac{1}{1-\beta},$$

which implies that

$$c_t^3 = c_0^3 = 1.$$

For consumer 1,

$$c_0^1 = \frac{\sum_{t=0}^{\infty} p_t w_t^1}{\sum_{t=0}^{\infty} p_t} = \frac{2\sum_{t=0}^{\infty} \beta^{2t} + \sum_{t=0}^{\infty} \beta^{2t+1}}{\sum_{t=0}^{\infty} \beta^t} = \frac{2\frac{1}{1-\beta^2} + \frac{\beta}{1-\beta^2}}{\frac{1}{1-\beta}}$$

$$c_t^1 = c_0^1 = \frac{2+\beta}{1+\beta}.$$

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Similarly, for consumer 2,

$$c_0^2 = \frac{\sum_{t=0}^{\infty} p_t w_t^2}{\sum_{t=0}^{\infty} p_t} = \frac{\sum_{t=0}^{\infty} \beta^{2t} + 2\sum_{t=0}^{\infty} \beta^{2t+1}}{\sum_{t=0}^{\infty} \beta^t} = \frac{\frac{1}{1-\beta^2} + \frac{2\beta}{1-\beta^2}}{\frac{1}{1-\beta}}$$

$$c_t^2 = c_0^2 = \frac{2\beta+1}{1+\beta}.$$

To calculate the sequential markets equilibrium, we just use the formulas from proposition 1 in part c. For example,

$$r_t = \frac{\hat{p}_{t-1}}{\hat{p}_t} - 1 = \frac{1}{\beta} - 1.$$

Notice that, in $t=0$,

$$\hat{b}_1^1 = 2 - \frac{2+\beta}{1+\beta} = \frac{\beta}{1+\beta}.$$

$$\hat{b}_1^2 = -\frac{\beta}{1+\beta}.$$

That is, in even periods, consumer 1 lends $\frac{\beta}{1+\beta}$ to consumer 2, who pays back $\frac{1}{1+\beta}$ in odd periods. Consumer 3 does not borrow or lend.

(e) A **sequential markets equilibrium** is sequences of rental rates on capital $\hat{r}_0^k, \hat{r}_1^k, \hat{r}_2^k, \ldots$; wages $\hat{w}_0, \hat{w}_1, \hat{w}_2, \ldots$; interest rates on bonds $\hat{r}_0^b, \hat{r}_1^b, \hat{r}_2^b, \ldots$, consumption levels $\hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots$; $\hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots$; $\hat{c}_0^3, \hat{c}_1^3, \hat{c}_2^3, \ldots$; capital holdings $\hat{k}_0^1, \hat{k}_1^1, \hat{k}_2^1, \ldots$; $\hat{k}_0^2, \hat{k}_1^2, \hat{k}_2^2, \ldots$; $\hat{k}_0^3, \hat{k}_1^3, \hat{k}_2^3, \ldots$; bond holdings $\hat{b}_0^1, \hat{b}_1^1, \hat{b}_2^1, \ldots$; $\hat{b}_0^2, \hat{b}_1^2, \hat{b}_2^2, \ldots$; $\hat{b}_0^3, \hat{b}_1^3, \hat{b}_2^3, \ldots$;and production plans $(\hat{y}_0, \hat{k}_0, \hat{\ell}_0), (\hat{y}_1, \hat{k}_1, \hat{\ell}_1), (\hat{y}_2, \hat{k}_2, \hat{\ell}_2), \ldots$ such that

- Given $\hat{r}_0^k, \hat{r}_1^k, \hat{r}_2^k, \ldots$, $\hat{w}_0, \hat{w}_1, \hat{w}_2, \ldots$, $\hat{r}_0^b, \hat{r}_1^b, \hat{r}_2^b, \ldots$, consumer i, $i = 1,2,3$, chooses $\hat{c}_0^i, \hat{c}_1^i, \hat{c}_2^i, \ldots, \hat{k}_0^i, \hat{k}_1^i, \hat{k}_2^i, \ldots, \hat{b}_0^i, \hat{b}_1^i, \hat{b}_2^i, \ldots$ to solve

$$\max \sum_{t=0}^{\infty} \beta^t \log c_t^i$$
$$\text{s.t. } c_t^i + k_{t+1}^i + b_{t+1}^i \le \hat{w}_t^i \bar{\ell}_t^i + (1 + \hat{r}_t^k - \delta) k_t^i + (1 + \hat{r}_t^b) b_t^i,\ t = 0,1,\ldots$$
$$c_t^i \ge 0,\ k_t^i \ge 0\ b_t^i \ge -B$$
$$k_0^i = \bar{k}_0^i,\ b_0^i = 0.$$

- Given $\hat{r}_0^k, \hat{r}_1^k, \hat{r}_2^k, \ldots$, $\hat{w}_0, \hat{w}_1, \hat{w}_2, \ldots$, firms choose $(\hat{y}_0, \hat{k}_0, \hat{\ell}_0), (\hat{y}_1, \hat{k}_1, \hat{\ell}_1), (\hat{y}_2, \hat{k}_2, \hat{\ell}_2), \ldots$ to minimize costs, and $\hat{r}_0^k, \hat{r}_1^k, \hat{r}_2^k, \ldots$, $\hat{w}_0, \hat{w}_1, \hat{w}_2, \ldots$ are such that firms earn 0 profits:

$$\hat{r}_t^k = \alpha \theta \hat{k}_t^{\alpha-1} \hat{\ell}_t^{1-\alpha}$$
$$\hat{w}_t = (1-\alpha)\theta \hat{k}_t^{\alpha} \hat{\ell}_t^{-\alpha}$$

- $\hat{c}_t^1 + \hat{c}_t^2 + \hat{c}_t^3 + \hat{k}_{t+1} - (1-\delta)\hat{k}_t = \hat{y}_t = \theta \hat{k}_t^{\alpha} \hat{\ell}_t^{1-\alpha}$, $t = 0,1,\ldots$
- $\hat{k}_t^1 + \hat{k}_t^2 + \hat{k}_t^3 = \hat{k}_t$, $t = 0,1,\ldots$
- $\bar{\ell}_t^1 + \bar{\ell}_t^2 + \bar{\ell}_t^3 = \hat{\ell}_t$, $t = 0,1,\ldots$
- $\hat{b}_t^1 + \hat{b}_t^2 + \hat{b}_t^3 = 0$, $t = 0,1,\ldots$

2. (a) With an Arrow-Debreu markets structure futures markets for goods are open in period 1. Consumers trade futures contracts among themselves.

An **Arrow-Debreu equilibrium** is a sequence of prices $\hat{p}_1, \hat{p}_2, \ldots$ and an allocation $\hat{c}_1^0, (\hat{c}_1^1, \hat{c}_2^1), (\hat{c}_2^2, \hat{c}_3^2), \ldots$ such that

- Given $\hat{p}_1$, consumer 0 chooses $\hat{c}_1^0$ to solve

$$\begin{aligned} &\max \ \log c_1^0 \\ &\text{s.t } \ \hat{p}_1 c_1^0 \le \hat{p}_1 w_2 + m \\ &c_1^0 \ge 0 . \end{aligned}$$

- Given $\hat{p}_t, \hat{p}_{t+1}$, consumer *t*, $t = 1, 2, \ldots$, chooses $(\hat{c}_t^t, \hat{c}_{t+1}^t)$ to solve

$$\begin{aligned} &\max \ c_t^t + \log c_{t+1}^t \\ &\text{s.t. } \ \hat{p}_t c_t^t + \hat{p}_{t+1} c_{t+1}^t \le \hat{p}_t w_1 + \hat{p}_{t+1} w_2 \\ &c_t^t, c_{t+1}^t \ge 0 . \end{aligned}$$

- $\hat{c}_t^{t-1} + \hat{c}_t^t = w_2 + w_1$, $t = 1, 2, \ldots$.

(b) With sequential market markets structure, there are markets for goods and assets open every period. The consumers in generations $t-1$ and t trade goods and assets among themselves.

A **sequential markets equilibrium** is a sequence of interest rates $\hat{r}_2, \hat{r}_3, \ldots$, an allocation $\hat{c}_1^0, (\hat{c}_1^1, \hat{c}_2^1), (\hat{c}_2^2, \hat{c}_3^2), \ldots$, and asset holdings $\hat{s}_2^1, \hat{s}_3^2, \ldots$ such that

- Consumer 0 chooses $\hat{c}_1^0$ to solve

$$\begin{aligned} &\max \ \log c_1^0 \\ &\text{s.t } \ c_1^0 \le w_2 + m \\ &c_1^0 \ge 0 . \end{aligned}$$

- Given $\hat{r}_{t+1}$, consumer *t*, $t = 1, 2, \ldots$, chooses $(\hat{c}_t^t, \hat{c}_{t+1}^t)$ and $\hat{s}_{t+1}^t$ to solve

$$\begin{aligned} &\max \ c_t^t + \log c_{t+1}^t \\ &\text{s.t. } \ c_t^t + s_{t+1}^t \le w_1 \\ &c_{t+1}^t \le w_2 + (1 + \hat{r}_{t+1}) s_{t+1}^t \\ &c_t^t, c_{t+1}^t \ge 0 . \end{aligned}$$

- $\hat{c}_t^{t-1} + \hat{c}_t^t = w_2 + w_1$, $t = 1, 2, \ldots$.

- $\hat{s}_2^1 = m$, $\hat{s}_{t+1}^t = \left[\prod_{\tau=2}^{t}(1+\hat{r}_\tau)\right]m$, $t = 2,3,\ldots$.

(c) Since there is no fiat money, there is only one good per period, there is only one consumer type in each generation, and consumers live for only two periods, the equilibrium allocation is autarky:

$$\hat{c}_1^0 = w_2$$
$$(\hat{c}_t^t, \hat{c}_{t+1}^t) = (w_1, w_2)$$

The first order conditions from the consumers' problems in the Arrow-Debreu equilibrium imply that

$$\hat{c}_{t+1}^t = \frac{\hat{p}_t}{\hat{p}_{t+1}}.$$

Normalizing $\hat{p}_1 = 1$, we obtain $\hat{p}_t = w_2^{1-t}$. Similarly, the first order conditions from the consumers' problems in the sequential markets equilibrium, imply that

$$1 + \hat{r}_{t+1} = \hat{c}_{t+1}^t = w_2$$

or $\hat{r}_t = w_2 - 1$. Since the equilibrium allocation is autarky, $\hat{s}_{t+1}^t = 0$.

(d) An allocation $\hat{c}_1^0, (\hat{c}_1^1, \hat{c}_2^1), (\hat{c}_2^2, \hat{c}_3^2), \ldots$ is **feasible** if

$$\hat{c}_t^{t-1} + \hat{c}_t^t \le w_2 + w_1,\ t = 1,\ 2,\ \ldots.$$

An allocation is **Pareto efficient** if it is feasible and there exists no other allocation $\bar{c}_1^0, (\bar{c}_1^1, \bar{c}_2^1), (\bar{c}_2^2, \bar{c}_3^2), \ldots$ that is also feasible and satisfies

$$\log \bar{c}_1^0 \ge \log \hat{c}_1^0$$
$$\bar{c}_t^t + \log \bar{c}_{t+1}^t \ge \hat{c}_t^t + \log \hat{c}_{t+1}^t,\ t = 1,2,\ldots,$$

with at least one inequality strict.

If $w_2 > 1$, the equilibrium allocation is Pareto efficient. Suppose not. Then there exists a feasible allocation that is Pareto superior. If

$$\bar{c}_t^t + \log \bar{c}_{t+1}^t > \hat{c}_t^t + \log \hat{c}_{t+1}^t,$$

then

$$\hat{p}_t \bar{c}_t^t + \hat{p}_{t+1} \bar{c}_{t+1}^t > \hat{p}_t w_1 + \hat{p}_{t+1} w_2.$$

Otherwise, $(\hat{c}_t^t, \hat{c}_{t+1}^t)$ would not solve the maximization problem of generation t. Similarly, $\log \bar{c}_1^0 > \log \hat{c}_1^0$ implies $\hat{p}_1 \bar{c}_1^0 > \hat{p}_1 w_2$.

Suppose that

$$\bar{c}_t^t + \log \bar{c}_{t+1}^t \geq \hat{c}_t^t + \log \hat{c}_{t+1}^t$$

but that

$$\hat{p}_t \bar{c}_t^t + \hat{p}_{t+1} \bar{c}_{t+1}^t < \hat{p}_t w_1 + \hat{p}_{t+1} w_2 .$$

Then let

$$\bar{\bar{c}}_t^t = \bar{c}_t^t + \frac{\hat{p}_t w_1 + \hat{p}_{t+1} w_2 - \hat{p}_t \bar{c}_t^t + \hat{p}_{t+1} \bar{c}_{t+1}^t}{\hat{p}_t} > \bar{c}_t^t$$

and $\bar{\bar{c}}_{t+1}^t = \bar{c}_{t+1}^t$. Then

$$\bar{\bar{c}}_t^t + \log \bar{\bar{c}}_{t+1}^t > \hat{c}_t^t + \log \hat{c}_{t+1}^t$$

but

$$\hat{p}_t \bar{\bar{c}}_t^t + \hat{p}_{t+1} \bar{\bar{c}}_{t+1}^t = \hat{p}_t w_1 + \hat{p}_{t+1} w_2 .$$

Once again, this would imply that $(\hat{c}_t^t, \hat{c}_{t+1}^t)$ would not solve the maximization problem of generation t, which is impossible. Consequently,

$$\hat{p}_t \bar{c}_t^t + \hat{p}_{t+1} \bar{c}_{t+1}^t \geq \hat{p}_t w_1 + \hat{p}_{t+1} w_2 .$$

Similarly, $\log \bar{c}_1^0 \geq \log \hat{c}_1^0$ implies $\hat{p}_1 \bar{c}_1^0 \geq \hat{p}_1 w_2$.
Therefore

$$\hat{p}_1 \bar{c}_1^0 \geq \hat{p}_1 w_2$$
$$\hat{p}_t \bar{c}_t^t + \hat{p}_{t+1} \bar{c}_{t+1}^t \geq \hat{p}_t w_1 + \hat{p}_{t+1} w_2 ,\ t = 1, 2, \ldots ,$$

with at least one inequality strict. Adding these inequalities up, we obtain

$$\sum_{t=1}^{\infty} \hat{p}_t (\bar{c}_t^{t-1} + \bar{c}_t^t) > \sum_{t=1}^{\infty} \hat{p}_t (w_1 + w_2) .$$

It is here that $\hat{p}_t = w_2^{1-t}$ plays its role in ensuring that these series converge.

$$\sum_{t=1}^{\infty} \hat{p}_t (w_1 + w_2) = \sum_{t=1}^{\infty} w_2^{1-t} (w_1 + w_2) = \frac{w_1 + w_2}{1 - w_2^{-1}} < \infty$$

Multiplying the feasibility condition in period t by $\hat{p}_t > 0$ and adding up yields

$$\sum_{t=1}^{\infty} \hat{p}_t (\bar{c}_t^{t-1} + \bar{c}_t^t) \leq \sum_{t=1}^{\infty} \hat{p}_t (w_1 + w_2) < \infty ,$$

which is a contradiction.

(e) An **Arrow-Debreu equilibrium** is a sequence of prices $\hat{p}_1, \hat{p}_2, \ldots$ and an allocation $\hat{c}_1^{10}, \hat{c}_1^{20}, (\hat{c}_1^{11}, \hat{c}_2^{11}), (\hat{c}_1^{21}, \hat{c}_2^{21}), (\hat{c}_2^{12}, \hat{c}_3^{12}), (\hat{c}_2^{22}, \hat{c}_3^{22})$ … such that

- Given $\hat{p}_1$, consumer $i0$ chooses $\hat{c}_1^{i0}$, $i = 1, 2$, to solve

$$\max \ u_{i0}(c_1^{i0})$$
$$\text{s.t} \quad \hat{p}_1 c_1^{i0} \le \hat{p}_1 w_2^i + m^i$$
$$c_1^{i0} \ge 0 .$$

- Given $\hat{p}_t, \hat{p}_{t+1}$, consumer it, $i = 1,2$, $t = 1,2,\ldots$, chooses $(\hat{c}_t^{it}, \hat{c}_{t+1}^{it})$ to solve

$$\max \ u_i(c_t^{it}, c_{t+1}^{it})$$
$$\text{s.t.} \quad \hat{p}_t c_t^{it} + \hat{p}_{t+1} c_{t+1}^{it} \le \hat{p}_t w_1^i + \hat{p}_{t+1} w_2^i$$
$$c_t^{it}, c_{t+1}^{it} \ge 0 .$$

- $\hat{c}_t^{1t-1} + \hat{c}_t^{2t-1} + \hat{c}_t^{1t} + \hat{c}_t^{2t} = w_2^1 + w_2^2 + w_1^1 + w_1^2$, $t = 1,2,\ldots$.

3. (a) With an Arrow-Debreu markets structure futures markets for goods are open in period 1. Consumers trade futures contracts among themselves.

An **Arrow-Debreu equilibrium** is a sequence of prices $\hat{p}_1, \hat{p}_2, \ldots$ and an allocation $\hat{c}_1^0, (\hat{c}_1^1, \hat{c}_2^1), (\hat{c}_2^2, \hat{c}_3^2), \ldots$ such that

- Given $\hat{p}_1$, consumer 0 chooses $\hat{c}_1^0$ to solve

$$\max \ \log c_1^0$$
$$\text{s.t} \ \ \hat{p}_1 c_1^0 \le \hat{p}_1 w_2 + m$$
$$c_1^0 \ge 0 .$$

- Given $\hat{p}_t, \hat{p}_{t+1}$, consumer t, $t = 1,2,\ldots$, chooses $(\hat{c}_t^t, \hat{c}_{t+1}^t)$ to solve

$$\max \ \log c_t^t + \log c_{t+1}^t$$
$$\text{s.t.} \ \ \hat{p}_t c_t^t + \hat{p}_{t+1} c_{t+1}^t \le \hat{p}_t w_1 + \hat{p}_{t+1} w_2$$
$$c_t^t, c_{t+1}^t \ge 0 .$$

- $\hat{c}_t^{t-1} + \hat{c}_t^t = w_2 + w_1$, $t = 1, 2, \ldots$.

(b) With sequential market markets structure, there are markets for goods and assets open every period. The consumers in generations $t-1$ and t trade goods and assets among themselves.

A **sequential markets equilibrium** is a sequence of interest rates $\hat{r}_2, \hat{r}_3, \ldots$, an allocation $\hat{c}_1^0, (\hat{c}_1^1, \hat{c}_2^1), (\hat{c}_2^2, \hat{c}_3^2), \ldots$, and asset holdings $\hat{s}_2^1, \hat{s}_3^2, \ldots$ such that

- Consumer 0 chooses $\hat{c}_1^0$ to solve

$$\begin{aligned} &\max \ \log c_1^0 \\ &\text{s.t } c_1^0 \le w_2 + m \\ &c_1^0 \ge 0 . \end{aligned}$$

- Given $\hat{r}_{t+1}$, consumer *t*, $t = 1, 2, \dots$, chooses $(\hat{c}_t^t, \hat{c}_{t+1}^t)$ and $\hat{s}_{t+1}^t$ to solve

$$\begin{aligned} &\max \ \log c_t^t + \log c_{t+1}^t \\ &\text{s.t. } c_t^t + s_{t+1}^t \le w_1 \\ &c_{t+1}^t \le w_2 + (1 + \hat{r}_{t+1}) s_{t+1}^t \\ &c_t^t, c_{t+1}^t \ge 0 . \end{aligned}$$

- $\hat{c}_t^{t-1} + \hat{c}_t^t = w_2 + w_1$, $t = 1, 2, \dots$.

- $\hat{s}_2^1 = m$

  $\hat{s}_{t+1}^t = \left[\prod_{\tau=2}^t (1 + \hat{r}_\tau) \right] m$, $t = 2, 3, \dots$.

(c) Since there is no fiat money, there is only one good per period, there is only one consumer type in each generation, and consumers live for only two periods, the equilibrium allocation is autarky:

$$\begin{aligned} \hat{c}_1^0 &= w_2 \\ (\hat{c}_t^t, \hat{c}_{t+1}^t) &= (w_1, w_2) \end{aligned}$$

The first order conditions from the consumers' problems in the Arrow-Debreu equilibrium imply that

$$\frac{\hat{p}_{t+1}}{\hat{p}_t} = \frac{\hat{c}_t^t}{\hat{c}_{t+1}^t} = \frac{w_1}{w_2} .$$

Normalizing $\hat{p}_1 = 1$, we obtain $\hat{p}_t = (w_1 / w_2)^{t-1}$. Similarly, the first order conditions from the consumers' problems in the sequential markets equilibrium, imply that

$$\frac{1}{1 + \hat{r}_{t+1}} = \frac{\hat{c}_t^t}{\hat{c}_{t+1}^t} = \frac{w_1}{w_2}$$

or $\hat{r}_t = (w_2 / w_1) - 1$. Since the equilibrium allocation is autarky, $\hat{s}_{t+1}^t = 0$.

(d) An allocation $\hat{c}_1^0, (\hat{c}_1^1, \hat{c}_2^1), (\hat{c}_2^2, \hat{c}_3^2), \ldots$ is **feasible** if

$$\hat{c}_t^{t-1} + \hat{c}_t^t \leq w_2 + w_1, \ t = 1,\ 2,\ \ldots.$$

An allocation is **Pareto efficient** if it is feasible and there exists no other allocation $\overline{c}_1^0, (\overline{c}_1^1, \overline{c}_2^1), (\overline{c}_2^2, \overline{c}_3^2), \ldots$ that is also feasible and satisfies

$$\log \overline{c}_1^0 \geq \log \hat{c}_1^0$$
$$\log \overline{c}_t^t + \log \overline{c}_{t+1}^t \geq \log \hat{c}_t^t + \log \hat{c}_{t+1}^t,\ t = 1,2,\ldots,$$

with at least one inequality strict.

If $w_2 > w_1$, the equilibrium allocation is Pareto efficient. Suppose not. Then using the same logic as in the proof of proposition 3 in question 1, we have

$$\hat{p}_1 \overline{c}_1^0 \geq \hat{p}_1 w_2$$
$$\hat{p}_t \overline{c}_t^t + \hat{p}_{t+1} \overline{c}_{t+1}^t \geq \hat{p}_t w_1 + \hat{p}_{t+1} w_2,\ t = 1,2,\ldots,$$

with at least one inequality strict. Adding these inequalities up, we obtain

$$\sum_{t=1}^{\infty} \hat{p}_t (\overline{c}_t^{t-1} + \overline{c}_t^t) > \sum_{t=1}^{\infty} \hat{p}_t (w_1 + w_2).$$

It is here that $\hat{p}_t = (w_1 / w_2)^{t-1}$ plays its role in ensuring that these series converge. Multiplying the feasibility condition in period t by $\hat{p}_t > 0$ and adding up yields

$$\sum_{t=1}^{\infty} \hat{p}_t (\overline{c}_t^{t-1} + \overline{c}_t^t) \leq \sum_{t=1}^{\infty} \hat{p}_t (w_1 + w_2),$$

which is a contradiction.

(e) A **sequential markets equilibrium** is a sequence of interest rates $\hat{r}_2, \hat{r}_3, \ldots$, an allocation $\hat{c}_1^{-1}, (\hat{c}_1^0, \hat{c}_2^0), (\hat{c}_1^1, \hat{c}_2^1, \hat{c}_3^1), (\hat{c}_2^2, \hat{c}_3^2, \hat{c}_4^2), \ldots$, and asset holdings $\hat{s}_2^0, (\hat{s}_2^1, \hat{s}_3^1), (\hat{s}_3^2, \hat{s}_4^2), \ldots$ such that

- Consumer -1 chooses $\hat{c}_1^{-1}$ to solve

$$\max \ \log c_1^{-1}$$
$$\text{s.t} \quad c_1^{-1} \leq w_3 + m^{-1}$$
$$c_1^{-1} \geq 0.$$

- Given $\hat{r}_2$, consumer 0 chooses $(\hat{c}_1^0, \hat{c}_2^0)$ and $\hat{s}_2^0$ to solve

$$\begin{gathered}
\max \ \log c_1^0 + \log c_2^0 \\
\text{s.t.} \quad c_1^0 + s_2^0 \le w_2 + m^0 \\
c_2^0 \le w_3 + (1+\hat{r}_2) s_2^0 \\
c_1^0, c_2^0 \ge 0 \,.
\end{gathered}$$

- Given $\hat{r}_{t+1}, \hat{r}_{t+2}$, consumer t, $t = 1, 2, \ldots$, chooses $(\hat{c}_t^t, \hat{c}_{t+1}^t, \hat{c}_{t+2}^t)$ and $(\hat{s}_{t+1}^t, \hat{s}_{t+2}^t)$ to solve

$$\begin{gathered}
\max \ \log c_t^t + \log c_{t+1}^t + \log c_{t+2}^t \\
\text{s.t.} \ c_t^t + s_{t+1}^t \le w_1 \\
c_{t+1}^t + s_{t+2}^t \le w_2 + (1+\hat{r}_{t+1}) s_{t+1}^t \\
c_{t+2}^t \le w_3 + (1+\hat{r}_{t+2}) s_{t+2}^t \\
c_t^t, c_{t+1}^t, c_{t+2}^t \ge 0 \,.
\end{gathered}$$

- $\hat{c}_t^{t-2} + \hat{c}_t^{t-1} + \hat{c}_t^t = w_3 + w_2 + w_1$, $t = 1, 2, \ldots$.

- $\hat{s}_2^0 + \hat{s}_2^1 = m^{-1} + m^0$
  $\hat{s}_{t+1}^{t-1} + \hat{s}_{t+1}^t = \left[\prod_{\tau=2}^{t} (1+\hat{r}_\tau) \right] (m^{-1} + m^0)$, $t = 2, 3, \ldots$.