A sequence $V_0, V_1, V_2, \ldots$ in $C(K)$ **converges** to $\hat{V} \in C(K)$ if for every $\varepsilon > 0$ there exists N_ε such that

$$d(V_n, \hat{V}) < \varepsilon \text{ for all } n \geq N_\varepsilon.$$

A sequence $V_0, V_1, V_2, \ldots$ in $C(K)$ is a **Cauchy sequence** if for every $\varepsilon > 0$ there exists N_ε such that

$$d(V_m, V_n) < \varepsilon \text{ for all } m, n \geq N_\varepsilon.$$

The space $C(K)$ is **complete** if every Cauchy sequence in $C(K)$ converges to a point in $C(K)$.

$C(K)$ is a **vector space** with vector addition defined by

$$(V+W)(k)=V(k)+W(k)$$

and scalar multiplication (over the field of real numbers) defined by

$$(\alpha V)(k)=\alpha V(k).$$

A vector space is **normed** if it has a metric given by a norm

$$d(V,W)=\|V-W\|$$

where

$$\|\alpha V-\alpha W\|=\alpha\|V-W\| \text{ for all } \alpha\geq 0.$$

$C(K)$ endowed with the sup norm

$$\|V - W\| = \sup_{k \in K} |V(k) - W(k)|$$

is a **Banach space**, a complete normed vector space.

Let

$$T: C(K) \to C(K).$$

Suppose that for any $V, W \in C(K)$,

$$\|T(V) - T(W)\| \leq \gamma \|V - W\|$$

for some fixed γ, $1 > \gamma > 0$.

Then we call T a contraction mapping with modulus γ.

We want to show that the mapping T defined by

$$
\begin{aligned}
T(V)(k) = \max\ & u(c) + \beta V(k') \\
\text{s.t.}\quad & c + k' - (1-\delta)k \le f(k) \\
& c, k' \ge 0
\end{aligned}
$$

maps continuous bounded functions into continuous bounded functions, that is,

$$T : C(K) \to C(K)$$

and that T is a contraction mapping with modulus β.

Then

$$\|V_{n+2} - V_{n+1}\| = \|T(V_{n+1}) - T(V_n)\| \le \beta \|V_{n+1} - V_n\|$$
$$\|V_{n+2} - V_{n+1}\| \le \beta^{n+1} \|V_1 - V_0\|.$$

The sequence of value functions $V_0, V_1, V_2, \ldots$ in $C(K)$ generated by value function iteration $V_{n+1} = T(V_n)$ is a Cauchy sequence and therefore converges to a value function $\hat{V} \in C(K)$ that satisfies the Bellman equation

$$\hat{V}(k) = \max\ u(c) + \beta \hat{V}(k')$$
$$\text{s.t.} \quad c + k' - (1-\delta)k \le f(k)$$
$$c, k' \ge 0.$$

How do we show that the mapping T defined by

$$\begin{aligned} T(V)(k) = \max \; & u(c) + \beta V(k') \\ \text{s.t.} \quad & c + k' - (1-\delta)k \le f(k) \\ & c, k' \ge 0 \end{aligned}$$

is a contraction mapping?

Blackwell's sufficient conditions

Theorem: Let $B(K)$ be the space of bounded functions $V : K \to R$ with the sup norm. Suppose that the mapping $T : B(K) \to B(K)$ satisfies that conditions

1.(**monotonicity**) If $V, W \in B(K)$ and $W(k) \geq V(k)$ for all $k \in K$, then

$$T(W)(k) \geq T(V)(k) \text{ for all } k \in K.$$

2.(**discounting**) There exists β, $0 < \beta < 1$, such that

$$T(V + \alpha)(k) \leq T(V)(k) + \beta\alpha \text{ for all } V \in B(K),\ \alpha \geq 0,\ k \in K.$$

Then T is a contraction mapping wit modulus β.