

## Profit maximization with constant returns-to-scale

Production function  $f(k, \ell)$ :

$$f(\theta k, \theta \ell) = \theta f(k, \ell)$$

$f(k, \ell)$  is concave.

Profit maximization

$$\hat{p}f(\hat{k}, \hat{\ell}) - \hat{r}\hat{k} - \hat{w}\hat{\ell} = 0$$

$$\hat{p}f(k, \ell) - \hat{r}k - \hat{w}\ell \leq 0 \text{ for all } k, \ell$$

These conditions are as much restrictions on the prices  $\hat{p}, \hat{r}, \hat{w}$  as they are on  $\hat{k}, \hat{\ell}$ .

Suppose that

$$\hat{r} = \hat{p}f_k(\hat{k}, \hat{\ell})$$

$$\hat{w} = \hat{p}f_\ell(\hat{k}, \hat{\ell})$$

Then we can show that  $\hat{k}, \hat{\ell}$  solves

$$\begin{aligned} & \min \hat{r}k + \hat{w}l \\ & \text{s.t. } f(k, l) \geq f(\hat{k}, \hat{\ell}) \end{aligned}$$

and

$$\hat{p}f(\hat{k}, \hat{\ell}) = \hat{r}\hat{k} + \hat{w}\hat{\ell}.$$

## Debreu's proof of the Pareto efficiency of equilibrium allocations

Suppose that  $\hat{p}_0, \hat{p}_1, \hat{p}_2, \dots, \hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \dots, \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \dots$  are an equilibrium:

- Given  $\hat{p}_0, \hat{p}_1, \hat{p}_2, \dots$ , consumer  $i, i = 1, 2$ , chooses  $\hat{c}_0^i, \hat{c}_1^i, \hat{c}_2^i, \dots$  to solve

$$\begin{aligned} & \max \sum_{t=0}^{\infty} \beta^t \log c_t^i \\ \text{s.t. } & \sum_{t=0}^{\infty} \hat{p}_t c_t^i \leq \sum_{t=0}^{\infty} \hat{p}_t w_t^i \\ & c_t^i \geq 0. \end{aligned}$$

- $\hat{c}_t^1 + \hat{c}_t^2 \leq w_t^1 + w_t^2, t = 0, 1, \dots$

Then  $\hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \dots, \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \dots$  is a Pareto efficient allocation:

It is feasible

- $\hat{c}_t^1 + \hat{c}_t^2 \leq w_t^1 + w_t^2, t = 0, 1, \dots$

and there exists no other feasible allocation  $\bar{c}_0^1, \bar{c}_1^1, \bar{c}_2^1, \dots, \bar{c}_0^2, \bar{c}_1^2, \bar{c}_2^2, \dots$  such that

$$\sum_{t=0}^{\infty} \beta^t \log \bar{c}_t^i \geq \sum_{t=0}^{\infty} \beta^t \log \hat{c}_t^i, \quad i = 1, 2,$$

with one inequality strict.

### **Steps in proof:**

Proof by contradiction. Suppose that there is a feasible allocation that is Pareto superior.

If  $\sum_{t=0}^{\infty} \beta^t \log \bar{c}_t^i > \sum_{t=0}^{\infty} \beta^t \log \hat{c}_t^i$ , then  $\sum_{t=0}^{\infty} \hat{p}_t \bar{c}_t^i > \sum_{t=0}^{\infty} \hat{p}_t w_t^i$ .

If  $\sum_{t=0}^{\infty} \beta^t \log \bar{c}_t^i \geq \sum_{t=0}^{\infty} \beta^t \log \hat{c}_t^i$ , then  $\sum_{t=0}^{\infty} \hat{p}_t \bar{c}_t^i \geq \sum_{t=0}^{\infty} \hat{p}_t w_t^i$ .

Consequently,

$$\sum_{i=1}^2 \sum_{t=0}^{\infty} \hat{p}_t \bar{c}_t^i > \sum_{i=1}^2 \sum_{t=0}^{\infty} \hat{p}_t w_t^i$$

Multiplying each feasibility constraint by the respective price and adding up produces

$$\sum_{i=1}^2 \sum_{t=0}^{\infty} \hat{p}_t \bar{c}_t^i \leq \sum_{i=1}^2 \sum_{t=0}^{\infty} \hat{p}_t w_t^i.$$

which is a contradiction.