MACROECONOMIC THEORY **T. J. KEHOE**
ECON 8105 **FALL 2017**

PROBLEM SET #3

1. Consider an overlapping generations economy in which there is one good in each period and each generation, except the initial one, lives for two periods. The representative consumer in generation t, $t = 1,2,\ldots$, has the utility function

$$\log\ c_t^t + \log\ c_{t+1}^t$$

and the endowment $(w_t^t, w_{t+1}^t) = (1,5)$. The representative consumer in generation 0 lives only in period 1, prefers more consumption to less, and has the endowment $w_1^0 = 5$. There is no fiat money.

a) Define an Arrow-Debreu equilibrium for this economy. Calculate the unique Arrow-Debreu equilibrium.

b) Define a sequential markets equilibrium for this economy. Calculate the unique sequential markets equilibrium.

c) Define a Pareto efficient allocation. Prove either that the equilibrium allocation in part a is Pareto efficient or prove that it is not.

d) Suppose now that $(w_t^t, w_{t+1}^t) = (5,1)$ and that $w_1^0 = 1$. Prove either that the equilibrium allocation is Pareto efficient or prove that it is not.

e) Suppose now that there is a continuum of measure 1 of two types of consumers in each generation t, $t = 1, 2, \ldots$. Both types of consumers have the utility function

$$\log\ c_t^{it} + \log\ c_{t+1}^{it},\ i = 1, 2.$$

Consumers of type 1 have the endowment $(w_t^{1t}, w_{t+1}^{1t}) = (1,5)$, while consumers of type 2 have the endowment $(w_t^{2t}, w_{t+1}^{2t}) = (5,5)$. The two representative consumers in generation 0 live only in period 1, prefer more to less, and have the endowment $w_1^{i0} = 5$, $i = 1, 2$. There is no fiat money. Define an Arrow-Debreu equilibrium for this economy.

d) In the equilibrium allocation is $c_t^{1t} = 1$? Explain carefully why or why not.

2. Consider an economy with an infinitely lived consumer who has the utility function

$$\sum_{t=0}^{\infty} \beta^t \log c_t$$

This consumer is endowed with $\ell_t = 1$ in every period and $\bar{k}_0$ period 0. Feasible consumption/production plans satisfy

$$c_t + k_{t+1} \leq \theta k_t^\alpha$$
$$c_t, k_t \geq 0 .$$

Here $1 > \alpha > 0$, $1 > \beta > 0$, $\theta > 0$. Notice that the rate of depreciation, δ, is implicitly equal to 1.

a) Define a sequential markets equilibrium of this economy.

b) Define a steady state of this economy. Calculate the steady state values of c and k.

c) Write down an optimal growth problem for this economy. Write down the Euler conditions and the transversality condition for this problem. Suppose that in the solution to the optimal growth problem

$$k_{t+1} = b\theta k_t^\alpha$$

for some constant b. Using the Euler conditions, solve for b. Verify that the resulting solution to the Euler conditions satisfies the transversality condition.

d) Use the solution to part c to calculate the sequential markets equilibrium for this economy.

e) Define an Arrow-Debreu equilibrium for this economy. Use the solution to part d to calculate this Arrow-Debreu equilibrium.

3. Consider the social planning problem

$$\begin{aligned} \max \ & \sum_{t=0}^{\infty} \beta^t \log \ c_t \\ \text{s.t.} \ & c_t + k_{t+1} \leq \theta k_t^\alpha \\ & c_t, k_t \geq 0 \\ & k_0 \leq \bar{k}_0 . \end{aligned}$$

a) Write down the Euler conditions and the transversality condition for this problem.

b) Let $v(k_t, k_{t+1})$ be the solution to

$$\begin{aligned} \max \ & \log c_t \\ \text{s.t.} \ & c_t + k_{t+1} \leq \theta k_t^\alpha \\ & c_t \geq 0 \end{aligned}$$

for fixed k_t, k_{t+1}. What is $v(k_t, k_{t+1})$? What conditions do k_t and k_{t+1} need to satisfy to ensure $c_t, k_{t+1}, \geq 0$? If we write these conditions as $k_{t+1} \in \Gamma(k_t)$, what is $\Gamma(k_t)$?

c) Write down the Euler conditions and the transversality condition for the problem

$$\max \sum_{t=0}^{\infty} \beta^t v(k_t, k_{t+1})$$
$$\text{s.t.} \quad k_{t+1} \in \Gamma(k_t)$$
$$k_0 \leq \overline{k}_0 .$$

d) Prove that a sequence $\hat{k}_0, \hat{k}_1, \ldots$ solves the conditions in part c if an only if there exist sequences of Lagrange multipliers $\hat{p}_0, \hat{p}_1, \ldots$ and of consumption $\hat{c}_0, \hat{c}_1, \ldots$ such that $(\hat{c}_0, \hat{c}_1, \ldots, \hat{k}_0, \hat{k}_1, \ldots, \hat{p}_{0,} \hat{p}_1, \ldots)$ satisfy the conditions in part a.

e) Prove that, if a sequence $\hat{k}_0, \hat{k}_1, \ldots$ satisfies the Euler conditions and transversality condition in part c, then it solves the related planning problem. (Hint: you can adapt the general proof on pp. 98–99 of Stokey, Lucas, and Prescott to these specific functions.)

4. Let $v(k_t, k_{t+1})$ and $\Gamma(k_t)$ be defined as in part b of question 1.

a) Consider the dynamic programming problem with the Bellman equation

$$V(k) = \max v(k, k') + \beta V(k')$$
$$\text{s.t.} \quad k' \in \Gamma(k) .$$

Guess that $V(k)$ has the form $a_0 + a_1 \log k$ and solve for a_0 and a_1.

b) What is the policy function $g(k)$ such that $k' = g(k)$? Verify that $k_{t+1} = g(k_t)$ satisfies the Euler equations and the transversality condition in part c of question 1.

c) Try to approximate $V(k)$: Guess that $V_0(k) = 0$ for all k and use the iterative updating rule

$$V_{n+1}(k) = \max \; v(k,k') + \beta V_n(k')$$
$$\text{s.t.} \quad k' \in \Gamma(k).$$

Calculate the functions V_1, V_2, V_3, and V_4.