Balanced Growth

Kaldor's "stylized facts"

1. Y_t / N_t (output per working age person) exhibits continual growth.

2. K_t / N_t (capital per working age person) exhibits continual growth.

3. $r_t - \delta$ (real interest rate) is roughly constant.

4. K_t / Y_t (capital-output ratio) is roughly constant.

5. $r_t K_t / Y_t$, $w_t L_t / Y_t$ (factor shares) are roughly constant.

6. There are wide differences in the rate of growth of Y_t / N_t across countries.

I have modified Kaldor's stylized facts to put things in terms of output per working age person and capital per working age person, rather than output per worker and capital per worker. This is the way Kehoe and Prescott do it.

N. Kaldor (1961), "Capital Accumulation and Economic Growth," in F. A. Lutz and D. C. Hague, editors, *The Theory of Capital*, St. Martin's Press, 177–222.

T. J. Kehoe and E. C. Prescott (2007), "Great Depressions of the Twentieth Century," in T. J. Kehoe and E. C. Prescott, editors, *Great Depressions of the Twentieth Century*, Federal Reserve Bank of Minneapolis, 1–20.

The growth model

Production function:

$$Y_t = A_0 K_t^{\alpha} (\gamma^t L_t)^{1-\alpha} .$$

Representative consumer's problem:

$$\begin{gathered}
\max \ \sum_{t=0}^{\infty} \beta^t \left[\theta \log C_t + (1-\theta) \log(N_t \bar{h} - L_t) \right] \\
\text{s. t. } \ C_t + K_{t+1} - K_t + B_{t+1} - B_t \leq w_t L_t + (r_t - \delta)(K_t + B_t) \\
C_t, K_t \geq 0 ,\ N_t \bar{h} \geq L_t \geq 0 ,\ B_t \geq -(\gamma\eta)^t \bar{B} \\
K_0 = \bar{K}_0 ,\ B_0 = 0 \\
N_t = \eta^t N_0 .
\end{gathered}$$

where $w_t = (1-\alpha)(\gamma^{1-\alpha})^t A_0 K_t^{\alpha} L_t^{-\alpha}$, $r_t = \alpha (\gamma^{1-\alpha})^t A_0 K_t^{\alpha-1} L_t^{1-\alpha}$.

We can solve

$$
\begin{gathered}
\max \ \sum_{t=0}^{\infty} \beta^t \left[\theta \log C_t + (1-\theta)\log(N_t\bar{h} - L_t)\right] \\
\text{s. t. } C_t + K_{t+1} - (1-\delta)K_t \le (\gamma^{1-\alpha})^t A_0 K_t^{\alpha} L_t^{1-\alpha} \\
C_t, K_t \ge 0,\ N_t\bar{h} \ge L_t \ge 0 \\
K_0 = \bar{K}_0 \\
N_t = \eta^t N_0 .
\end{gathered}
$$

First-order conditions:

$$
\begin{gathered}
\frac{\beta^t \theta}{C_t} = p_t \\
\frac{\beta^t (1-\theta)}{N_t\bar{h} - L_t} = p_t (1-\alpha)(\gamma^{1-\alpha})^t A_0 K_t^{\alpha} L_t^{-\alpha} \\
p_{t-1} = p_t \left(\alpha(\gamma^{1-\alpha})^t A_0 K_t^{\alpha-1} L_t^{1-\alpha} + 1 - \delta\right).
\end{gathered}
$$

Feasibility:

$$
C_t + K_{t+1} - (1-\delta)K_t = (\gamma^{1-\alpha})^t A_0 K_t^{\alpha} L_t^{1-\alpha} .
$$

Suppose that C_t / N_t and K_t / N_t grow at the same constant rate:

$$
\frac{C_{t+1} / N_{t+1}}{C_t / N_t} = \frac{K_{t+1} / N_{t+1}}{K_t / N_t} = g .
$$

Let us argue that $g = \gamma$ and that $L_t / N_t = L_{t-1} / N_{t-1}$. From feasibility, we obtain

$$
\begin{gathered}
\frac{C_t}{N_t} + \frac{K_{t+1}}{N_t} - (1-\delta)\frac{K_t}{N_t} = (\gamma^{1-\alpha})^t A_0 \left(\frac{K_t}{N_t}\right)^{\alpha} \left(\frac{L_t}{N_t}\right)^{1-\alpha} \\
\frac{C_t}{N_t} + (g\eta - 1 + \delta)\frac{K_t}{N_t} = (\gamma^{1-\alpha})^t A_0 \left(\frac{K_t}{N_t}\right)^{\alpha} \left(\frac{L_t}{N_t}\right)^{1-\alpha} \\
\frac{\dfrac{C_t}{N_t} + (g\eta - 1 + \delta)\dfrac{K_t}{N_t}}{\dfrac{C_{t-1}}{N_{t-1}} + (g\eta - 1 + \delta)\dfrac{K_{t-1}}{N_{t-1}}} = \frac{(\gamma^{1-\alpha})^t A_0 \left(\dfrac{K_t}{N_t}\right)^{\alpha} \left(\dfrac{L_t}{N_t}\right)^{1-\alpha}}{(\gamma^{1-\alpha})^{t-1} A_0 \left(\dfrac{K_{t-1}}{N_{t-1}}\right)^{\alpha} \left(\dfrac{L_{t-1}}{N_{t-1}}\right)^{1-\alpha}} \\
g = \gamma^{1-\alpha} g^{\alpha} \left(\frac{L_t / N_t}{L_{t-1} / N_{t-1}}\right)^{1-\alpha}
\end{gathered}
$$

$$g = \gamma^{1-\alpha} g^{\alpha} g_{\ell}^{1-\alpha}, \tag{1}$$

where

$$g_{\ell} = \frac{L_t / N_t}{L_{t-1} / N_{t-1}},$$

which is necessarily constant.

The first-order conditions also imply that

$$\frac{(1-\theta)C_t}{\theta(N_t \bar{h} - L_t)} = (1-\alpha)(\gamma^{1-\alpha})^t A_0 K_t^{\alpha} L_t^{-\alpha}$$

$$\frac{\dfrac{(1-\theta)C_t / N_t}{\theta(\bar{h} - L_t / N_t)}}{\dfrac{(1-\theta)C_{t-1} / N_{t-1}}{\theta(\bar{h} - L_{t-1} / N_{t-1})}} = \frac{(1-\alpha)(\gamma^{1-\alpha})^t A_0 \left(\dfrac{K_t}{N_t}\right)^{\alpha} \left(\dfrac{N_t}{L_t}\right)^{\alpha}}{(1-\alpha)(\gamma^{1-\alpha})^{t-1} A_0 \left(\dfrac{K_{t-1}}{N_{t-1}}\right)^{\alpha} \left(\dfrac{N_t}{L_{t-1}}\right)^{\alpha}}$$

$$\frac{g(\bar{h} - L_{t-1} / N_{t-1})}{\bar{h} - L_t / N_t} = \gamma^{1-\alpha} g^{\alpha} g_{\ell}^{-\alpha}. \tag{2}$$

Dividing the expression (1) that we obtained using the feasibility condition by this expression (2), we obtain

$$\frac{\bar{h} - L_t / N_t}{\bar{h} - L_{t-1} / N_{t-1}} = g_{\ell}$$

$$\bar{h} - L_t / N_t = g_{\ell}\left(\bar{h} - L_{t-1} / N_{t-1}\right)$$

$$\bar{h} = g_{\ell} \bar{h}$$

$$g_{\ell} = 1,$$

which implies that

$$g = \gamma$$

$$\frac{C_{t+1} / N_{t+1}}{C_t / N_t} = \frac{K_{t+1} / N_{t+1}}{K_t / N_t} = \frac{Y_{t+1} / N_{t+1}}{Y_t / N_t} = \gamma.$$

Redefine variables C_t, K_t, and Y_t by dividing by effective working age persons $\tilde{N}_t = \gamma^t N_t = (\gamma\eta)^t N_0$. Divide L_t by N_t:

$$\tilde{c}_t = C_t / \tilde{N}_t = \gamma^{-t}(C_t / N_t)$$

$$\tilde{k}_t = K_t / \tilde{N}_t = \gamma^{-t}(K_t / N_t)$$
$$\tilde{y}_t = Y_t / \tilde{N}_t = \gamma^{-t}(Y_t / N_t)$$
$$\tilde{\ell}_t = L_t / N_t$$

Notice that

$$\theta \log\ C_t / N_t + (1-\theta)\log(\bar{h}N_t - L_t) / N_t = \theta \log\ \gamma^t \tilde{c}_t + (1-\theta)\log(\bar{h} - \tilde{\ell}_t)$$
$$\theta \log\ C_t / N_t + (1-\theta)\log(\bar{h}N_t - L_t) / N_t = \theta \log\ \tilde{c}_t + (1-\theta)\log(\bar{h} - \tilde{\ell}_t) + t\theta \log\ \gamma ,$$

where $t\theta \log\ \gamma$ is a constant that we can ignore in the maximization problem.

(A more consistent notation might be to write $\ell_t = L_t / N_t$ because we are not dividing by γ^t.)

Notice that the balanced growth path is the steady state $\tilde{c}_t = \tilde{c}$, $\tilde{k}_t = \tilde{k}$, $\tilde{\ell}_t = \tilde{\ell}$ of the redefined problem

$$\max\ \sum_{t=0}^{\infty} \beta^t \left[\theta \log\ \tilde{c}_t + (1-\theta)\log(\bar{h} - \tilde{\ell}_t) \right]$$
$$\text{s. t. } \tilde{c}_t + \gamma\eta\tilde{k}_{t+1} - (1-\delta)\tilde{k}_t \le A_0 \tilde{k}_t^{\alpha} \tilde{\ell}_t^{1-\alpha}$$
$$\tilde{c}_t,\ \tilde{k}_t \ge 0,\ \bar{h} \ge \tilde{\ell}_t \ge 0$$
$$\tilde{k}_0 = \bar{K}_0 / N_0 .$$

The balanced growth path matches Kaldor's stylized facts (although the explanation for fact 6 is not very interesting):

1. $Y_t / N_t = (\gamma^{1-\alpha})^t A_0 (K_t / N_t)^{\alpha} (L_t / N_t)^{1-\alpha} = \gamma^t A_0 \tilde{k}^{\alpha} \tilde{\ell}^{1-\alpha}$ grows at rate $\gamma - 1$.

2. $K_t / N_t = \gamma^t \tilde{k}$ grows at rate $\gamma - 1$.

3. $r_t - \delta = \alpha(\gamma^{1-\alpha})^t A_0 K_t^{\alpha-1} L_t^{1-\alpha} - \delta = \alpha A_0 \tilde{k}^{\alpha-1} \tilde{\ell}^{1-\alpha} - \delta = \gamma\eta / \beta - 1$ is constant.

4. $K_t / Y_t = \tilde{k} / (A_0 \tilde{k}^{1-\alpha} \tilde{\ell}^{1-\alpha})$ is constant.

5. $r_t K_t / Y_t = \alpha$, $w_t L_t / Y_t = 1 - \alpha$ are constant.

6. rate of growth of Y_t / N_t is determined solely by γ.