A Ricardian model with a continuum of goods

Consider an economy in which there are two countries and a continuum of goods in indexed $z \in [0,1]$. Goods are produced using labor:

$$y_j(z) = \ell_j(z) / a_j(z).$$

where

$$a_1(z) = e^{\alpha z}$$
$$a_2(z) = e^{\alpha(1-z)}.$$

Here $y_j(z)$ is the production of good z in country j and $\ell_j(z)$ is the input of labor. The stand-in consumer in each country has the utility function

$$\int_0^1 \log c_j(z)\, dz.$$

This consumer is endowed with $\overline{\ell}_j$ unites of labor where $\overline{\ell}_1 = \overline{\ell}_2 = \overline{\ell}$.

Definition of equilibrium: An equilibrium is
a price function $\hat{p}(z)$,
wage rates $\hat{w}_1$, $\hat{w}_2$,
consumption functions $\hat{c}_1(z)$, $\hat{c}_2(z)$,
and production plans $\hat{y}_1(z)$, $\hat{\ell}_1(z)$, $\hat{y}_2(z)$, $\hat{\ell}_2(z)$
such that

- Given $\hat{p}(z)$, $\hat{w}_j$, the consumer in country j, $j = 1,2$, chooses $\hat{c}^j(z)$ to solve

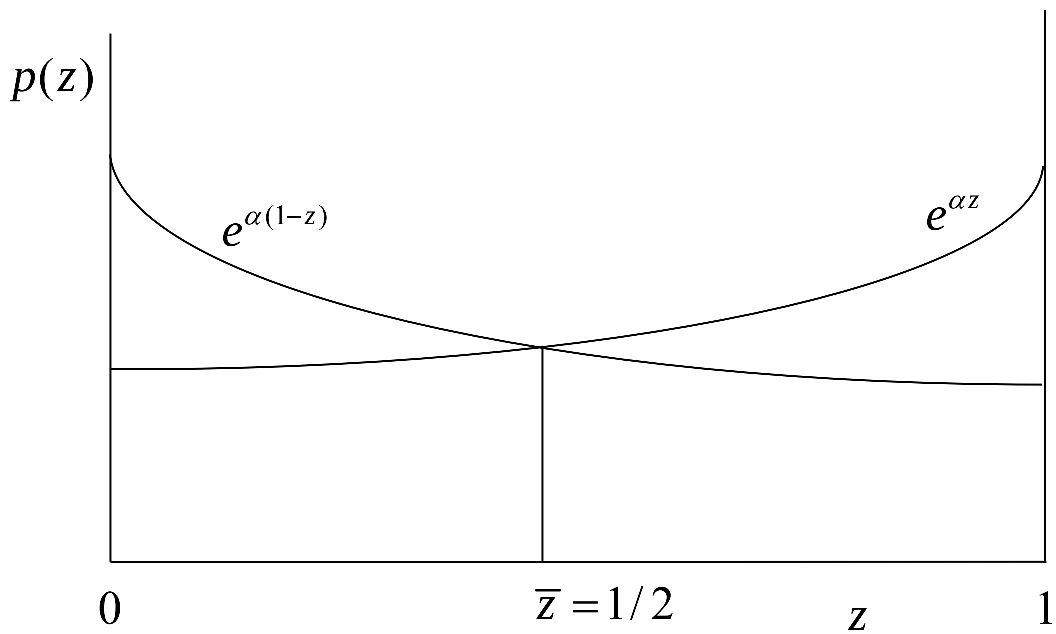
$$\max \int_0^1 \log c_j(z)\, dz$$
$$\text{s.t.} \quad \int_0^1 \hat{p}(z) c_j(z)\, dz \le \hat{w}_j \overline{\ell}_j$$
$$c_j(z) \ge 0.$$

- $\hat{p}(z) - a_j(z)\hat{w}_j \le 0$, $= 0$ if $\hat{y}_j(z) > 0$, $j = 1,2,\ z \in [0,1]$

- $\hat{c}_1(z) + \hat{c}_2(z) = \hat{y}_1(z) + \hat{y}_2(z)$, $z \in [0,1]$.

- $\int_0^1 \hat{\ell}_j(z)\, dz = \overline{\ell}$, $j = 1,2$.

Because of symmetry, we know that there is an equilibrium in which $\hat{w}_1 = \hat{w}_2 = 1$. This implies that the pattern of production, trade, and specialization is



Country 1 produces and exports the goods in the interval $[0, \overline{z}]$ while country 2 produces and exports the goods in the interval $(\overline{z}, 1]$.

The prices of the goods are

$$\hat{p}(z) = \begin{cases} e^{\alpha z} & z \in [0, \overline{z}] \\ e^{\alpha(1-z)} & z \in (\overline{z}, 1] \end{cases}.$$

The consumption levels are

$$\hat{c}_1(z) = \hat{c}_2(z) = \frac{\overline{\ell}}{\hat{p}(z)}.$$

The production plans are

$$\hat{y}_1(z) = \frac{2\overline{\ell}}{\hat{p}(z)},\ \hat{\ell}_1(z) = 2\overline{\ell},\ \hat{y}_2(z) = \hat{\ell}_2(z) = 0,\ z \in [0, \overline{z}]$$

$$\hat{y}_1(z) = \hat{\ell}_1(z) = 0,\ \hat{y}_2(z) = \frac{2\overline{\ell}}{\hat{p}(z)},\ \hat{\ell}_2(z) = 2\overline{\ell},\ z \in (\overline{z}, 1].$$

Model with tariffs

An **equilibrium** is
producer price functions $\hat{p}_1(z)$, $\hat{p}_2(z)$,
wage rates $\hat{w}_1$, $\hat{w}_2$,
consumption functions $\hat{c}_1^1(z)$, $\hat{c}_2^1(z)$, $\hat{c}_1^2(z)$, $\hat{c}_2^2(z)$,
production plans $\hat{y}_1^1(z)$, $\hat{\ell}_1^1(z)$, $\hat{y}_2^1(z)$, $\hat{\ell}_2^1(z)$, $\hat{y}_1^2(z)$, $\hat{\ell}_1^2(z)$, $\hat{y}_2^2(z)$, $\hat{\ell}_2^2(z)$,
and tariff revenues $\hat{T}_1$, $\hat{T}_2$
such that

- Given $\hat{p}_1(z)$, $\hat{p}_2(z)$, $\hat{w}_1$, the consumer in country 1 chooses $\hat{c}_1^1(z)$, $\hat{c}_2^1(z)$ to solve

$$\max \int_0^1 \log\,(c_1^1(z)+c_2^1(z))\;dz$$
$$\text{s. t.}\quad \int_0^1 (\hat{p}_1(z)c_1^1(z)+(1+\tau)\hat{p}_2(z)c_2^1(z))dz \le \hat{w}_1\overline{\ell}_1+\hat{T}_1$$
$$c_i^1(z)\ge 0\,.$$

  Similarly for the consumer in country 2.

- $\hat{p}_j(z)-a_j(z)\hat{w}_j \le 0$, $=0$ if $\hat{y}_j(z)>0$, $j=1,2$, $z\in[0,1]$

- $\hat{c}_j^1(z)+\hat{c}_j^2(z)=\hat{y}_j(z)$, $j=1,2$, $z\in[0,1]$

- $\int_0^1 \hat{\ell}_j(z)dz=\overline{\ell}$, $j=1,2$.

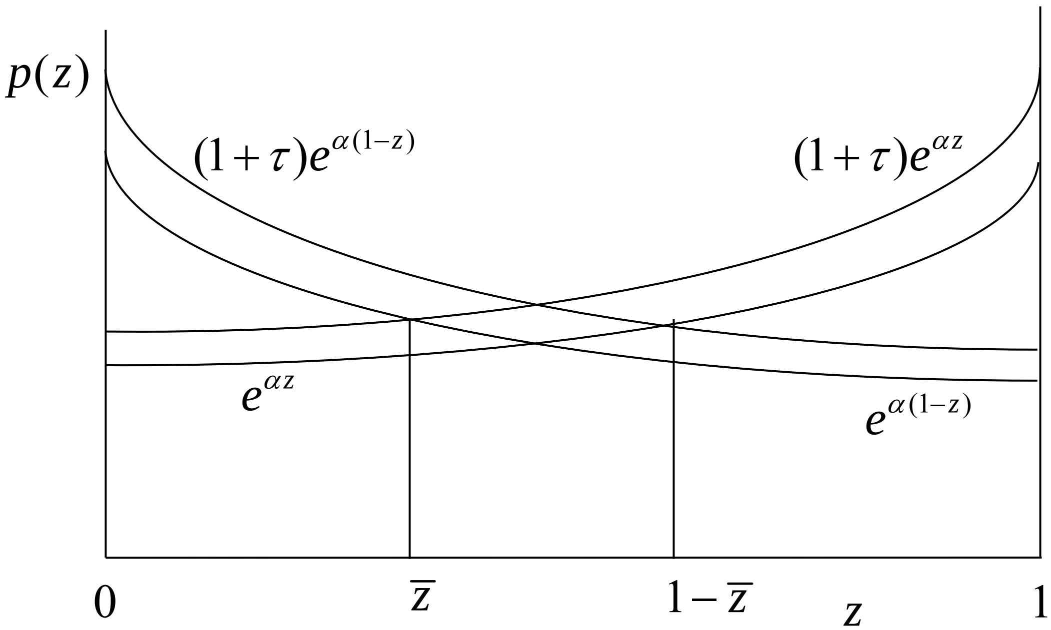
- $\hat{T}_i=\int_0^1 \tau\hat{p}_j\hat{c}_j^i(z)dz$, $i,j=1,2$, $i\ne j$

Once again, because of symmetry, we know that there is an equilibrium in which $\hat{w}_1=\hat{w}_2=1$.

There are two possibilities: either there is no trade in equilibrium or there is trade in trade in equilibrium.

First, τ is so large and/or α is so small that there is no trade in equilibrium because $(1+\tau)>e^{\alpha}$, which implies that $a_1(z)\hat{w}_1<(1+\tau)a_2(z)\hat{w}_2$ and $(1+\tau)a_1(z)\hat{w}_1>a_2(z)\hat{w}_2$ for all $z\in[0,1]$.

Second, if $(1+\tau)<e^{\alpha}$, then the pattern of production, trade, and specialization is



Country 1 produces the goods in the interval $[0,1-\overline{z}]$ and exports the goods in the interval $[0,\overline{z}]$. Country 2 produces the goods in the interval $[\overline{z},0]$ and exports the goods in the interval $[1-\overline{z},0]$. The goods in the interval $[\overline{z},1-\overline{z}]$ are not traded.

$$
\begin{gathered}
(1+\tau)e^{\alpha\overline{z}} = e^{\alpha(1-\overline{z})} \\
\log(1+\tau)+\alpha\overline{z} = \alpha(1-\overline{z}) \\
\overline{z} = \frac{1}{2} - \frac{\log(1+\tau)}{2\alpha} .
\end{gathered}
$$

The prices are

$$\hat{p}_1(z) = e^{\alpha z} , \ \hat{p}_2(z) = e^{\alpha(1-z)}$$

The consumption levels are

The consumption levels are

$$
\begin{gathered}
\hat{c}_1^1(z) = \frac{\overline{\ell}+\hat{T}_1}{e^{\alpha z}} , \ \hat{c}_2^1(z) = 0 , \ \hat{c}_1^2(z) = \frac{\overline{\ell}+\hat{T}_2}{(1+\tau)e^{\alpha z}} , \ \hat{c}_2^2(z) = 0 , z \in [0,\overline{z}] \\
\hat{c}_1^1(z) = \frac{\overline{\ell}+\hat{T}_1}{e^{\alpha z}} , \ \hat{c}_2^1(z) = 0 , \ \hat{c}_1^2(z) = 0 \ \hat{c}_2^2(z) = \frac{\overline{\ell}+\hat{T}_2}{e^{\alpha(1-z)}} , \ z \in (\overline{z},1-\overline{z}] \\
\hat{c}_1^1(z) = 0 , \ \hat{c}_2^1(z) = \frac{\overline{\ell}+\hat{T}_1}{(1+\tau)e^{\alpha(1-z)}} , \ \hat{c}_1^2(z) = 0 , \ \hat{c}_2^2(z) = \frac{\overline{\ell}+\hat{T}_2}{e^{\alpha(1-z)}} , z \in (1-\overline{z},\overline{z}] .
\end{gathered}
$$

The tariff revenue in country 1 is

$$T_1 = \int_{1-\overline{z}}^{1} \tau p_2(z) \frac{\overline{\ell} + T_1}{(1+\tau)p_2(z)}\, dz$$

$$\hat{T}_1 = \hat{T}_2 = \hat{T} = \frac{\tau \overline{z}\,\overline{\ell}}{1+\tau(1-\overline{z})}.$$

The production and labor levels are

$$\hat{y}_1(z) = \frac{(2+\tau)(\overline{\ell}+\hat{T})}{(1+\tau)\hat{p}_1(z)},\ \hat{\ell}_1(z) = \frac{(2+\tau)(\overline{\ell}+\hat{T})}{(1+\tau)},\ \hat{y}_2(z) = \hat{\ell}_2(z) = 0,\ z \in [0,\overline{z}]$$

$$\hat{y}_1(z) = \frac{\overline{\ell}+\hat{T}}{\hat{p}_1(z)},\ \hat{\ell}_1(z) = \overline{\ell}+\hat{T},\ \hat{y}_2(z) = \frac{\overline{\ell}+\hat{T}}{\hat{p}_2(z)},\ \hat{\ell}_2(z) = \overline{\ell}+\hat{T},\ \ z \in (\overline{z}, 1-\overline{z}]$$

$$\hat{y}_1(z) = \hat{\ell}_1(z) = 0,\ \hat{y}_2(z) = \frac{(2+\tau)(\overline{\ell}+\hat{T})}{(1+\tau)\hat{p}_2(z)},\ \hat{\ell}_2(z) = \frac{(2+\tau)(\overline{\ell}+\hat{T})}{(1+\tau)},\ z \in (1-\overline{z}, 1].$$

Check labor allocation:

$$\int_0^1 \hat{\ell}_1(z)dz = \overline{z}\,\frac{(2+\tau)(\overline{\ell}+\hat{T})}{(1+\tau)} + (1-\overline{z}-\overline{z})(\overline{\ell}+\hat{T})$$

$$\int_0^1 \hat{\ell}_1(z)dz = \frac{\overline{\ell}+\hat{T}}{(1+\tau)}(\overline{z}(2+\tau) + (1-2\overline{z})(1+\tau))$$

$$\int_0^1 \hat{\ell}_1(z)dz = \frac{\overline{\ell}+\hat{T}}{(1+\tau)}((1+\tau) + \overline{z}(2+\tau) - 2\overline{z}(1+\tau))$$

$$\int_0^1 \hat{\ell}_1(z)dz = \frac{\overline{\ell}+\hat{T}}{(1+\tau)}(1+\tau(1-\overline{z}))$$

$$\int_0^1 \hat{\ell}_1(z)dz = \frac{\overline{\ell} + \dfrac{\tau\overline{z}\,\overline{\ell}}{1+\tau(1-\overline{z})}}{(1+\tau)}(1+\tau(1-\overline{z}))$$

$$\int_0^1 \hat{\ell}_1(z)dz = \frac{\overline{\ell}(1+\tau(1-\overline{z})+\tau\overline{z})}{(1+\tau)} = \overline{\ell}.$$

A Heckscher-Ohlin Model with a Continuum of Goods

Suppose now that goods are produced using both capital and labor:

$$y_j(z) = k_j(z)^{\alpha(z)} \ell_j(z)^{1-\alpha(z)},$$

where $\alpha(z) = z$, $z \in [0,1]$. Notice that production technologies are now identical across countries. Endowments, however, are different. Specifically,

$$\bar{\ell}_1 = \bar{k}_2 > \bar{\ell}_2 = \bar{k}_1.$$

Definition of equilibrium: An equilibrium is
a price function $\hat{p}(z)$,
factor prices $\hat{r}_1$, $\hat{w}_1$, $\hat{r}_2$, $\hat{w}_2$,
consumption functions $\hat{c}_1(z)$, $\hat{c}_2(z)$,
and production plans $\hat{y}_1(z)$, $\hat{k}_1(z)$, $\hat{\ell}_1(z)$, $\hat{y}_2(z)$, $\hat{k}_2(z)$, $\hat{\ell}_2(z)$
such that

- Given $\hat{p}(z)$, $\hat{w}_j$, the consumer in country j, $j = 1,2$, chooses $\hat{c}_1(z)$ to solve

$$\max \int_0^1 \log c_j(z)\, dz$$
$$\text{s.t.} \quad \int_0^1 \hat{p}(z) c_j(z)\, dz \le \hat{r}_j \bar{k}_j + \hat{w}_j \bar{\ell}_j$$
$$c_j(z) \ge 0.$$

- $\hat{p}(z)\alpha(z)\hat{k}_j^{\alpha(z)-1}\hat{\ell}_j^{1-\alpha(z)} - \hat{r}_j \le 0, \ = 0 \text{ if } \hat{y}_j(z) > 0, \ j = 1,2, \ z \in [0,1]$
  $\hat{p}(z)(1-\alpha(z))\hat{k}_j^{\alpha(z)}\hat{\ell}_j^{-\alpha(z)} - \hat{w}_j \le 0, \ = 0 \text{ if } \hat{y}_j(z) > 0, \ j = 1,2, \ z \in [0,1]$

- $\hat{c}_1(z) + \hat{c}_2(z) = \hat{y}_1(z) + \hat{y}_2(z), \ z \in [0,1]$.

- $\int_0^1 \hat{\ell}_j(z)\, dz = \bar{\ell}, \ j = 1,2$.

Because of symmetry, we know that there is an equilibrium on which $\hat{r}_1 = \hat{w}_2$ and $\hat{w}_1 = \hat{r}_2$. There are two possibilities: either $\hat{r}_1 = \hat{w}_2 > \hat{w}_1 = \hat{r}_2 = 1$ or $\hat{r}_1 = \hat{w}_2 = \hat{w}_1 = \hat{r}_2 = 1$. If $\hat{r}_1 = \hat{w}_2 > \hat{w}_1 = \hat{r}_2 = 1$, then country 1 specializes in all of the goods less capital intensive then a specific level $\bar{z}$, that is all $z \le \bar{z}$, and country 2 specializes in all goods more capital intensive than the same $\bar{z}$, that is all $z > \bar{z}$. Because of symmetry, $\bar{z} = 1/2$. The graph looks like that for the Ricardian model without tariff:

On the other hand, if $\hat{r}_1 = \hat{w}_2 = \hat{w}_1 = \hat{r}_2 = 1$, then the structure of production and trade is indeterminate.

We use the first-order conditions for firm z in country j to obtain

$$\ell_j(z) = \left(\frac{r_j(1-z)}{w_j z}\right)^z y_j(z)$$
$$k_j(z) = \left(\frac{w_j z}{r_j(1-z)}\right)^{1-z} y_j(z)$$
$$p(z) = \frac{r_j^z w_j^{1-z}}{z^z(1-z)^{1-z}}.$$

To see which of the two cases that we are in, we suppose that $\hat{r}_1 = \hat{w}_2 = \hat{w}_1 = \hat{r}_2 = 1$. Let us calculate the demand for labor in country 1 under the assumption that that country 1 produces all of the goods $z \le 1/2$. If this amount of labor is less than $\bar{\ell}$, then we know that we are in the other case, where $\hat{r}_1 = \hat{w}_2 > \hat{w}_1 = \hat{r}_2 = 1$.

$$\ell_1(z) = \left(\frac{(1-z)}{z}\right)^z y_1(z)$$
$$p(z) = \frac{1}{z^z(1-z)^{1-z}}$$
$$c_1(z) = \frac{\bar{k}_1 + \bar{\ell}_1}{p(z)}$$
$$c_2(z) = \frac{\bar{k}_2 + \bar{\ell}_2}{p(z)},$$

which imply that

$$y_1(z) = c_1(z) + c_2(z) = \frac{\bar{k}_1 + \bar{\ell}_1 + \bar{k}_2 + \bar{\ell}_2}{p(z)} = z^z(1-z)^{1-z}\left(\bar{k}_1 + \bar{\ell}_1 + \bar{k}_2 + \bar{\ell}_2\right)$$
$$\ell_1(z) = \left(\frac{(1-z)}{z}\right)^z z^z(1-z)^{1-z}\left(\bar{k}_1 + \bar{\ell}_1 + \bar{k}_2 + \bar{\ell}_2\right) = (1-z)\left(\bar{k}_1 + \bar{\ell}_1 + \bar{k}_2 + \bar{\ell}_2\right).$$

The total demand for labor in country 1 is

$$\int_0^{1/2}(1-z)\left(\bar{k}_1 + \bar{\ell}_1 + \bar{k}_2 + \bar{\ell}_2\right)\,dz = \left(\bar{k}_1 + \bar{\ell}_1 + \bar{k}_2 + \bar{\ell}_2\right)\left(z - \frac{z^2}{2}\right)\Bigg|_0^{1/2} = \frac{3}{8}\left(\bar{k}_1 + \bar{\ell}_1 + \bar{k}_2 + \bar{\ell}_2\right).$$

If

$$\overline{\ell}_1 < \frac{3}{8}\left(\overline{k}_1 + \overline{\ell}_1 + \overline{k}_2 + \overline{\ell}_2\right),$$

then we know that we are in the case where $\hat{r}_1 = \hat{w}_2 > \hat{w}_1 = \hat{r}_2 = 1$. Since $\overline{k}_2 = \overline{\ell}_1$ and $\overline{\ell}_2 = \overline{k}_1$, this condition is

$$\overline{\ell}_1 < \frac{3}{4}\left(\overline{k}_1 + \overline{\ell}_1\right)$$
$$\overline{\ell}_1 < \frac{1}{3}\overline{k}_1.$$

If $\overline{\ell}_1 < \frac{1}{3}\overline{k}_1$, let us solve for $\hat{w}_2 = \hat{r}_1 = r$

$$p(z) = \frac{r^z}{z^z(1-z)^{1-z}}$$
$$y_1(z) = c_1(z) + c_2(z) = \frac{2\left(r\overline{k}_1 + \overline{\ell}_1\right)}{p(z)} = \frac{2z^z(1-z)^{1-z}\left(r\overline{k}_1 + \overline{\ell}_1\right)}{r^z}$$
$$\ell_1(z) = \left(\frac{r(1-z)}{z}\right)^z \frac{2z^z(1-z)^{1-z}\left(r\overline{k}_1 + \overline{\ell}_1\right)}{r^z}$$
$$\ell_1(z) = 2(1-z)\left(r\overline{k}_1 + \overline{\ell}_1\right)$$
$$\int_0^{1/2} 2(1-z)\left(r\overline{k}_1 + \overline{\ell}_1\right)\, dz = 2\left(r\overline{k}_1 + \overline{\ell}_1\right)\left(z - \frac{z^2}{2}\right)\Bigg|_0^{1/2} = \frac{3}{4}\left(r\overline{k}_1 + \overline{\ell}_1\right).$$

Solving for $\hat{w}_2 = \hat{r}_1 = r$, we obtain

$$\frac{3}{4}\left(r\overline{k}_1 + \overline{\ell}_1\right) = \overline{\ell}_1$$
$$\hat{w}_2 = \hat{r}_1 = r = \frac{\overline{\ell}_1}{3\overline{k}_1}.$$