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AN OBSERVATION ON GROSS SUBSTITUTABILITY AND THE WEAK AXIOM OF REVEALED PREFERENCE

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It is shown that if there are three commodities or less, an excess demand function that fulfills the property of Gross Substitutability satisfies also the Weak Axiom of Revealed Preference.

Suppose that the consumption side of a constant returns production economy is described by an excess demand function f: $R^l_{++} \to R^l$ satisfying the usual properties of continuity, homogeneity of degree zero, and $p \cdot f(p) = 0$ for all p. A property of central importance is the *Weak Axiom of Revealed Preference* (WA). The excess demand function f satisfies WA if $f(p) \neq f(q)$ and $q \cdot f(p) \leqslant 0$ implies $p \cdot f(q) > 0$. It is not difficult to verify that the WA property is necessary and sufficient if we are interested in guaranteeing that the equilibrium price set be convex (hence there is a unique equilibrium in the regular case) for any production technology. This was pointed out to us years ago by H. Scarf. The sufficiency part was originally demonstrated by Wald (1951).

A familiar property in applications is *Gross Substitutability* (GS). The excess demand function f satisfies GS if $q = p + \alpha e_i$, $a > 0$, implies $f^i(q) < f^i(p)$ and $f^j(q) > f^j(p)$ for all $j \neq i$ (here e_i is the ith unit vector). There is an extensive literature on the GS property [see Arrow and Hahn (1971, chs. 8 and 9)]. Note that, in contrast to the WA, the GS condition is preserved under addition; that is, a collection of GS consumers has a GS aggregate excess demand.

What are the relationships between the WA and the GS properties? Of course, the WA, which is always satisfied by an individual excess demand function, is compatible with complementarities. So, no implication from WA to GS should be sought. In the other direction, it is trivially verified that, for $l = 2$, GS implies WA. An example of Kehoe (1982) shows that, for $l \geqslant 4$, GS does not imply WA. It has been known to us for some time [see Kehoe (1982)] that for $l = 3$ the WA property is implied by GS. Since this does not seem to be widely known and models with three commodities are not a rarity in applications we have decided to put this observation on record.

Proposition. In a three commodity world a Gross Substitute excess demand function satisfies the Weak Axiom.

Proof. It is quite straightforward. Let p, q with $f(p) \neq f(q)$ be given. Consider the following monotonicity property (M). There is a normalization factor $a > 0$ such that $(p - \alpha q) \cdot (f(p) - f(q)) < 0$.

Property (M) implies the WA. Indeed, $-p \cdot f(q) - \alpha q \cdot f(p) = (p - \alpha q) \cdot (f(p) - \mathrm{f}(q)) < 0$. Hence, $q \cdot f(p) \leqslant 0$ yields $p \cdot f(q) > 0$. When $n = 3$ the GS property implies (M). Suppose, without loss of generality, that $p^1/q^1 \geqslant p^2/q^2 \geqslant p^3/q^3$, with one of the two inequalities strict. By reiterated application of the GS definition, we obtain $f^1(p) < f^1(q)$ and $f^3(p) > f^3(q)$. If we set $\alpha = p^2/q^2$, we have $(p - \alpha q) \cdot (f(p) - f(q)) = p^2((p^1/p^2) - (q^1/q^2))\ (f^1(p) - f^1(q)) - p^2((p^3/p^2) - (q^3/q^2))\ (f^3(p) - f^3(q)) < 0$. Q.E.D.

Remark. For $n = 2$ the WA property implies the existence of a representative consumer [the Strong Axiom of Revealed Preference (SA) is satisfied]. Hence, so does the GS property. For $l = 3$ this is no longer so. Under GS the WA is satisfied but there is no reason for the SA to be. Therefore, we have here an example where the usefulness of the WA is not derivative from the theory of the representative consumer.

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