MULTIPLICITY OF EQUILIBRIA AND COMPARATIVE STATICS*

TIMOTHY J. KEHOE

Conditions that guarantee the uniqueness of equilibrium in models of economic competition are crucial to applications of these models in exercises of comparative statics. Until now, most of the attention given to the uniqueness question has been focused on pure exchange economies. In this paper we use a topological index theorem to derive necessary and sufficient conditions for the uniqueness of equilibrium in economies with production. Unfortunately, conditions that imply uniqueness appear to be too restrictive to have much applicability. We argue, for example, that the only economically interpretable restrictions that imply uniqueness are either that the demand side of an economy behaves like a single consumer or that the supply side is an input-output system. Our results suggest a need for reformulation of the comparative statics method.

I. INTRODUCTION

Conditions that guarantee the uniqueness of equilibrium in models of economic competition are crucial to applications of these models in exercises of comparative statics. The fundamental hypothesis underlying this type of analysis is that the state of the economic system can be completely specified by the solution to a mathematical model, which is the equilibrium of the system. If, for a given vector of parameters, there is more than one solution to the model, then the comparative statics method breaks down. Lacking conditions that guarantee uniqueness, we must resort to considerations of historical conditions and dynamic stability, which greatly complicate the analysis.

Because of the importance of this issue, there have been many approaches to answering the question of when an equilibrium is unique (see Arrow and Hahn [1971, Ch. 9] for a survey). Until recently, however, these approaches have been marked by two shortcomings: they have considered sufficient rather than necessary conditions; and they have focused on pure exchange economies rather than economies that allow production. The devel-

*I would like to thank my dissertation advisors, Herbert Scarf and Andreu Mas-Colell, for their advice and guidance. Although both have had a heavy impact on the ideas presented here, neither should be held responsible for any of my personal prejudices that creep through. David Backus, Robert Dorfman, Franklin Fisher, and an anonymous referee read earlier versions of this paper and provided me with useful advice. I am also grateful to the participants in seminars at Yale University, University of Western Ontario, Harvard University, Instituto Technologico Autonomo de Mexico, Wesleyan University, and M.I.T. for helpful comments.

The Quarterly Journal of Economics, February 1985 CCC 0033-5533/85/010119-29$04.00

opment of a topological index theorem for pure exchange economies by Dierker [1972] and Varian [1974] and its extension to economies with production by Mas-Colell [1978] and Kehoe [1980] have provided us with tools more powerful than any previously available for examining the uniqueness question.

In this paper we explore the question of when an economy with production has a unique equilibrium. We begin by dealing with economies with activity analysis production technologies. Later we indicate how our results can be extended to more general technologies, including those that exhibit decreasing returns. Our approach employs the index theorem developed by Kehoe [1980], who utilizes a single-valued, continuous function whose fixed points are equivalent to equilibria of the model. Each fixed point of this function is associated with an index that is an integer determined by the local properties of this function at that point. The index theorem makes a statement about the sum of all the indices of equilibria that allows us to establish conditions sufficient for uniqueness. Furthermore, the mathematical conditions sufficient for uniqueness are necessary in almost all economies.

The index theorem can be easily motivated by the same diagram that is typically used to motivate Brouwer's fixed point theorem. Suppose that $g(\pi)$ is a continuous function from the unit interval into itself; that is, $0 \leqslant g(\pi) \leqslant 1$ for any $0 \leqslant \pi \leqslant 1$. Brouwer's fixed point theorem says that g has a fixed point $\hat{\pi} = g(\hat{\pi})$, in other words, that the graph of g must cross the diagonal as Figure I illustrates. Notice, however, that more can be said: suppose that all fixed points lie in the interior of the interval. Then the graph of g must cross the diagonal once from above. After that, in general, it crosses once from above for every time it crosses from below. Let us associate an index $+1$ with a fixed point $\hat{\pi}$ if the graph of g crosses the diagonal from above at $\hat{\pi}$, and an index -1 if it crosses from below. In the case where g is continuously differentiable, index$(\hat{\pi})$ can be computed simply by finding the sign of the expression $1 - dg/d\pi(\hat{\pi})$. The index theorem says that the sum of the indexes of all the equilibria is $+1$. Consequently, there are an odd number of equilibria, and if index$(\hat{\pi}) = +1$ at every equilibrium, there is only one equilibrium. Furthermore, if index$(\hat{\pi}) = -1$ at any equilibrium, then there must be multiple equilibria.

Unfortunately, the conditions required for uniqueness of equilibrium in production economies appear to be more restrictive

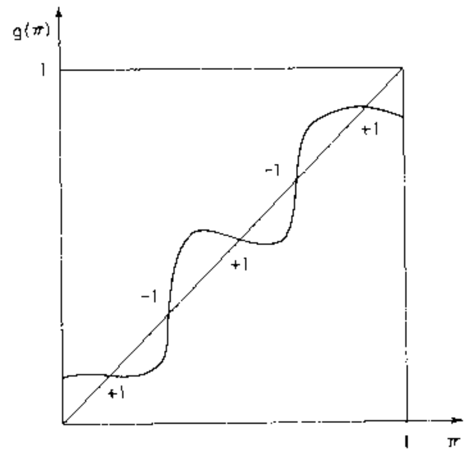

FIGURE I

than those in pure exchange economies. For example, it is well-known that, if either the weak axiom of revealed preference or gross substitutability is satisfied by the consumer excess demand function, then a pure exchange economy has a unique equilibrium. This is not the case for an economy with production. In fact, if the production technology is arbitrary, although the weak axiom is both necessary and sufficient for uniqueness, gross substitutability is neither. Since the weak axiom is an extremely restrictive assumption to impose on the aggregate excess demand function of consumers, this observation suggests that non-uniqueness of equilibrium is a less pathological situation than sometimes thought. A more subtle and far-reaching suggestion of our results is the need for a reformulation of the comparative statics method itself.

II. AN EXAMPLE OF NON-UNIQUENESS OF EQUILIBRIA

Let us begin by considering a simple economy with multiple equilibria. In this example there are four commodities and four consumers. An interesting feature is that the aggregate excess demand function exhibits gross substitutability.

Consumer j maximizes a Cobb-Douglas utility function,

$$u^j(x_1,x_2,x_3,x_4) = x_1^{\alpha_1^j} x_2^{\alpha_2^j} x_3^{\alpha_3^j} x_4^{\alpha_4^j}$$

subject to the constraints $\Sigma_{i=1}^{4} \pi_i x_i \leqslant \Sigma_{i=1}^{4} \pi_i w_i^j$ and $x_i \geqslant 0$, $i = 1, 2, 3, 4$. Here the parameter w_i^j denotes the initial endowment of good i held by j. The vector of initial endowments of consumer j is given by the jth column of the following matrix:

	Consumer			
Commodity	1	2	3	4
1	50	0	0	0
2	0	50	0	0
3	0	0	400	0
4	0	0	0	400

Similarly, the vector of utility parameters $\alpha^j = (\alpha_1^j, \alpha_2^j, \alpha_3^j, \alpha_4^j)$ for consumer j is given by the jth column of the following matrix:

	Consumer			
Commodity	1	2	3	4
1	0.52	0.86	0.5	0.06
2	0.4	0.1	0.2	0.25
3	0.04	0.02	0.2975	0.0025
4	0.04	0.02	0.0025	0.6875

Consumer j has an excess demand function $\xi_i^j(\alpha)$ for commodity i given by the rule,

$$\xi_i^j(\pi) = \alpha_i^j \frac{\Sigma_{l=1}^{4} \pi_l w_l^j}{\pi_i} - w_i^j. \tag{1}$$

The vector of aggregate excess demands $\xi(\pi)$ is formed by summing the individual excess demands: $\xi(\pi) = \Sigma_{j=1}^{4} \xi^j(\pi)$. Notice that ξ satisfies the typical properties of aggregate excess demand functions. First, it is continuous as long as prices are strictly positive. Second, it is bounded from below by the negative of the vector of aggregate initial endowments $w = \Sigma_{j=1}^{4} w^j$, $\xi(\pi) \geqslant -w$. Third, it is homogeneous of degree zero, $\xi(t\pi) \equiv \xi(\pi)$ for all $t > 0$; only relative prices matter to consumers' decision making. Fourth, it satisfies Walras' law, $\pi'\xi(\pi) \equiv 0$; since all consumers satisfy their budget constraints, the aggregate excess demand function satisfies an aggregate budget constraint. ξ also satisfies the very restrictive property of gross substitutability:

$$\frac{\partial \xi_i}{\partial \pi_l}(\pi) = \sum_{j=1}^{4} \frac{\alpha_i^j}{\pi_i} w_l^j > 0 \text{ for } l \neq i. \tag{2}$$

The production side of this economy is given by a 4×6 activity analysis matrix A. As is usual with constant-returns technologies, the delineation of individual producers or firms does not matter in the study of equilibria; all that matters is the aggregate technology specified by A. Each column of A represents an activity, or known technological process, which transforms inputs taken from the vector of aggregate initial endowments or from the outputs of other activities into outputs, which are either consumed or further used as inputs. Positive entries in an activity denote quantities of outputs produced by the activity; negative entries denote quantities of inputs consumed. Aggregate production is denoted Ay, where y is a 6×1 vector of nonnegative activity levels:

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 & 6 & -1 \\ 0 & -1 & 0 & 0 & -1 & 3 \\ 0 & 0 & -1 & 0 & -4 & -1 \\ 0 & 0 & 0 & -1 & -1 & -1 \end{bmatrix}. \tag{3}$$

The first four columns of this matrix are, of course, free disposal activities.

An equilibrium of this economy is a price vector $\hat{\pi} = (\hat{\pi}_1,\hat{\pi}_2,\hat{\pi}_3,\hat{\pi}_4)$ that satisfies the following three properties: first, $\hat{\pi}'A \leqslant 0$; second, there exists a nonnegative vector of activity levels $\hat{y} = (\hat{y}_1,\hat{y}_2,\hat{y}_3,\hat{y}_4,\hat{y}_5,\hat{y}_6)$ such that $A\hat{y} = \xi(\hat{\pi})$; and third, $\sum_{i=1}^{4} \hat{\pi}_i = 1$. The first condition requires that there be no excess profits available. The second requires that supply equal demand. When these two conditions are combined with Walras' law, they imply that profits are, in fact, maximized by the production plan $A\hat{y}$, since $\hat{\pi}'A\hat{y} = \hat{\pi}'\xi(\hat{\pi}) = 0$ but $\hat{\pi}'Ay \leqslant 0$ for any $y \geqslant 0$. The third condition is just a price normalization that we are permitted by the homogeneity of ξ; if $\hat{\pi}$ satisfies the first two equilibrium conditions, then so does $t\hat{\pi}$ for any $t > 0$.

Unfortunately, even though ξ exhibits gross substitutability, the economy specified by ξ and A has multiple equilibria. The three equilibria of (ξ,A), together with activity levels, consumption allocations, and utility levels, are listed below. Each of the three equilibria is, of course, Pareto optimal, although there are wide differences in allocations across equilibria:

Equilibrium 1

$$\pi^1 = (0.25000, 0.25000, 0.25000, 0.25000)$$
$$y^1 = (0, 0, 0, 0, 52.000, 69.000)$$

	Consumer			
Commodity	1	2	3	4
1	26.000	43.000	200.000	24.000
2	20.000	5.000	80.000	100.000
3	2.000	1.000	119.000	1.000
4	2.000	1.000	1.000	275.000
u^i	19.067	29.832	140.802	181.909

Equilibrium 2

$$\pi^2 = (0.15942, 0.25000, 0.03865, 0.55193)$$
$$y^1 = (0, 0, 0, 0, 42.701, 81.198)$$

	Consumer			
Commodity	1	2	3	4
1	26.000	67.431	48.490	83.089
2	12.754	5.000	12.368	220.771
3	8.249	6.468	119.000	14.280
4	0.578	0.453	0.070	275.000
u^i	16.039	44.880	47.410	240.484

Equilibrium 3

$$\pi^3 = (0.27514, 0.25000, 0.30865, 0.16621)$$
$$y^3 = (0, 0, 0, 0, 53.180, 65.148)$$

	Consumer			
Commodity	1	2	3	4
1	26.000	39.072	224.362	14.499
2	22.011	5.000	98.768	66.485
3	1.783	0.810	119.000	0.539
4	3.311	1.504	1.857	275.000
u^i	20.123	27.581	155.794	159.115

III. THE INDEX THEOREM

In this section we present mathematical conditions necessary and sufficient for uniqueness of equilibrium in economies similar to that of the previous section. An economy with n goods is specified by an aggregate excess demand function ξ and an activity analysis matrix A. ξ is assumed to be continuously differentiable at all positive prices, to be bounded from below by some vector $-w, w > 0$, to be homogeneous of degree zero, and to satisfy Walras' law. Recent studies have suggested that assuming excess demand to be a continuously differentiable, single-valued function, rather than the more general upper-semi-continuous, set-valued correspondence is not overly restrictive [Debreu, 1972; Mas-Colell, 1974]. The central point of these studies is that arbitrarily small perturbations in the underlying preferences of individual consumers suffice to make excess demand a differentiable function. Other studies have demonstrated that any excess demand function that satisfies these assumptions can be generated by an economy of utility-maximizing consumers (see McFadden, Mas-Colell, Mantel, and Richter [1974]).

We assume that the $n \times m$ activity analysis matrix A has n free disposal activities, one for each commodity, and that there can be no outputs without any inputs, $\{x \in R^n | x = Ay > 0, y > 0\} = \{0\}$. This latter condition is equivalent to the assumption that there exists $\pi > 0$ such that $\pi' A < 0$. A pure exchange economy is one in which the only production activities are the disposal ones, in other words, in which $A = -I$, where I is the $n \times n$ identity matrix. An equilibrium of an economy (ξ, A) is defined as before.

To derive the index theorem, we define a single-valued continuous function g whose fixed points $\hat{\pi} = g(\hat{\pi})$ are equivalent to equilibria of (ξ, A). Letting $S_A = \{\pi \in R^n | \pi' A \leqslant 0, \pi' e = 1\}$, we define p^{S_A} as the projection map that takes any point $q = R^n$ into the point $p^{S_A}(q) = S_A$ that is closest in terms of Euclidean distance. See Figure II. The function g is defined by the rule $g(\pi) = p^{S_A}[\pi + \xi(\pi)]$. (Similar least-distance mappings have been used by Eaves [1971] and Todd [1979].)

THEOREM 1. Fixed points of the function g and equilibria of the economy (ξ, A) are equivalent.

Proof of Theorem 1. $g(\pi)$ is the unique solution to the problem,

$$\text{minimize } 1/2[g - \pi - \xi(\pi)]'[g - \pi - \xi(\pi)] \tag{4}$$

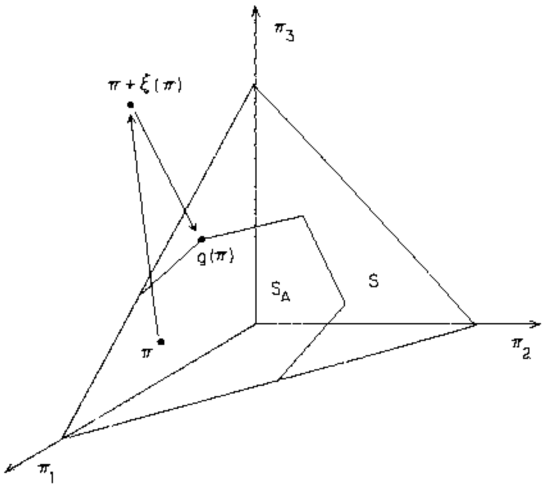

FIGURE II

subject to

$$\begin{aligned} g'A &\leqslant 0 \\ g'e &= 1. \end{aligned}$$

Here $e = (1, 1, \ldots, 1)$. The Kuhn-Tucker theorem implies that g is a solution to this problem if and only if there exist a nonnegative vector y and a scalar γ such that

$$g - \pi - \xi(\pi) + Ay + \lambda e = 0 \tag{5}$$

$$g'Ay = 0. \tag{6}$$

Suppose that $g(\hat{\pi}) = \hat{\pi}$. Then $\xi(\hat{\pi}) = \hat{y} + \hat{\lambda}e$. Walras' law and (6) imply that $\hat{\lambda} = 0$. Consequently, $\hat{\pi}$ is an equilibrium. Conversely, we can set $y = \hat{y}$ and $\lambda = 0$ in (5) and (6) to demonstrate that, if $\hat{\pi}$ is an equilibrium, then it is also a fixed point of g.

Q.E.D.

Since S_A is non-empty, compact, and convex, p^{S_A} is a continuous function. To define g for all nonnegative prices, we need to bound ξ without disturbing it in some neighborhood of any equilibrium. Kehoe [1982b] presents a simple method for doing this. Since g is then a continuous mapping of S_A into itself, Brouwer's fixed point theorem implies that it has a fixed point and, hence, that (ξ,A) has an equilibrium.

In the following analysis we focus our attention on the derivatives of g at its fixed points. Unfortunately, the function g is not everywhere differentiable. To ensure that it is differentiable at its fixed points, we impose two additional restrictions: first, no column of A can be expressed as a linear conbination of fewer than n other columns; second, every activity that earns zero profit at equilibrium is associated with a positive activity level. An economy (ξ,A) is a *regular economy* if it satisfies these two assumptions, and the additional restriction that $I - Dg(\hat{\pi})$ is nonsingular at every equilibrium $\hat{\pi}$. Here $Dg(\hat{\pi})$ denotes that $n \times n$ matrix of partial derivatives of g evaluated at $\hat{\pi}$. Economies that are not regular are *critical economies*. This definition of a regular economy is equivalent to that given by Debreu [1970] for the special case of a pure exchange economy with all equilibria strictly positive.

We focus our attention on regular economies because, first, almost all economies are regular, and second, regular economies possess very desirable properties. To make a statement about how common regularity is in the space of economies, we can take either a topological or a measure-theoretic approach. We can specify a topological structure on the space of economies by defining the concept of distance between two economies. If we do this in a suitable manner, we can prove that the set of regular economies is an open dense subset of the space of all economies. In other words, if a regular economy is subjected to any small perturbation, it remains regular, but, if an economy is not regular, an arbitrarily small perturbation can make it regular. On the other hand, if we give the space of economies a suitable parameterization, we can prove that regular economies form a subset of full measure. Since critical economies then have zero measure, the probability of choosing a critical economy at random from the space of all economies is zero.

The same properties of regular economies studied by Debreu [1970] in the context of the pure exchange model hold for the model with constant-returns production. If (ξ,A) is regular, then it has a finite number of isolated equilibria that vary continuously with its underlying parameters. Furthermore, the index theorem developed by Dierker [1972] for pure exchange economies can be extended to the model with production. We define the *index* of an equilibrium $\hat{\pi}$ of a regular economy to be $\operatorname{sgn}(\det[I - Dg(\hat{\pi})])$. Kehoe [1980] proves the following theorem:

THEOREM 2. If (ξ, A) is a regular economy, then

$$\Sigma_{\pi = g(\pi)} \text{ index } (\pi) = +1.$$

This result is illustrated in Figure I, where π and $g(\pi)$ may be interpreted as the first coordinates of a two-commodity economy, where $\pi_2 = 1 - \pi_1$ and $g_2(\pi) = 1 - g_1(\pi)$. That $I - Dg(\hat{\pi})$ is nonsingular rules out situations where the graph of g is tangent to the diagonal. A simple extension of the domain of g keeps all of its fixed points in the interior of the domain.

Employing this index theorem, we are able to count the number of equilibria of a regular economy. The formula that we have developed for this purpose, however, lacks any clear economic interpretation. To apply our results to specific economic models, we must find alternative expressions for the index of an equilibrium. One way to develop such expressions is to manipulate the matrix $I - Dg(\hat{\pi})$ without changing the sign of its determinant. Using elementary row and column operations with this property, we would find it a straightforward, if slightly laborious, task to demonstrate that

$$\text{(7)} \qquad \operatorname{index}(\hat{\pi}) = (-1)^n \operatorname{sgn}\left(\det \begin{bmatrix} 0 & e' & 0 \\ e & D\xi(\hat{\pi}) & B(\hat{\pi}) \\ 0 & B'(\hat{\pi}) & 0 \end{bmatrix} \right).$$

Here $B(\hat{\pi})$ denotes the submatrix of A whose columns are all those activities (possibly none) that earn zero profit at $\hat{\pi}$. Another formula for index $(\hat{\pi})$ can be computed as follows: choose any $n \times (n - k)$ matrix V whose columns span the null space of the $n \times k$ matrix $B(\hat{\pi})$. Let E be the $n \times n$ matrix whose every element is unity. Then it is possible to demonstrate that

$$\text{(8)} \qquad \operatorname{index}(\hat{\pi}) = \operatorname{sgn}[\det(V'[E - D\xi(\hat{\pi})]V)].$$

For the derivations of these and other formulas for index$(\hat{\pi})$, see Kehoe [1979].

Yet another useful formula can be derived by choosing some price, say the first, as numeraire. Let $\bar{J}$ be formed by deleting the first row and column from $D\xi(\hat{\pi})$, and let $\bar{B}$ be formed by deleting the first row from $B(\hat{\pi})$. If $\hat{\pi}_1 > 0$, then

$$\text{index}(\hat{\pi}) = \operatorname{sgn}\left(\det\begin{bmatrix} -\bar{J} & \bar{B} \\ -\bar{B}' & 0 \end{bmatrix}\right). \tag{9}$$

Since the disposal activity for commodity i is a column of $B(\hat{\pi})$ if $\hat{\pi}_i = 0$, we can also delete any row and column from $\bar{J}$ and the row and disposal activity from $\bar{B}$ that correspond to a commodity that is a free good at equilibrium. If only one price is positive at an equilibrium, then index$(\hat{\pi}) = +1$.

To motivate this expression for the index consider the equations that locally determine an equilibrium:

$$\xi(\pi) - By = 0 \tag{10}$$

$$\pi' B = 0. \tag{11}$$

We can set $\pi_1 = \hat{\pi}_1$ by homogeneity and use Walras' law to ignore the first equation, $\xi_1(\pi) - \Sigma b_{1j} y_j = 0$. Differentiating this system with respect to π and y yields the Jacobian matrix,

$$\begin{bmatrix} \bar{J} & -\bar{B} \\ \bar{B}' & 0 \end{bmatrix}.$$

If this matrix is nonsingular, then the inverse function theorem tells us that $\hat{\pi}$ is an isolated equilibrium and the implicit function theorem tells us that $\hat{\pi}$ varies continuously with the parameters of (ξ, A). These are the regularity conditions. The index theorem says that the sign of the determinant of this matrix is crucial for ensuring uniqueness of equilibrium.

The most significant consequence of the index theorem is that it permits us to establish conditions sufficient for uniqueness of equilibria. If the parameters of an economy (ξ, A) are such that at every equilibrium the index is equal to $+1$, then there is a unique equilibrium. Conversely, if an economy (ξ, A) has a unique equilibrium $\hat{\pi}$, it cannot be the case that index$(\hat{\pi}) = -1$. Thus, an economy with a unique equilibrium either is a critical economy or else is such that index$(\hat{\pi}) = +1$. The condition that index$(\hat{\pi}) = +1$ at every equilibrium is, therefore, necessary as well as sufficient for uniqueness in almost all cases.

Indeed, the example in the preceding section was found by constructing an economy with an equilibrium with index -1. The parameters of this economy were chosen so that

$\hat{\pi} = (0.25,0.25,0.25,0.25)$ would be an equilibrium. We easily evaluate the Jacobian matrix,

$$D\xi(\hat{\pi}) = \begin{bmatrix} -1{,}068 & 172 & 800 & 96 \\ 80 & -800 & 320 & 400 \\ 8 & 4 & -16 & 4 \\ 8 & 4 & 4 & -16 \end{bmatrix}.$$

Therefore, since activities a^5 and a^6 are in use at equilibrium $\hat{\pi}$,

$$\text{index}(\hat{\pi}) = \text{sgn}\left(\det\begin{bmatrix} -\bar{J} & \bar{B} \\ -\bar{B}' & 0 \end{bmatrix}\right)$$

$$= \text{sgn}\left(\det\begin{bmatrix} 800 & -320 & -400 & -1 & 3 \\ -4 & 16 & -4 & -4 & -1 \\ -4 & -4 & 16 & -1 & -1 \\ 1 & 4 & 1 & 0 & 0 \\ -3 & 1 & 1 & 0 & 0 \end{bmatrix}\right)$$

$$= \text{sgn}(-1{,}292) = -1.$$

This implies that (ξ,A) has multiple equilibria.

In fact, this economy has only the three equilibria reported. When an economy is regular, we know that its equilibria are finite and odd in number. If we cannot prove that there is a unique equilibrium, this is usually all we can say about the number of equilibria. The fixed point algorithm developed by Scarf [1973] is able to find some, but not necessarily all, of the equilibria of an economy. Indeed, a conventional fixed point algorithm never terminates at a fixed point whose index is -1 (see Eaves and Scarf [1976]). Although such algorithms can be modified to get around this particular difficulty, it remains true in general that it is impossible to find all of the equilibria of an economy if there is no guarantee that there is only one. Here, however, the task can be simplified by demonstrating that (ξ,A) has the structure of the input-output model with two scarce factors described by Kehoe

[1981]. The search for equilibria of (ξ, A) can then be reduced to a one-dimensional line search, and an exhaustive search can be used to find all of the equilibria.

IV. Gross Substitutability

Since the index theorem provides necessary and sufficient conditions for the uniqueness of equilibrium, it is not surprising to find that all previous theorems dealing with uniqueness are special cases of this theorem. As we have noted, the bulk of work done on the uniqueness question has dealt with the pure exchange model, which, from our perspective, is an easy case to analyze. The index theorem impies that, if $\det[-\bar{J}] > 0$ at every equilibrium of a pure exchange economy, then there is a unique equilibrium. When we restrict ourselves to economies with differentiable excess demand functions, this is the most general result possible. Indeed, any conditions that imply uniqueness must also imply that $\det[-\bar{J}] \geqslant 0$ at every equilibrium.

Previous to the development of the index theorem by Dierker [1972], the most general conditions for uniqueness of equilibrium were those given by Gale and Nikaido [1965]. The basic result was that, if every equilibrium of a pure exchange economy is strictly positive and if every principal minor of $-\bar{J}$ has the same sign on the domain of all positive price vectors, then the economy has a unique equilibrium. Obviously, this global univalence theorem is a special case of our index theorem since, if $\det[-\bar{J}]$ has the same sign at every equilibrium, then it must indeed be positive at every equilibrium. The index theorem is, however, more general than the global univalence theorem. It imposes conditions only on $\det[-\bar{J}]$, rather than on all the principal minors of $-\bar{J}$; it imposes conditions on $-\bar{J}$ only at equilibria, rather than at all positive price vectors; it holds at equilibria with free goods; and it is necessary as well as sufficient.

To illustrate the consequences of the index theorem, let us consider the case of gross substitutability. An excess demand function ξ is said to exhibit *gross substitutability* if $(\partial\xi_i/\partial\pi_j)(\pi) > 0$ for all $i \neq j$. It is well-known that gross substitutability in ξ implies uniqueness of equilibrium in a pure exchange economy. This same result is trivial to demonstrate using our index theorem: we first note that gross substitutability implies that any equilibrium price vector is strictly positive (see Arrow and Hahn [1971, pp. 221–22]).

The homogeneity assumption, when differentiated, implies that $D\xi(\pi)\pi \equiv 0$. Therefore, the matrix $-\bar{J}$ has positive diagonal elements and negative off-diagonal elements. Furthermore,

$$-\sum_{j=2}^{n} \pi_j \frac{\partial \xi_i}{\partial \pi_j} \pi = \pi_1 \frac{\partial \xi_i}{\partial \pi_1} \pi > 0, \quad i = 2, \ldots, n,$$

implies that $-\bar{J}$ has the same form as a productive Leontief matrix; that is, $-\bar{J}$ is a P matrix, a matrix with all principal minors positive. Thus, $\det[-\bar{J}] > 0$ at every equilibrium, and the index theorem implies that there is a unique equilibrium. The gross substitutability conditions can easily be weakened to $(\partial\xi_i/\partial\pi_j)(\pi) \geqq 0$ for all $i \neq j$ if we make provisions to rule out critical economies.

Unfortunately, in economies with production, the gross substitutability conditions do not seem to play a significant role. However, if $-\bar{J}$ is a positive definite matrix, a special type of P matrix, then $\text{index}(\hat{\pi}) = +1$. To demonstrate this point, we need the following lemma.

LEMMA 1. Let $\bar{J}$ and $\bar{B}$ to be defined as previously. If $\bar{J}$ is nonsingular, then $\text{index}(\hat{\pi}) = \text{sgn}(\det[-\bar{J}]\det[-\bar{B}'\bar{J}^{-1}\bar{B}])$.

Proof of Lemma 1. We base our argument on one of Gantmacher [1959, pp. 45–46]. The expression

$$\det\begin{bmatrix} -\bar{J} & \bar{B} \\ -\bar{B}' & 0 \end{bmatrix}$$

has the same sign as $\text{index}(\hat{\pi})$. If we pre-multiply the first row of this matrix by $\bar{B}'\bar{J}^{-1}$ and add it to the last row, the determinant stays the same. The sign of $\text{index}(\hat{\pi})$ is then the same as that of

$$(12)\quad \det\begin{bmatrix} -\bar{J} & -\bar{B} \\ 0 & -\bar{B}'\bar{J}^{-1}\bar{B} \end{bmatrix} = \det[-\bar{J}]\det[-\bar{B}'\bar{J}^{-1}\bar{B}].$$

Q.E.D.

That $-\bar{J}$ is positive definite implies that $-\bar{J}^{-1}$ is positive definite, which in turn implies that $-\bar{B}'\bar{J}^{-1}\bar{B}$ is positive definite. Therefore, if $-\bar{J}$ is positive definite, $\text{index}(\hat{\pi}) = +1$. Recall, however, that, although a positive definite matrix is a P matrix, the converse does not necessarily hold unless the P matrix is symmetric.

There is, of course, no reason for $-\bar{J}$ to be symmetric. That $-\bar{J}$ is a P matrix does not, therefore, seem to imply anything about index($\hat{\pi}$).

Nonetheless, it should be admitted that an example of non-uniqueness coupled with gross substitutability in demand cannot be constructed with fewer than four commodities. To demonstrate this somewhat curious result, we employ the following lemma.

LEMMA 2. Let $\bar{J}$ and $\bar{B}$ be defined as previously. If $\bar{B}$ has $n-1$ columns, then index($\hat{\pi}$) $= +1$.

Proof of Lemma 2. If an equilibrium $\hat{\pi}$ has an $n \times (n-1)$ matrix of activities associated with it, the matrix $\bar{B}$ is square and, by our nondegeneracy assumption, nonsingular. Therefore,

$$\det\begin{bmatrix} -\bar{J} & \bar{B} \\ -\bar{B}' & 0 \end{bmatrix} = \det[\bar{B}'\bar{B}]. \tag{13}$$

Since $\bar{B}'\bar{B}$ is positive definite, index($\hat{\pi}$) $= +1$.

Q.E.D.

There are two cases to investigate: that of $n = 2$, and that of $n = 3$. Suppose first that $n = 2$. Either production takes place at equilibrium, or it does not. If it does, there can be only $1 = n - 1$ activity in use, and index($\hat{\pi}$) is therefore $+1$. If it does not, index($\hat{\pi}$) $= +1$ because $-\bar{J}$ is a P matrix, in fact, a positive scalar. A similar argument works for the case $n = 3$. Either there are two or one or zero activities in use at equilibrium. If there are two or zero, then we can demonstrate that index($\hat{\pi}$) $= +1$ using the same reasoning as above. If there is only one activity in use at equilibrium, we can choose the commodity whose corresponding row is deleted from the 3×1 matrix $B(\hat{\pi})$ so that both elements of $\bar{B}$ have the same sign. It follows that the 3×3 matrix,

$$\begin{bmatrix} \bar{J} & \bar{B} \\ \bar{B}' & 0 \end{bmatrix},$$

has one of two sign patterns:

$$\begin{bmatrix} + & - & - \\ - & + & - \\ + & + & 0 \end{bmatrix} \text{ or } \begin{bmatrix} + & - & + \\ - & + & + \\ - & - & 0 \end{bmatrix}.$$

In either case, the determinant is positive, and index$(\hat{\pi}) = +1$.

Our arguments have produced the following theorem:

THEOREM 3. Suppose an economy (ξ,A) is such that ξ exhibits gross substitutability. Then (ξ,A) has a unique equilibrium if either of the following two additional conditions are satisfied:

a. $A = -I$; that is, (ξ,A) is a pure exchange economy;

b. $n \leq 3$.

As our example illustrates, if neither of these two conditions holds, then (ξ,A) does not necessarily have a unique equilibrium even if ξ exhibits gross substitutability.

V. THE WEAK AXIOM OF REVEALED PREFERENCE

That ξ exhibits gross substitutability does not, in general, imply that (ξ,A) has a unique equilibrium. However, if ξ satisfies another condition, the weak axiom of revealed preference, then (ξ,A) does not have a unique equilibrium. An excess demand function ξ is said to satisfy the *weak axiom of revealed preference* if, for every price vectors π^1 and π^2, $\xi(\pi^1) \neq \xi(\pi^2)$ and $\pi^{2\prime}\xi(\pi^1) \leq 0$ imply $\pi^{1\prime}\xi(\pi^2) > 0$. In an economy with a single consumer, this condition has a simple interpretation: if the consumer can afford to make the net trades $\xi(\pi^1)$ at prices π^2, then he cannot afford to make the net trades $\xi(\pi^2)$ at prices π^1 except in the trivial case where the two vectors of net trades are identical. The net trade vector $\xi(\pi^2)$ is said to be revealed preferred to $\xi(\pi^1)$. Although this interpretation breaks down in an economy with heterogeneous consumers, the condition itself has been known since the time of Wald [1951] to imply uniqueness of equilibrium in production economies.

Not surprisingly, it is possible to prove this result using the index theorem. To do so, we rely on a result whose proof follows closely one given by Kihlstrom, Mas-Colell, and Sonnenschein [1976, Lemma 1]:

LEMMA 3. Suppose that ξ satisfies the weak axiom of revealed preference. Then $D\xi(\pi)$ is negative semi-definite on the null space of $\xi(\pi)$; in other words, $v'D\xi(\pi)v \leq 0$ for any vector v such that $v'\xi(\pi) = 0$.

If $\hat{\pi}$ is an equilibrium, then this implies that $v'D\xi(\hat{\pi})v \leq 0$ for any v such that $v'B(\hat{\pi}) = 0$, since $B(\hat{\pi})\hat{y} = \xi(\hat{\pi})$. Recall that

$$\text{index}(\hat{\pi}) = \text{sgn}[\det(V'[E - D\xi(\hat{\pi})]V)], \tag{14}$$

where V is any matrix whose columns span the null space of the columns of $B(\hat{\pi})$ and E is a matrix of ones. If ξ satisfies the weak axiom, then $V'D\xi(\hat{\pi})V$ is a negative semi-definite matrix. Adding the positive semi-definite matrix $-V'D\xi(\hat{\pi})V$ to the positive semi-definite matrix $V'EV$ produces a positive semi-definite matrix. Our arguments have, therefore, yielded the desired result:

THEOREM 4. Suppose that (ξ,A) is a regular economy. If ξ satisfies the weak axiom of revealed preferences, then it has a unique equilibrium.

This is, of course, a well-known result, which could be proved with a far simpler argument.

Imposing the weak axiom of revealed preference on ξ implies, in general, that every principal minor of $-V'[D\xi(\hat{\pi}) - E]V$ is positive. To ensure uniqueness, however, all we really need is that the determinant of this matrix is positive. We might, therefore, be tempted to look for weaker conditions to impose on ξ that imply uniqueness. Unfortunately, the following result, which was shown to the writer by Herbert Scarf, indicates that such a search would be fruitless.

THEOREM 5. Suppose that ξ is an excess demand function that violates the weak axiom of revealed preference in the sense that there exist distinct price vectors π^1 and π^2, one of which is strictly positive, such that $\pi^{2\prime}\xi(\pi^1) \leqslant 0$ and $\pi^{1\prime}\xi(\pi^2) \leqslant 0$. Then there exists an activity analysis matrix A such that (ξ,A) has multiple equilibria.

Proof of Theorem 5. Let $a^1 = \xi(\pi^1)$, $a^2 = \xi(\pi^2)$, and $A = [-I\ a^1\ a^2]$. This matrix obviously satisfies our free disposal assumption. It also satisfies the assumption that there exist some $\pi > 0$ such that $\pi'A \leqslant 0$, since either π^1 or π^2 is strictly positive and both satisfy $\pi'A \leqslant 0$. Since both $\pi^i/\Sigma_{j=1}^{n} \pi_j^i$, $i = 1,2$, satisfy the equilibrium conditions, (ξ,A) has multiple equilibria.

Q.E.D.

We require that either π^1 or π^2 is positive to ensure that there exists a vector $\pi > 0$ such that $\pi'A \leqslant 0$. If the only price vectors that violate the weak axiom have the same elements equal to zero, then it may not be possible to construct a matrix A that satisfies this assumption. Thus, a necessary condition for uniqueness is a slightly weaker version of the weak axiom. Notice, however, even if ξ satisfies the weak axiom, if $\xi(\pi^1) = \xi(\pi^2)$ at distinct price vectors π^1 and π^2, one of which is strictly positive, then an economy (ξ,A) with multiple equilibria can be constructed. In this case, the economy must be critical, since it is easy to show that the set of equilibria is convex, which implies that there is a continuum of equilibria. Thus, a sufficient condition for uniqueness is a slightly stronger version of the weak axiom. These qualifications are, however, mere technicalities. Both conditions are equivalent to a weak axiom in almost all cases: if ξ violates the weak axiom, then it almost always does so for strictly positive price vectors, and if ξ satisfies the weak axiom, then (ξ,A) is almost always a regular economy. In general, therefore, the weak axiom is both necessary and sufficient for uniqueness of equilibria when the activity analysis technology is arbitrary.

It is difficult, however, to give much of an economic interpretation to the condition that the aggregate excess demand function ξ satisfies the weak axiom. Although any individual excess demand function, consistent with maximization of a strictly quasi-concave utility function over a non-empty budget set, satisfies the weak axiom, this property is not preserved by aggregation. In other words, even if ξ^1 and ξ^2 satisfy the weak axiom, $\xi^1 + \xi^2$ may not. It has been shown by Arrow, Block, and Hurwicz [1959] that gross substitutability in ξ implies that the weak axiom holds at least in comparisons between the equilibrium of a pure exchange economy and any non-equilibrium price vector. Now gross substitutability, a fairly restrictive property, is preserved by aggregation. When production takes place at equilibrium, however, this relationship between gross substitutability and the weak axiom breaks down. It is nonetheless true that gross substitutability implies that the weak axiom holds if there are fewer than four goods. This curious result is an immediate consequence of Theorems 3 and 5. It should, of course, be possible to demonstrate this result by simpler means.

We have demonstrated that, if the excess demand function ξ is such that it could be derived by the maximization of utility by a single consumer, then there is a unique equilibrium. There is an interesting interpretation of this observation that relates the formula for index$(\hat{\pi})$ to the bordered Hessian of an optimization problem whose solution is an equilibrium of a single consumer economy. We specify the consumption side of such an economy using the concepts of utility function and initial endowment rather than that of excess demand. We retain the activity analysis characterization of the production side.

We impose differentiability, quasi concavity, and monotonicity conditions on the utility function u and a positivity condition on the initial endowment vector w sufficient to guarantee that the solution to the problem of maximizing $u(x)$ subject to $\pi'x \leqslant \pi'w$, $x \geqslant 0$, varies smoothly with π and $\pi'w$ when prices and income are strictly positive. See Debreu [1972] and Mas-Colell [1974; 1976, pp. 87–90] for a discussion and justification of such conditions. Denote the solution to this problem as $x(\pi,\pi'w) \equiv \xi(\pi) + w$. The indirect utility function is defined as $v(\pi,I) \equiv u[x(\pi,I)]$, where $I = \pi'w$. These same conditions on u and w imply that v is continuous with continuous first- and second-order partial derivatives, strictly quasi-convex in π, monotonically decreasing in π, and monotonically increasing in I (see Diewert [1974, pp. 120–24], Varian [1978, pp. 89–91]). Moreover, v is related to the excess demand function ξ by Roy's identity,

$$\xi_i(\pi) \equiv -\frac{\partial v}{\partial \pi_j}(\pi,\pi'w) \Big/ \frac{\partial v}{\partial I}(\pi,\pi'w) - w_i. \tag{15}$$

THEOREM 6. The unique equilibrium of a single consumer economy (u,w,A) is equivalent to the solution to the problem of minimizing $v(\pi,\pi'w)$ subject to $\pi'A \leqslant 0$ and $\pi'e = 1$.

Proof of Theorem 6. If $\hat{\pi}$ is a solution to the optimization problem, then, by the Kuhn-Tucker theorem,

$$\begin{aligned} &\frac{\partial v}{\partial \pi_i}(\hat{\pi},\hat{\pi}'w) + w_i \frac{\partial v}{\partial I}(\hat{\pi},\hat{\pi}'w) \\ &\qquad + \sum_{j=1}^{m} a_{ij}\hat{\mu}_j + \hat{\lambda} = 0, \qquad i = 1, \ldots, n, \end{aligned} \tag{16}$$

for some $\hat{\mu}_j \geqslant 0$, $j = 1, \ldots, m$, and $\hat{\lambda}$. Dividing each equation through by $(\partial v/\partial I)(\hat{\pi},\hat{\pi}' w) > 0$, we use Roy's identity to establish that

$$\begin{aligned} &-\xi_i(\hat{\pi}) + \left(\sum_{j=1}^{m} a_{ij}\hat{\mu}_j + \hat{\lambda}\right) \Big/ \frac{\partial v}{\partial I}(\hat{\pi},\hat{\pi}'w) = 0, \\ &\qquad\qquad i = 1, \ldots, n. \end{aligned} \tag{17}$$

If we define $\hat{y}_j$ as $\hat{\mu}_j/(\partial v/\partial I)(\hat{\pi},\hat{\pi}'w) \geqslant 0$, $j = 1, \ldots, m$, we can use Walras' law and $\hat{\pi}'A\hat{\mu} = 0$ to establish that $\xi(\hat{\pi}) = A\hat{y}$. Therefore, since $\hat{\pi}'A \leq 0$ and $\hat{\pi}'e = 1$, $\hat{\pi}$ is an equilibrium. Conversely, if $\hat{\pi}$ is an equilibrium, then it satisfies the conditions sufficient for a minimum to the optimization problem.

The strict quasi convexity of v implies that, if a solution to the minimization problem exists, it is unique. We already know that an equilibrium exists. Therefore, the single consumer economy (u,w,A) has a unique equilibrium.

Q.E.D.

It is the strict quasi convexity of v that inplies the uniqueness of equilibrium for (u,w,A). The differentiable version of the quasi convexity at a solution $\hat{\pi}$ is a set of second-order conditions on the bordered Hessian,

$$\begin{bmatrix} H & C \\ C' & 0 \end{bmatrix}.$$

Here H is the matrix of second partial derivatives of $v(\pi,\pi'w)$ and C is the matrix of coefficients of those constraints that hold with equality at $\hat{\pi}$, $C = [e \quad B(\hat{\pi})]$. If $B(\hat{\pi})$ is $n \times k$, then second-order conditions require that

$$(18) \qquad (-1)^{k+1} \det\begin{bmatrix} H & C \\ C' & 0 \end{bmatrix} = (-1)^{k+1} \det\begin{bmatrix} 0 & e' & 0 \\ e & H & B(\hat{\pi}) \\ 0 & B'(\hat{\pi}) & 0 \end{bmatrix} \geqq 0.$$

Using Roy's identity, we can express the typical element in H as

$$(19) \qquad h_{ij} = -\frac{\partial v}{\partial I}\frac{\partial \xi_i}{\partial \pi_j} - \xi_i(\hat{\pi})\frac{\partial^2 v}{\partial I\,\partial \pi_j} + w_j\frac{\partial^2 v}{\partial I^2}.$$

Here all partial derivatives are evaluated at $(\hat{\pi},\hat{\pi}'w)$ and $\hat{\pi}$. We can multiply each of the final k columns of the second matrix in (18) by $y_b([\partial^2 v/\partial I \partial \pi_j] + w_j[\partial^2 v/\partial I^2])$ and add it to the jth column, $j = 1, \ldots, n$, without changing its determinant. The equilibrium condition $\xi(\hat{\pi}) = B(\hat{\pi})y$ then implies that

$$(20)\quad (-1)^{k+1}\left(\frac{\partial v}{\partial I}\right)^{n-k-1} \det\begin{bmatrix} 0 & e' & 0 \\ e & -\,D\xi(\hat{\pi}) & B(\hat{\pi}) \\ 0 & B'(\hat{\pi}) & 0 \end{bmatrix} \geqslant 0.$$

Consequently, since $\partial v/\partial I > 0$, the second-order conditions imply that

$$(21)\qquad (-1)^n \det\begin{bmatrix} 0 & e' & 0 \\ e & D\xi(\hat{\pi}) & B(\hat{\pi}) \\ 0 & B'(\hat{\pi}) & 0 \end{bmatrix} \geqslant 0.$$

Thus, when the economy (ξ,A) is regular, the second-order conditions for our optimization problem imply that index$(\hat{\pi}) = +1$ at every equilibrium. The restriction that ξ behaves like the excess demand of a single consumer is obviously stringent. There are two different sets of conditions that imply that this restriction holds. Both require that all demands are homogeneous of degree one in $\pi' w$; that is, $x^j(\pi, t\pi' w) \equiv t x^j(\pi,\pi' w)$ for all $t > 0$ and all consumers j. If, in addition, either all utility functions are identical [Gorman, 1953] or all initial endowments are proportional [Chipman, 1974], then the aggregate excess demand function behaves like that of a single consumer. Consequently, our example of non-uniqueness relies heavily on having consumers being heterogeneous, in terms of both utility functions and endowments.

We should note, however, that the single consumer assumption is not necessary to ensure that the weak axiom of revealed preference is satisfied. Utility maximization by a single consumer is equivalent to a stronger restriction on ξ, the strong axiom of revealed preference. An example due to Gale [1960] demonstrates that the strong axiom and the weak axiom are not equivalent. Unfortunately, there does not appear to be an obvious economic interpretation for excess demand functions that satisfy the weak axiom but not the strong axiom. Our goal is to find conditions for uniqueness that can be checked for by examining the parameters of (ξ,A). Searching for such conditions on the demand side of the economy alone has produced only the extremely restrictive single consumer assumption.

VI. The Nonsubstitution Theorem

We now turn our search for conditions that ensure uniqueness of equilibrium to the production side of (ξ,A). Unfortunately, with the exception of the restriction that there is only one positive price vector π for which $\pi'A \leqslant 0$, no condition on A implies that (ξ,A) has a unique equilibrium if the only restrictions placed on ξ are differentiability, homogeneity, and Walras' law. For example, $\xi(\pi) \equiv 0$ produces an infinite number of equilibria as long as there is more than one π for which $\pi'A \leqslant 0$. The economy in this instance is critical. Suppose that A satisfies the regularity condition that no column can be expressed as a linear combination of fewer than n other columns. This rules out the possibility of reversibility in production, and implies that there exists $\pi > 0$ such that $\pi'A < 0$. If this is the case, then we can find an open set of demand functions such that (ξ,A) has multiple equilibria. In other words, no small perturbations can eliminate the multiplicity of equilibria.

The condition that there is only one $\bar{\pi}$ such that $\bar{\pi}'A \leqslant 0$ is, unfortunately, unpalatable. It implies that the set of efficient points of the production set $Y = \{x \in R^n | x = Ay, y \geqslant 0\}$ is a hyperplane; in other words, there is complete reversibility of production in every direction (see Figure III).

To guarantee the uniqueness of equilibrium, we must somehow combine conditions from the consumption side with ones from the production side. Lemma 2 implies that, if (ξ,A) has $n - 1$

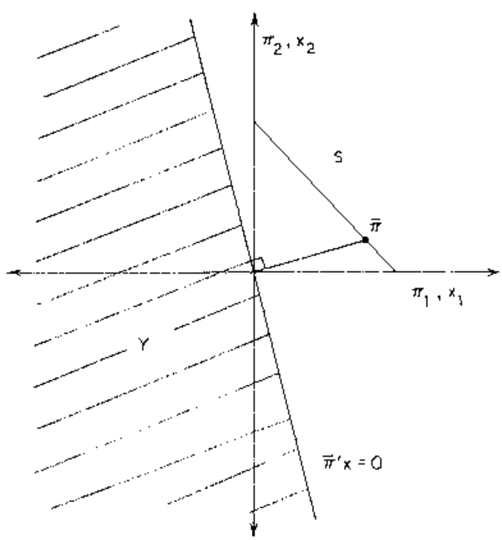

FIGURE III

linearly independent activities in use at every equilibrium, then it is a regular economy with a unique equilibrium. To gain some insight into this observation, recall that the conditions of the well-known nonsubstitution theorem of input-output analysis imply that there are always $n - 1$ activities in use at equilibrium (see Samuelson [1951]). An *input-output economy* is an economy (ξ, A) that satisfies the following four conditions. First, there is one nonproduced good; $a_{nj} \leq 0$ for $j = 1, \ldots, m$. Second, $\xi_n(\pi) > 0$, $i = 1, \ldots, n - 1$, at every equilibrium π. Third, there is no joint production; $a_{ij} > 0$ for at most one $i, j = 1, \ldots, m$. Fourth, there exists some vector of nonnegative activity levels y such that $\sum_{j=1}^{m} a_{ij} y_j > 0$, $i = 1, \ldots, n - 1$. One way to ensure that the second condition holds is to allow initial endowments only of the nonproduced good, traditionally labor. Since it must be the case that all $n - 1$ produced goods are produced at equilibrium, and since we rule out joint production, $n - 1$ activities must be run at positive levels at every equilibrium $\hat{\pi}$. If no activity in A can be expressed as a linear combination of fewer than n other columns, then any activity that earns zero profit at an equilibrium must be used at a positive level, since $n - 1$ is the maximum number of activities that can earn zero profit. We have therefore derived the following theorem with little effort:

THEOREM 7. Suppose that (ξ, A) is an input-output economy and no column of A can be expressed as a linear combination of fewer than n other columns. Then (ξ, A) is regular and has a unique equilibrium.

The nonsubstitution theorem implies that the unique equilibrium of (ξ, A) is determined independently of ξ, a conclusion that our theorem misses. To offset this shortcoming, we should note that Lemma 2 is a very general mathematical result. All we need to know is that the maximum possible number of activities are in use at every equilibrium to guarantee uniqueness. Unfortunately, the conditions of an input-output economy seem to be the only conditions on the parameters of (ξ, A) which ensure that this premise holds.

VIII. SMOOTH PRODUCTION TECHNOLOGIES

Although we have worked with activity analysis production to simplify the exposition, the results presented in the previous sections are applicable to economies with very general production technologies. In this section we indicate how our results can be extended to production technologies specified by smooth profit functions. To gain some understanding of the concept of profit function that we employ, let us consider an example in which any vector x that satisfies the constraints,

$$(22)\qquad \begin{aligned} f(x) &= 0 \\ x_i &\leqslant 0, \qquad i = 1, \ldots, l \\ x_i &\geqslant 0, \qquad i = l+1, \ldots, n, \end{aligned}$$

is a feasible net-output combination. Here f is a constant-returns production function that employs the first l commodities as inputs and produces the final $n - l$ commodities as outputs. We assume that f is homogeneous of degree one and concave. For example, $f(x_1,x_2,x_3) = \beta(-x_1)^{\alpha}(-x_2)^{1-\alpha} - x_3$ is the familiar Cobb-Douglas production function where $1 \geqslant \alpha \geqslant 0$ and $\beta > 0$. To derive the profit function a, we find a vector $x(\pi)$ that solves the problem,

maximize

$$(23)\qquad \pi' x$$

subject to

$$\begin{aligned} f(x) &= 0 \\ x'x &= 1 \\ x_i &\leqslant 0, \qquad i = 1, \ldots, l \\ x_i &\geqslant 0, \qquad i = l+1, \ldots, n. \end{aligned}$$

We then set $a(\pi) = \pi' x(\pi)$. Thus, the profit function tells us the maximum profit that can be earned at prices π when we are constrained by $x'x = 1$. Given our assumption of constant returns to scale, such a constraint is necessary since the profit that can be earned at π is unbounded if $a(\pi) > 0$. It is well-known that $a(\pi)$ is homogeneous of degree one, convex, and continuous even if the vector $x(\pi)$ is not unique. If a is continuously differentiable, Hotelling's lemma says that $Da(\pi) = x(\pi)$ (see, for example, Diewert [1974]).

Suppose that production is specified by the profit functions $a_j(\pi)$, $j = 1, \ldots, m$. We define the set $S_a = \{\pi \in R^n | a_j(\pi) \leqslant 0, j = 1, \ldots, m, \sum_{i=1}^n \pi_i = 1\}$. Clearly, this definition is a generalization of the activity analysis case, where $a_j(\pi)$ is the linear function, $\sum_{i=1}^n a_{ij}\pi_i$. We can define the concepts of regularity and fixed point index as before. Define $B(\hat{\pi})$ as the $n \times k$ matrix whose columns are the gradients of the k profit functions that satisfy $a_j(\hat{\pi}) = 0$. Hotelling's lemma and the assumption of constant returns allows us to interpret $B(\hat{\pi})$ as a matrix of activities. Further define $H(\hat{\pi})$ as the $n \times n$ weighted sum of the Hessian matrices of the same k functions; the weights are the appropriate activity levels. The calculation of index$(\hat{\pi})$ becomes

(24)

$$\text{index}(\hat{\pi}) = (-1)^n \operatorname{sgn}\left(\det\begin{bmatrix} 0 & e' & 0 \\ e & D\xi(\hat{\pi}) - H(\hat{\pi}) & B(\hat{\pi}) \\ 0 & B'(\hat{\pi}) & 0 \end{bmatrix}\right).$$

The activity analysis technology is, of course, the special case where $H(\pi) \equiv 0$. Utilizing the principle of duality, we can completely specify the production technology by a vector of profit functions $a(\pi) = (a_1(\pi), \ldots, a_m(\pi))$. An economy would thus be specified as a pair (ξ, a). A more detailed analysis of this type of economy, including the calculation of the index, is given by Kehoe [1983].

An advantage to this more general approach is that it can easily be extended to economies with production technologies that exhibit decreasing returns to scale. In such an environment we have to specify production functions for individual firms and make provisions to distribute the profit of these firms to consumers. The situation can then be treated as a special case of constant-returns production, where we define an additional primary good to account for the profits of each firm; see McKenzie [1959] for details of the construction. We can directly apply the results that we have derived here to such economies.

For an economy with decreasing-return production the index can be computed as

(25)

$$\text{index}(\hat{\pi}) = (-1)^n \operatorname{sgn}\left(\det\begin{bmatrix} 0 & e' \\ e & D\xi_\pi(\hat{\pi},\hat{r}) + D\xi_r(\hat{\pi},\hat{r})\, B'(\hat{\pi}) - H(\hat{\pi}) \end{bmatrix}\right).$$

Here $D\xi_r(\pi, r)$ is the matrix of partial derivatives of the excess demand function with respect to the vector of profits of the individual firms and $H(\pi)$ can be thought of as the Jacobian matrix of the excess supply function $B(\pi)e$.

Although this formula looks much like that for a pure exchange economy, it is difficult to interpret such considerations as gross substitutability in this context. Rader [1968], for example, has argued that factors of production are rarely gross substitutes; in other words, the off-diagonal elements of $-H(\pi)$ are rarely all nonnegative.

Suppose, however, that $D\xi_\pi(\hat{\pi},\hat{r}) + D\xi_r(\hat{\pi},\hat{r})B'(\hat{\pi})$ has all of its off-diagonal elements positive. Differentiating the homogeneity assumption,

$$\xi[t\pi,r(t\pi)] \equiv \xi[\pi,r(\pi)], \tag{26}$$

we can demonstrate, as we did in the pure exchange case, that the negative of this matrix, with any row and column deleted, is a productive Leontief matrix. In other words, $\overline{J\pi} > 0$, where $\overline{\pi}$ is the vector formed by deleting the same element from $\hat{\pi}$ as row and column deleted from $D\xi_\pi(\hat{\pi},\hat{r}) + D\xi_r(\hat{\pi},\hat{r})B'(\hat{\pi})$ to form $\overline{J}$. Furthermore, differentiating Walras' law,

$$\pi'\xi[\pi,r(\pi)] \equiv e'r(\pi), \tag{27}$$

we can demonstrate the same for the transpose of this matrix. Consequently, $1/2(\overline{J} + \overline{J}')\overline{\pi} > 0$, which implies that $1/2(\overline{J} + \overline{J}')$ is itself a productive Leontief matrix and hence positive definite. This implies that $D\xi_\pi(\hat{\pi},\hat{r}) + D\xi_r(\hat{\pi},\hat{r})B'(\hat{\pi})$ is negative semi-definite. Consequently, since $H(\hat{\pi})$ is the sum of Hessians of convex functions, and therefore positive semi-definite, we can argue that $\text{index}(\hat{\pi}) = +1$.

This result was originally discovered by Rader [1972], who did not use an index theorem. The interpretation that he gave it was that gross substitutability in demand implies uniqueness of equilibrium regardless of what the production technology looks like. The example of non-uniqueness of equilibria presented earlier should make us suspicious of such an interpretation. The problem is that the term $D\xi_r(\hat{\pi},\hat{r})B'(\hat{\pi})$ does not depend on consumer demand alone; it involves a complex interaction between income effects from the demand side of the economy and activities from the production side. It seems impossible to develop easily checked conditions to guarantee that $D\xi_\pi(\hat{\pi},\hat{r}) + D\xi_r(\hat{\pi},\hat{r})B'(\hat{\pi})$ has the required sign pattern.

VIII. Concluding Remarks

Previous researchers are to be admired for developing most of the significant economic theorems dealing with uniqueness without having such a powerful tool as the index theorem at their disposal. Indeed, as Andreu Mas-Colell once remarked to the writer, it would be surprising if there were many significant economic theorems to be derived after so many powerful minds had turned their attention to the question of uniqueness. Nonetheless, many of these researchers are to be faulted for two basic defects in approach: First, that previous approaches dealt with sufficient, rather than necessary, conditions has led many to believe optimistically that more general conditions could be derived. Second, they have given too much attention to the pure exchange model, which misses many of the subtleties of the more general situation.

Even our approach has its shortcomings, however. In the effort to translate the mathematically necessary and sufficient conditions of the index theorem into restrictions on (ξ, A) with economic interpretations, many conditions lose their necessity. Nevertheless, we have derived some powerful negative results: no condition on the consumption side of the economy that does not imply the weak axiom of revealed preference, and no condition on the production side except complete reversibility of production, is sufficient for uniqueness. Moreover, through the use of an example, we have demonstrated that gross substitutability in demand does not imply uniqueness. If there is any hope for ensuring the uniqueness of equilibrium in applied models, it would seem to be in finding conditions, besides the representative consumer conditions, that imply the weak axiom. The gross substitutability conditions for $n = 3$ is an example, albeit not a very general one, of such a set of conditions.

Non-uniqueness of equilibrium does not seem so pathological a situation as to warrant unqualified use of the simple comparative statics method when dealing with general equilibrium models. At present, many researchers are using variants of Scarf's algorithm to evaluate the implications of policy decisions in empirical general equilibrium models. Yet, for most of these models no method now exists to determine whether the equilibrium found by the algorithm is unique. To make matters worse, it appears that non-uniqueness of equilibria is an even more common situation in applied models that allow for such distortions as taxes, price rigidities, and unemployment than it is in the simple model we have investigated here. There is a suggestion of this result in the work of Foster and Sonnenschein [1970]; Kehoe [1982a] presents a demonstration using tools similar to those used in this paper.

Some justification can surely be given for working with a historically given equilibrium situation. Care must be taken, however, since the parameters of the economy shift in response to a policy change. If there is more than one equilibrium, then the problem becomes a dynamic one. If nothing else, a general set of conditions must be developed to ensure that the original equilibrium is locally stable under some realistic dynamic adjustment process. An even more complex problem arises as the parameters move through critical economies, where mathematical catastrophes can occur. Much work on these problems obviously remains to be done.

CLARE COLLEGE, CAMBRIDGE, ENGLAND

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