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Indeterminacy in Applied Intertemporal General Equilibrium Models

Timothy J. Kehoe and David K. Levine

1 Introduction

In recent years large-scale applied general equilibrium models have increasingly gained acceptance as a tool for policy analysis. To study issues such as social security schemes, the national debt, monetary policy and international exchange rates it is essential that these models have an explicitly dynamic component. Roughly, the goal is to build more realistic descriptive models incorporating the features found in more stylized dynamic models, such as those in Diamond (1965), Lucas (1972) and Kareken and Wallace (1981).

The most ambitious effort in this direction is the work of Auerbach, Kotlikoff and Skinner (1983). To analyze public finance issues, they build an empirical overlapping generations model. To calculate equilibria the model is truncated to effectively have a long finite horizon. Since this procedure is rather artificial, an obvious question that arises is how sensitive the calculated equilibrium is to the way in which the model is truncated. As we point out here, there is a close connection between this question and the question of whether the underlying infinite horizon model is determinate. Unfortunately, and in contrast with models with a finite number of infinitely lived agents, overlapping generations models may not have determinate equilibria (Kehoe and Levine, 1985).

That an overlapping generations model might have a continuum of equilibria is well known. When counting the equations and unknowns in his equilibrium conditions, Samuelson (1958) himself has noted that "we never seem to get enough equations: lengthening our time period turns out always to add as many new unknowns as it supplies equations" (see also Samuelson, 1960). Gale (1973) has extensively studied the overlapping generations model with a single two-period-lived consumer in each generation and one good in each period. In such a model he finds that indeterminacy is always associated with equilibria that have nonzero amounts of nominal debt. Such indeterminacy is always one dimensional; in other words the equilibria can be indexed by a single number, e.g. the price of fiat money. Balasko and Shell (1981) have extended these results to a model in which there are many goods in each period but a single two-period-lived consumer in each generation, in fact, a consumer with a Cobb–Douglas utility function. Geanakoplos and Polemarchakis (1984) and Kehoe and Levine (1984a) have extended these results to a model with a single two-period-lived consumer with intertemporally separable preferences in each generation. Kehoe et al. (1986) have extended them to a model in which demand satisfies the assumption of gross substitutes. Calvo (1978) has constructed examples in which the indeterminacy is still one dimensional, indexed by the price of an asset such as land or capital.

In this paper we consider pure exchange overlapping generations models with n goods in each period. We argue that for a model with a nonzero stock of nominal debt there is potentially an n-dimensional indeterminacy, while for a model with no nominal debt there is potentially an $(n - 1)$-dimensional indeterminacy. Thus relative prices within a period can be indeterminate. Although our results agree with those previously known for the case where there is one good in every period, they indicate that indeterminacy does not depend on the existance of fiat money or other assets. Furthermore, even in pure exchange models with no aggregate debt or assets, our results indicate that equilibria may be indeterminate or not whether or not they are Pareto efficient.

How far do we have to go to construct examples in which there are indeterminate equilibria without fiat money or indeterminate equilibria that are Pareto efficient? We present an example in which the only departure from the simple model considered by Gale is that the single consumer in each generation lives three, rather than two, periods. Gale himself considers such models and conjectures that the results he obtains for the two-period-lived model carries over to them. Unfortunately, we provide a robust example that demonstrates that this is not the case. This stands in fundamental contrast with the static pure exchange model, where, although it is always possible to construct examples with continua of equilibria, such examples cannot be robust. As we shall see, our three-period-lived consumer model can also be viewed as a model with two-period-lived consumers in each generation and two goods in each period.

Indeterminacy of relative prices is possible for very plausible parameter values. In fact, the value of the crucial parameter in our example, the elasticity of substitution in consumption over time, has been chosen to agree with the empirical evidence. (See Auerbach, Kotlikoff and Skinner (1983) and Mankiw, Rotemberg and Summers (1985) for summaries of this evidence.) Since the representative consumer in our example has a constant elasticity of substitution utility function, the value we have chosen, 0.2, allows some goods to be gross complements at some prices. In situations where goods are always gross substitutes we argue that indeterminacy of the type discussed in this paper is impossible. To guarantee gross substitutability, however, we need to set the elasticity of substitution greater than or equal to 1.0, which is an implausibly high value.

What does this mean for applied research such as that of Auerbach, Kotlikoff and Skinner (1983)? An obvious recommendation is to test the sensitivity of the model to terminal conditions. Our results indicate that this may be a substantial problem.

A second issue is empirical: indeterminacy is a property of overlapping generations models but not of models with finitely many infinitely lived dynasties (see Kehoe, Levine and Romer, forthcoming). Consequently the importance of the bequest motive, as discussed, for example, by Darby (1979) or Kotlikoff and Summers (1981), is crucial in whether or not sensitivity to endpoint conditions is likely to be important in practice. In addition, liquidity constraints, such as those of Bewley (1980, 1983), Scheinkman and Weiss (1986) or Levine (1989), tend to lead to overlapping-generations-type implications.

We begin by describing a simple stationary model and examining its steady state. We then study the behavior of equilibrium price paths around a steady state and characterize the dimensionality of paths that converge to the steady state. The second and third sections of this paper constitute a relatively nontechnical summary of the results of Gale (1973) and Kehoe and Levine (1984b, 1985). The fourth section presents a simple example of indeterminacy of relative prices. In the fifth section we argue that indeterminacy of equilibrium in the infinite model corresponding to acute sensitivity to terminal conditions in any truncated version of the model. In the sixth section we prove that indeterminacy is impossible if all goods are gross substitutes. In the seventh section we indicate how our results can be extended to models that have growing populations, models that are nonstationary for any finite number of periods and models that have equilibrium cycles. We conclude with a short discussion of some possible extensions of our results and their implications for applied work.

2 The Model and its Steady States

We begin by considering a model in which each generation lives two periods. As we shall explain, a model in which each generation lives more than two periods can be viewed as a special case of this model. Each generation $t \geq 1$ is identical and lives in periods t and $t+1$. There are n goods in each period. The vector $p_t = (p_t^1, \ldots, p_t^n)$ denotes prices in t. The consumption and savings decisions of the (possibly many different types of) consumers in generation t are aggregated into excess demand functions $y(p_t, p_{t+1})$ when young and $z(p_t, p_{t+1})$ when old; y and z are of course, n-dimensional vectors. Excess demands are assumed to be homogeneous of degree zero,

$$\begin{aligned} y(\theta p_t, \theta p_{t+1}) &\equiv y(p_t, p_{t+1}) \\ z(\theta p_t, \theta p_{t+1}) &\equiv z(p_t, p_{t+1}) \end{aligned} \tag{4.1}$$

for any $\theta > 0$, and to obey Walras's law

$$p_t' y(p_t, p_{t+1}) + p_{t+1}' z(p_t, p_{t+1}) \equiv 0 \tag{4.2}$$

Although the model is specified in terms of the excess demand functions y and z, it may be helpful to think of them as being derived from solutions to utility maximization problems of the form

$$\begin{aligned} &\text{maximize } u_h(y^h + w_1^h z^h + w_2^h) \\ &\text{subject to } p_t' y^h + p_{t+1}' z^h \leq 0 \\ &\qquad y^h \geq -w_1^h \qquad z^h \geq -w_2^h \end{aligned} \tag{4.3}$$

Here y^h is the vector of net trades made by consumer h in generation t when young, z^h is the vector of net trades when old, and w_1^h and w_2^h are endowment vectors. The aggregate excess demand functions are defined as

$$\begin{aligned} y(p_t, p_{t+1}) &= \sum_h y^h(p_t, p_{t+1}) \\ z(p_t, p_{t+1}) &= \sum_h z^h(p_t, p_{t+1}) \end{aligned} \tag{4.4}$$

The form of the budget constraint in (4.3), and the assumptions of homogeneity and Walras's law that correspond to it, are implicitly equivalent to the assumption that consumers are allowed to trade goods with each other, even if the goods are consumed in different time periods. One institutional story to go with this assumption is that we allow creation of private debt, or inside money, i.e. consumer h can be thought of as having two budget constraints, $p_t' y^h + m_t^h \leq 0$ and $p_{t+1}' z^h - m_t^h \leq 0$, where m_t^h is the amount of money that he carries over from period t to period $t+1$. We allow m_t^h to be negative, thus allowing borrowing. Adding these two budget constraints together eliminates m_t^h and yields the constraint in (4.3). The presence of public debt, or outside money, is a different matter, however, which, as we shall see, depends on initial conditions.

Viewing each consumer as facing a sequence of budget constraints provides us with an alternative way of viewing relative prices. The prices (p_t, p_{t+1}) considered above are the same prices that consumers face if there is a complete set of futures markets. Suppose instead that the consumer faces two budget constraints of the form

$$\begin{aligned} q_t'y^h + m_t^h &\leq 0 \\ q_{t+1}'z^h - m_t^h(1 + r_t) &\leq 0 \end{aligned} \tag{4.5}$$

where q_t and q_{t+1} are vectors of spot prices and r_t is the interest rate on borrowing and lending between periods t and $t + 1$. Again the budget constraints in (4.5) reduce to that in (4.3) if we set $p_t = q_t/(1 + r_1)(1 + r_2)\ldots(1 + r_{t-1})$.

We assume that excess demands are continuously differentiable for all strictly positive price pairs (p_t, p_{t+1}), which, as Debreu (1972) and Mas-Colell (1974) have shown, entails little loss of generality. We further assume that y and z are bounded from below and are such that some, but not all, prices approach zero, the sum of excess demand becomes unbounded; i.e. $e'[y(p_t, p_{t+1}) + z(p_t, p_{t+1})] \to \infty$ where e denotes the n vector whose every element is one. These assumptions are naturally satisfied when y and z are derived from utility maximization: if consumption of every good by every consumer must be non-negative, then an obvious lower bound for (y, z) is $(-w_1, -w_2)$ where w_1 and w_2 are the aggregate endowment vectors. If preferences are monotonically increasing in consumption, then when a single price goes to zero the excess demand for that good becomes infinite. Furthermore, if more than one price goes to zero, then excess demand for some, but perhaps not all, of the corresponding goods becomes infinite (see Arrow and Hahn, 1971, pp. 29–31).

Debreu (1974) has demonstrated that, for any y and z that satisfy the assumptions of homogeneity and Walras's law, there exists a generation of $2n$ utility maximizing consumers whose aggregate excess demand functions y^* and z^* agree with y and z on that set of positive relative prices uniformly bounded away from zero. There is a minor technical complication in that y^* and z^* may not agree with y and z as some relative prices approach zero. Utilizing a result due to Mas-Colell (1977), however, Kehoe and Levine (1984b) argue that we can ignore this qualification when studying the behavior of the excess demand functions near steady states. Consequently, for our purposes we are jusified in viewing our assumptions as both necessary and sufficient for demand functions derived from utility maximization by heterogeneous consumers. As we shall see, however, the possibility of indeterminacy of relative prices in overlapping generations models does not depend on implausible aggregate excess demand functions or even on heterogeneity among consumers within a generation.

In addition to the consumers born in periods 1, 2, . . ., there is an initial old generation alive in period 1. Its excess demand function $z_0(p_1, m)$ depends on prices p_1 in the first period and the nominal savings m of the initial old generation. These savings can be thought of as outside or fiat money, at least if $m \geqslant 0$. Implicitly, as m changes, the distribution of savings between consumers in the initial old generation changes in a fixed way. We assume that z_0 is continuously differentiable, is homogeneous of degree zero,

$$z_0(\theta p_1, m) \equiv z_0(p_1, m) \tag{4.6}$$

for any $\theta > 0$, and obeys Walras's law,

$$p_1' z_0(p_1, m) \equiv m \tag{4.7}$$

An equilibrium for this economy is an initial level of nominal savings m and a price path $(p_1, p_2, \ldots)$ for which excess demand vanishes in each period:

$$z_0(p_1, m) + y(p_1, p_2) = 0 \tag{4.8}$$

for $t = 1$ and

$$z(p_{t-1}, p_t) + y(p_t, p_{t+1}) = 0 \tag{4.9}$$

for $t > 1$. Repeated application of the equilibrium conditions and Walras's law implies that $-p_t' y(p_t, p_{t+1}) = p_{t+1}' z(p_t, p_{t+1}) = m$ at all times. Consequently, m is the fixed nominal net savings made by the young generation in each period. If m is non-negative it is easy to interpret it as fiat money. Even if it is negative, however, there are institutional stories to accompany it: think of an institution that makes loans to consumers when they are young and uses the repayment of these loans to make loans to consumers in the next generation.

A steady state of this economy is a relative price vector p and an inflation factor β such that $p_t = \beta^{t-1} p$ satisfies

$$z(\beta^{t-2} p, \beta^{t-1} p) + y(\beta^{t-1} p, \beta^t p) = z(p, \beta p) + y(p, \beta p) = 0 \tag{4.10}$$

Here $r = 1/\beta - 1$ is the steady state rate of interest. Notice that a steady state is not necessarily an equilibrium price path because it may not satisfy the equilibrium condition (4.8) in the first period. There are two types of steady states: real steady states in which $m = -p'y(p, \beta p) = 0$ and monetary, or nominal, steady states in which $m \neq 0$. On the one hand, Walras's law implies that $p'(y + \beta z) = 0$ and, consequently, that $\beta p'z = m$. On the other hand, the steady state condition (4.10) implies that $p'(y + z) = 0$ and, consequently, that $p'z = m$. Therefore $(\beta - 1)m = 0$, and any nominal steady state must have $\beta = 1$. Gale calls steady states in which $\beta = 1$ golden rule steady states because they maximize a weighted sum of individual utility functions subject to the constraint of stationary consumption over time. He calls real steady states balanced.

It is possible to construct examples in which a golden rule steady state is also balanced, i.e. in which $m = 0$ and $\beta = 1$ simultaneously. Such a steady state must satisfy $z(p, p) + y(p,p) = 0$ and $-p'y(p, p) = 0$. Walras's law implies that this is a system of n independent equations; homogeneity implies that there are $n - 1$ independent unknowns. Consequently we would expect this system of equations to have a solution only by coincidence. In fact, Kehoe and Levine (1984b) prove that almost all economies do not have a steady state where both $m = 0$ and $\beta = 1$. They give the space of economies (y, z) that satisfy the assumptions of differentiability, homogeneity, Walras's law and the boundary condition a topological structure: two economies are close to each other if the values of demand functions and the values of their partial derivatives are close. The phrase "almost all" in this context means that the property holds for a subset that is open and dense: any sufficiently small perturbation of an economy that does not have a steady state where $m = 0$ and $\beta = 1$ results in an economy that still does not have such a steady state; for any economy that has such a steady state, however, there exist arbitrarily small perturbations that result in economies that do not have such steady states. A property that holds for almost all economies is called a generic property.

Gale proves that the model with a single two-period-lived consumer in each generation has a unique nominal steady state and, generically, a unique real steady state. The unique nominal steady state is where the price of the single good is constant over time. Walras's law implies that this situation does indeed satisfy the steady state condition (4.10). At this steady state the savings of the young person are not, in general, zero. Since there is only one consumer in each generation any trade that takes place must be between generations. Consequently, since there is only one good in each period, there can be trade only if there is a corresponding transfer of nominal debt from period to period. Any real steady state must therefore be given by a relative price ratio $\beta = p_{t+1}/p_t$ that makes the consumer prefer not to trade. Such a price ratio obviously exists; generically there is only one.

With many consumers in each generation but only one good in each period, nominal steady states are still unique but real steady states need not be: consider a static pure exchange economy with two consumers and two goods that has multiple, but determinate, equilibria. Robust examples of this sort are, of course, easy to construct (see, for example, Shapley and Shubik, 1977). Now construct an overlapping generations economy by assigning two such consumers to each generation and by letting one of the goods by available in the first period of their lives and the other in the second. Each of the different equilibria of the static ecomony now corresponds to a different real steady state of the overlapping generations economy in which the two consumers in each generation trade with each other but not with other generations. With many goods and many consumers neither real steady states nor nominal steady states need by unique. Kehoe and Levine (1984b) prove, however, that generically there exists an odd number of each type. Their arguments are similar to those used to prove that the number of equilibria of pure exchange economy is odd (see, for example, Varian, 1974). In similar vein, Kehoe et al. (1986) show that, if y and z satisfy the assumption of gross substitutes, then there is a unique steady state of each type.

Using a result due to Balasko and Shell (1980), we are able to examine the efficiency properties of steady states (see also Burke, 1987). Balasko and Shell consider models with a single consumer in each generation that satisfy a uniform curvature condition on indifference surfaces that is quite natural in a stationary setting such as ours. They demonstrate that a necessary and sufficient condition for an equilibrium price path of such a model to be Pareto efficient is that the infinite sum $\Sigma 1/\|p_t\|$ does not converge. (Here, of course, $\|\cdot\|$ is the ordinary Euclidean norm: $\|p_t\| = (p_t'p_t)^{1/2}$.) This result can easily be extended to models with many consumers in every generation. Consequently, a steady state of our model is Pareto efficient if and only if $\beta \leq 1$, in other words, if and only if the interest rate is non-negative. Price paths that converge to steady states where $\beta \leq 1$ are Pareto efficient; those that converge to steady states where $\beta > 1$ are not.

Every economy has a Pareto-efficient steady state since it always has a steady state where $\beta = 1$. In the model with one two-period-lived consumer with gross substitutes excess demands in each generation and one good in each period, Gale finds that the unique real state has $\beta < 1$ if and only if the unique nominal steady state has $m < 0$. Similarly, $\beta > 1$ at the real steady state if and only if $m > 0$ at the nominal steady state. In the more general model we cannot make such strong statements. We can, however, demonstrate that every economy has an odd number of steady states where $\beta \leq 1$ and $m \leq 0$. We sketch an argument below; details are given by Kehoe and Levine (1984b). This argument also makes it clear why every economy has an odd number of real steady states and an odd number of nominal steady states.

Consider the n functions

$$f_i(p, \beta) = y^i(p, \beta p) + z^i(p, \beta p) - \frac{p'[y(p, \beta p) + z(p, \beta p)]}{e'p} \quad (4.11)$$

Notice that, for any fixed $\beta > 0$, that the functions $f_i(p, \beta)$ have the formal properties of excess demand functions of a static pure exchange economy: they are homogeneous of degree zero in p and satisfy Walras's law, $p'f(p, \beta) \equiv 0$. Consequently, for any $\beta > 0$ there exists at least one value of p that solves the equations

$$f_i(p, \beta) = 0 \qquad i = 1, \ldots, n \quad (4.12)$$

Walras's law implies that this is a system of $n - 1$ independent equations; homogeneity implies that there are $n - 1$ independent unknowns. Consequently, it can easily be shown that solutions to this system of equations are generally smooth implicit functions of β. Our assumptions on the behavior of excess demand as some prices tend toward zero guarantee that there exists $\underline{\beta} > 0$ and $\overline{\beta} > \underline{\beta}$ such that any steady state value of β satisfies $\underline{\beta} < \beta < \overline{\beta}$.

There are two distinct ways for a pair (p, β) that satisfies (4.12) to be a steady state: if $-p'y(p, \beta) = 0$ or if $\beta = 1$. In either case Walras's law implies that $p'[y(p, \beta p) + \beta z(p, \beta p)] = 0$. Let

$$m(p, \beta) = -p'y(p, \beta p) \quad (4.13)$$

for any (p, β), and consider pairs m and β such that $m = m(p, \beta)$ and (p, β) satisfies (4.12). Unfortunately, m is not, in general, a well-defined function of β since for any β there may be more than one p such that (4.12) is satisfied. We are justified, however, in drawing diagrams like that in figure 4.1. There are a finite number of paths of pairs m and β that satisfy our conditions. Some of them are loops that do not intersect the boundary $\beta = \underline{\beta}$ or $\beta = \overline{\beta}$. It is possible, however, to demonstrate that generically there are an odd number of points of the form $(p, \underline{\beta})$ where (4.12) is satisfied and, similarly, an odd number of the form $(p, \overline{\beta})$. To make this plausible recall that the functions $f_i(p, \beta)$ have the properties of excess demand functions of a static pure exchange economy. It is well known that generically (in this case, for almost all β) there are an odd number of equilibria of such an economy. An even number, possibly zero, of pairs m and $\underline{\beta}$ are associated with paths that return to the same boundary $\beta = \underline{\beta}$. An odd number, at least one, therefore cannot return. Our boundary assumption implies that for $\underline{\beta} \leq \beta \leq \overline{\beta}$, the prices that satisfy (4.12) are uniformly bounded away from zero. Consequently, $m(p, \beta)$ remains bounded, and paths that start at $\beta = \underline{\beta}$ and do not return must eventually reach the boundary $\beta = \overline{\beta}$. Any path or loop may intersect itself, but it does not, in general, do so where $\beta = 1$ (or $\underline{\beta}$ or $\overline{\beta}$) or where $m = 0$.

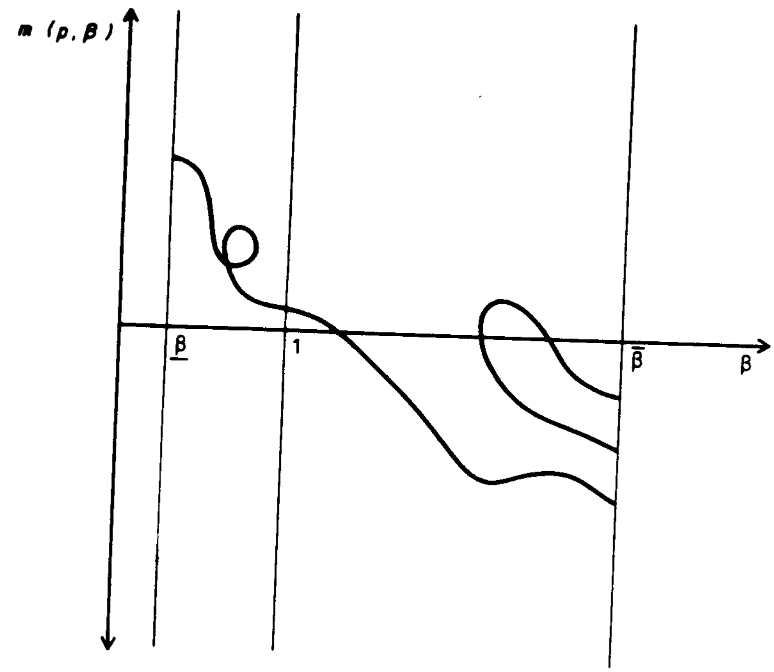


Figure 4.1

The boundary assumptions also imply that $m(p, \beta) > 0$ when $\beta = \underline{\beta}$ and $m(p, \beta) < 0$ when $\beta = \overline{\beta}$. Consequently, any path that starts at $\underline{\beta}$ and ends at $\overline{\beta}$ must intersect the line $m = 0$ an odd number of times. Similarly, any such path must intersect the line $\beta = 1$ an odd number of times. On the other hand, every loop or path that starts and ends at the same boundary intersects both $m = 0$ and $\beta = 1$ an even, possibly zero, number of times. Each of these intersections corresponds to a steady state. Generically, there is none where both $m = 0$ and $\beta = 1$. Experimenting with different possibilities, we can easily verify that any admissible graph must share with that in figure 4.1 the property that there are an odd number of steady states where $\beta \leq 1$ and $m \geq 0$ and an odd number where $\beta \geq 1$ and $m \leq 0$.

This theory has particularly strong implications when both types of steady states are unique. As we have remarked above, Kehoe et al. shows that this is always the case with gross substitutes. From the diagram we see that the situation is the same as Gale's one good case. The real steady state has $\beta < 1$ if and only if $m < 0$ at the nominal steady state, and $\beta > 1$ at the real steady state if and only if $m > 0$ at the nominal steady state.

3 Determinacy of Equilibrium Price Paths

We now focus our attention on the behavior of equilibrium price paths near a steady state. In addition to the requirement that markets clear in every period we require that prices converge to the steady state, i.e. that $(p_t, p_{t+1})/\|(p_t, p_{t+1})\| \rightarrow (p, \beta p)/\|(p, \beta p)\|$ at $t \rightarrow \infty$. We do this for two reasons. First, price paths that begin near and converge to a steady state are the most plausible perfect foresight equilibria. Agents can compute future prices using only local information. If prices are not converging to a steady state, however, then agents need global information and very large computers to compute future prices. Second, these price paths are the easiest to study: to determine the qualitative behavior of price paths near a steady state we can linearize the equilibrium conditions. Paths that do not converge may display very complex periodic, or even chaotic, behavior. It may be difficult to distinguish such paths from price sequences that satisfy the equilibrium conditions for a very long time but eventually lead to prices that are zero or negative, where excess demands explode, making a continuation of the sequence impossible.

Determinacy of equilibrium price paths that converge to a steady state may still leave room for indeterminacy. There may be paths that do not converge to a steady state but nevertheless always remain strictly positive and are therefore legitimate equilibria. Whether a model has a determinate path that converges to a steady state is a weak test. We shall establish, however, that there are robust examples of economies that fail even this test.

Consider again the conditions that an equilibrium price path must satisfy:

$$z_0(p_1, m) + y(p_1, p_2) = 0 \tag{4.14}$$

for $t = 1$ and

$$z(p_{t-1}, p_t) + y(p_t, p_{t+1}) = 0 \tag{4.15}$$

for $t > 1$. Once p_1 and p_2 are determined (4.15) acts as a nonlinear difference equation determining the rest of the price path. We begin by asking how many pairs (p_1, p_2) give rise to the price path that converges to a steady state $(p, \beta p)$. The stable manifold theorem from the theory of dynamic systems described, for example, by Irwin (1980) implies that, near $(p, \beta p)$, these questions can generically by answered by linearizing (4.15). We then ask how many pairs (p_1, p_2) are consistent with equilibrium in the first period. This question can be answered by linearizing (4.14). Pairs (p_1, p_2) that lie in the intersection of these two sets correspond to equilibrium price paths.

The dimension of this intersection can be deduced by a simple counting argument. If this dimension is greater than zero, there is a continuum of equilibrium price paths. If it is greater than unity, relative prices are indeterminate: not even by exogenously specifying the price level can we make price paths determinate. Details of the arguments presented below are given by Kehoe and Levine (1985).

Making use of the fact that derivatives of excess demand are homogeneous of degree minus one and that $(p, \beta p)$ is a steady state, we can write the linearized system as

$$z_0(p, m) + y(p, \beta p) + D_2y(p_2 - \beta p) + (D_1y + D_1z_0)(p_1 - p) = 0 \tag{4.16}$$

$$D_2y(p_{t+1} - \beta^t p) + (D_1y + \beta D_2z)(p_t - \beta^{t-1}p) + \beta D_1z(p_{t-1} - \beta^{t-2}p) = 0 \tag{4.17}$$

where D_1y is, for example, the matrix of partial derivatives of y with respect to its first vector of arguments and where all derivatives are evaluated at $(p, \beta p)$. β shows up as a coefficient of D_1z and D_2z in (4.17) because homogenity implies that $D_1z(\beta^{t-2}p, \beta^{t-1}p) = \beta^{2-t}D_1z$ $(p, \beta p)$, for example, while $D_1y(\beta^{t-1}p, \beta^t p) = \beta^{1-t}D_1y$ $(p, \beta p)$. Differentiating the homogeneity assumption (4.1) with respect to θ and evaluating the result at $\theta = 1$ and $(p_t, p_{t+1}) = (p, \beta p)$ results in

$$\begin{aligned} D_1yp + \beta D_2yp &= 0 \\ D_1zp + \beta D_2zp &= 0 \end{aligned} \tag{4.18}$$

Consequently, the linearized equilibrium conditions (4.16) and (4.17) can be rewritten as

$$D_2yp_2 + (D_1y + D_1z_0)p_1 = D_1z_0p - z_0(p, m) - y(p, \beta p) \tag{4.19}$$

$$D_2yp_{t+1} + (D_1y + \beta D_2z)p_t + \beta D_1zp_{t-1} = 0 \tag{4.20}$$

Kehoe and Levine (1984b) have shown that D_2y is generically nonsingular. Consequently, (4.20) can be solved for an explicit second-order difference equation. Using a standard trick, we can write this equation as the first-order system $q_t = Gq_{t-1}$ where $q_t = (p_t, p_{t+1})$ and

$$G = \begin{bmatrix} 0 & I \\ -\beta D_2y^{-1}D_1z & -D_2y^{-1}(D_1y + \beta D_2z) \end{bmatrix} \tag{4.21}$$

The stability properties of this difference equation are governed by the eigenvalues of G. The homogeneity assumption (4.18) implies that β is an eigenvalue of G since

$$G = \begin{bmatrix} p \\ \beta p \end{bmatrix} = \begin{bmatrix} \beta p \\ \beta^2 p \end{bmatrix} \tag{4.22}$$

Differentiating Walras's law (4.2) and evaluating the result at $(p_t, p_{t+1}) = (p, \beta p)$ implies that

$$\begin{aligned} y(p, \beta p)' + p'D_1 y + \beta p' D_1 z &= 0 \\ z(p, \beta p)' + p'D_2 y + \beta p' D_2 z &= 0 \end{aligned} \tag{4.23}$$

Consequently, unity is another eigenvalue since (4.23) and the steady state condition (4.10) imply that

$$p'[-\beta D_1 z \quad D_2 y]G = p'[-\beta D_1 z \quad D_2 y] \tag{4.24}$$

In the case where $\beta = 1$ these are generically the same restriction, and we have information about only one eigenvalue.

It should now be clear why the case where $n = 1$ is so special: if homogeneity and Walras's law are the only restrictions that economic theory imposes on the dynamical system (4.15), then we can expect to be able to pin down at most two eigenvalues of (4.21). In the case where $n = 1$ this is all the eigenvalues, but in general there are $2n - 2$ left to be determined by the parameters of the model. In fact, Kehoe and Levine (1984b) prove that for any pattern of $2n$ eigenvalues, as long as (4.22) and (4.24) are satisfied and complex numbers come in conjugate pairs, there exists (y, z) that satisfies all our assumptions and gives rise to a matrix G with those eigenvalues.

We are interested in the set of initial conditions $q_1 = (p_1, p_2)$ for the linear difference equation $q_t = Gq_{t-1}$ for which $q_t/\|q_t\| \to q/\|q\|$ where $q = (p, \beta p)$. This set is a subspace of $\mathbb{R}^{2n}$ determined by the eigenvectors of G. Suppose that $(x_i, y_i) \in \mathbb{C}^{2n}$ is the eigenvector in complex $2n$-dimensional space associated with the eigenvalue λ_i:

$$\begin{bmatrix} 0 & I \\ -\beta D_2 y^{-1} d_1 z & -D_2 y^{-1}(D_1 y + \beta D_2 z) \end{bmatrix} \begin{bmatrix} x_i \\ y_i \end{bmatrix} = \lambda_i \begin{bmatrix} x_i \\ y_i \end{bmatrix} \tag{4.25}$$

The first n equations state that $y_i = \lambda_i x_i$. If the eigenvalues of G are distinct, a condition that Kehoe and Levine (1984b) prove holds generically, then solutions to the difference equation $q_t = G(q_{t-1})$ take the form

$$\begin{bmatrix} p_t \\ p_{t+1} \end{bmatrix} = \sum_{i=1}^{2n} c_i \lambda_i^{t-1} \begin{bmatrix} x_i \\ \lambda_i x_i \end{bmatrix} \tag{4.26}$$

where the complex constants $c_1, \ldots, c_{2n}$ are determined by the initial conditions

$$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \sum_{i=1}^{2n} c_i \begin{bmatrix} x_i \\ \lambda_i x_i \end{bmatrix} \tag{4.27}$$

To ensure convergence to the steady state $(p, \beta p)$, we need to put a positive weight c_1 on the eigenvector $(x_1, \lambda_1 x_1) = (p, \beta p)$ and zero weights c_i on eigenvectors $(x_i, \lambda_i x_i)$ for which the modulus $|\lambda_i|$ is greater than β, i.e. we require that β is the dominant eigenvalue in (4.26).

Let n^s be the number of eigenvalues of G that are less than β in modulus. Reorder the eigenvalues so that $(x_2, \lambda_2 x_2), \ldots, (x_{n^s+1}, \lambda_{n^s+1} x_{n^s+1})$ are the associated eigenvectors. Since eigenvectors associated with distinct eigenvalues are linearly independent and since complex eigenvectors show up in conjugate pairs, the n^s+1 eigenvectors $(p, \beta p), (x_2, \lambda_2 x_2), \ldots, (x_{n^s+1}, \lambda_{n^s+1} x_{n^s+1})$ span an $(n^s + 1)$-dimensional subspace of $\mathbb{R}^{2n}$. Initial values (p_1, p_2) that lead to convergence to the steady state have the form

$$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = c_1 \begin{bmatrix} p \\ \beta p \end{bmatrix} + \sum_{i=2}^{n^s+1} c_i \begin{bmatrix} x_i \\ \lambda_i x_i \end{bmatrix} \tag{4.28}$$

Besides yielding a path that converges to the steady state ray, (p_1, p_2) must also satisfy the linearized equilibrium condition (4.19) in the first period. Let us first examine the situation where $m \neq 0$. In this case z_0 is not homogeneous of degree zero in p_1 alone, and, since $p_1' D_1 z_0(p_1, m) p_1 \equiv -p_1 z_0(p_1, m) \equiv -m \neq 0$, $D_1 z_0 p \neq 0$; fiat money, in fact, operates as numeraire. Consequently, (4.19) defines an n-dinemsional affine subset of prices (p_1, p_2) consistent with equilibrium in the first period. The intersection of this subset with the subspace of prices that yield a path that converges to the steady state generically has dimension $(n^s + 1) + n - 2n = n^s + 1 - n \; (\leqslant n)$. Roughly speaking, we say that $2n - n^s - 1$ of the $2n$ variables (p_1, p_2) are pinned down by the requirement of convergence to a steady state in (4.20) and n are pinned down by the equilibrium condition (4.19); this leaves $n^s + 1 - n$ variables free, which can be as many as n if $n^s = 2n - 1$.

These are several cases of interest. First, if $n^s < n - 1$, then generically there are no equilibrium paths that converge to this steady state. We call such a steady state unstable. Second, if $n^s = n - 1$, then stable equilibrium price paths are locally unique and, in a small enough neighborhood of the steady state, actually unique. We call such a steady state determinate. Third, if $n^s < n - 1$, then there is a continuum of locally stable paths. In fact, the (p_1, p_2) that generate these paths form a manifold of dimension $n^s + 1 - n$. We call such a steady state indeterminate. The $(n^s + 1 - n)$-dimensional affine subset of the corresponding linear system is, in fact, the tangent space to this manifold at $(p, \beta p)$, i.e. its best linear approximation.

Let us now consider the situation where $m = 0$. Since z_0 is now homogeneous of degree zero in p_1, $D_1 z_0 p = 0$. There are two considerations that reduce the dimension of the subspace of initial conditions that we are concerned with. First, since the equilibrium conditions (4.19) and (4.20) are now homogeneous of degree zero in p, p_1 and p_2, we can impose a price normalization and work in a $(2n-1)$-dimensional affine subset of $\mathbb{R}^{2n}$, for example, by setting $p_1^1 = 1$. This allows us to fix the weight put on the eigenvector $(p, \beta p)$ in (4.27). Second, since $m = 0$, the initial price pair (p_1, p_2), and all subsequent pairs (p_t, p_{t+1}), must satisfy $p_1'y(p_1, p_2) = 0$. This implies that (p_1, p_2) cannot put any weight on the eigenvector (x_i, x_i) associated with the eigenvalue $\lambda_i = 1$ in (4.27).

To illustrate the latter point, let us linearize the restriction $-p_1'y(p_1, p_2) = m$ at $(p, \beta p)$:

$$-p'y - (y' + p'D_1y)(p_1 - p) - p'D_2y(p_2 - \beta p) = m \quad (4.29)$$

Homogeneity (4.18) and Walras's law (4.23) imply that we can simplify this to

$$\beta p'D_1zp_1 - p'D_2yp_2 = m \quad (4.30)$$

Suppose that $(x_i, \lambda_i x_i)$ is an eigenvector of G. Then (4.24) implies that

$$\begin{aligned}[\beta p'D_1z \quad -p'D_2y]\begin{bmatrix} x_i \\ \lambda_i x_i \end{bmatrix} &= [\beta p'D_1z \quad -p'D_2y]G\begin{bmatrix} x_i \\ \lambda_i x_i \end{bmatrix} \\ &= \lambda_i[\beta p'D_1z \quad -p'D_2y]\begin{bmatrix} x_i \\ \lambda_i x_i \end{bmatrix} \end{aligned} \quad (4.31)$$

Consequently, for all $\lambda_i \neq 1$,

$$[\beta p'D_1z \quad -p'D_2y]\begin{bmatrix} x_i \\ \lambda_i x_i \end{bmatrix} = 0 \quad (4.32)$$

Premultiplying (4.27) by $[\beta p'D_1z \quad -p'D_2y]$, we obtain

$$\begin{aligned}[\beta p'D_1z \quad -p'D_2y]\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} &= \sum_{i=1}^{2n} c_i[\beta_p'D_1z \quad -p'D_2y]\begin{bmatrix} x_i \\ \lambda_i x_i \end{bmatrix} \\ m &= c_j[\beta p'D_1z \quad -p'D_2y]\begin{bmatrix} x_j \\ x_j \end{bmatrix} \end{aligned} \quad (4.33)$$

where (x_j, x_j) is the eigenvector associated with $\lambda_j = 1$. This implies that $c_j = 0$ if and only if $m = 0$. (Notice that in case where $m \neq 0$ and $\beta = 1$ it implies that $c_j = 1$ since $(x_j, x_j) = (p, p)$.)

Suppose that $m = 0$. Let $\bar{n}^s$ denote the number of eigenvalues of G that satisfy $|\lambda_i| < \beta$ excluding $\lambda_i = 1$ if $\beta > 1$. The set of prices $q_1 = (p_1, p_2)$ that satisfy $-p_1'y(p_1, p_2) = 0$ and the price normalization and give rise to a price path that converges to $q = (p_1, \beta p_2)$ forms an $\bar{n}^s$-dimensional set. The set of prices $q_1 = (p_1 \quad p_1)$ that satisfy $-p_1'y(p_1, p_2) = 0$ and the price normalization and are consistent with equilibrium in the first period forms an $(n - 1)$-dimensional set. Equilibrium price paths are associated with points in the intersection of these two sets, which generically has dimension $\bar{n}^s + (n - 1) - (2n - 2) = \bar{n}^s = 1 - n\ (< n - 1)$.

Although the eigenvalue $\lambda_i = 1$ is irrelevant for price paths in which $m = 0$, it is crucial for the behavior of paths where $m \neq 0$: if $m \neq 0$ initially, then the price path cannot converge to a steady state where $\beta < 1$, since $\lambda_i = 1$ must receive nonzero weight in (4.2) but λ_i is an unstable root. This makes good economic sense: there can be no equilibrium in which $m \neq 0$ that converges to a steady state with $\beta < 1$. The exponential deflation would cause the constant nominal money stock to become infinite in real terms. Although paths with $m \neq 0$ can converge to a steady state where $\beta > 1$, asymptotically the real money stock disappears because of inflation.

A warning should be given about the generic nature of our results. Although they hold for almost all economies, it is possible to think of examples that violate them: when there is a single two-period-lived consumer with an intertemporally separable utility function in each generation, for example, both D_2y and D_1z have rank one. Consequently, if there are two or more goods, we cannot invert D_2y. In this case, Kehoe and Levine (1984a) demonstrate that the situation is essentially the same as that in a model with only one good in each period: with nominal initial conditions there is at most a one-dimensional indeterminacy, and with real initial conditions no indeterminacy is possible. These results are, of course, closely related to those of Balasko and Shell (1981) and Geanakoplos and Polemarchakis (1984) cited earlier.

4 An Example with Three-period-lived Consumers

To construct an example of overlapping generations model with relative price indeterminacy, we cannot look at a model with one good per period and two-period-lived consumers, at a model with a single two-period-lived consumer with intertemporally separable utility in each generation, nor at a model with gross substitutes. In this section we consider an example with the simplest possible structure that allows relative price indeterminacy: there is a single good in each period and a single consumer with additively separable preferences in each generation who lives for three rather that two periods and who has constant elasticity of substitution (CES) preferences with an elasticity of substitution of 0.2. Balasko, Cass and Shell (1980) present a simple procedure for converting such a model into one in which consumers live two periods. Suppose that consumers live for k periods. Redefine generations $-k + 2, -k + 1, \ldots, 0$ to be generation 0, generations 1, 2, . . ., $k-1$ to be generation 1, and so on. Similarly, redefine periods. This procedure is illustrated in figure 4.2 for the case where $k = 3$. Notice that the number of goods in each period and the number of consumers in each generation both increase by a factor of $k - 1$.

In this transformation $k - 1$ period cycles become steady states. In fact, we can use this transformation to reduce the study of paths that converge to cycles of any finite length to the study of paths that converge to steady states. It is still the case in the transformed model that there are generically an odd number of steady states of each type, real and nominal. By performing the transformation one period at a time, we are able to demonstrate that the original model generically has an even number of cycles of each type of every length: we know that there are an odd number of steady states; performing the transformation for $k = 3$, we know that there are an

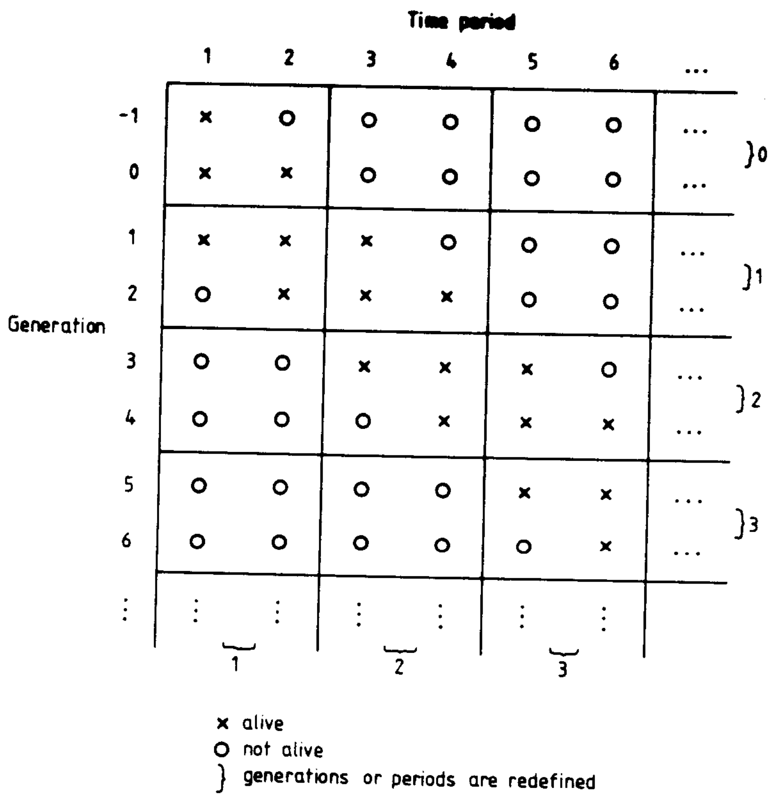


Figure 4.2

odd number of steady states and two cycles, and hence an even number, possibly zero, of two cycles; and so on. This result tells us nothing about the cycles of even length. If (p_1, p_2, p_1, p_2) is a nominal steady state, for example, then so is (p_2, p_1, p_2, p_1). It does, however, place restrictions on cycles of odd length.

Consider an economy in which the single consumer born in period t, where $t = 1, 2, \ldots$, lives for three periods and has the utility function

$$u(c_1, c_2, c_3) = \frac{1}{b}(a_1 c_1^b + a_2 c_2^b + a_2 c_3^b) \tag{4.34}$$

where $a_1, a_2, a_3 > 0$ and $b < 1$. This is, of course, the constant elasticity of substitution utility function with elasticity of substitution $\eta = 1/(1-b)$. If the consumer faces the budget constraint

$$p_t c_1 + p_{t+1} c_2 + p_{t+2} c_3 = p_t w_1 + p_{t+1} w_2 + p_{t+2} w_3 \tag{4.35}$$

where (w_1, w_2, w_3) is his endowment stream, then his excess demand functions are

$$x_j(p_t, p_{t+1}, p_{t+2}) = \frac{a_j^\eta \sum_{i=1}^{3} p_{t+i-1} w_i}{p_{t+j-1}^\eta \sum_{i=1}^{3} a_i^\eta p_{t+i-1}^{1-\eta}} - w_j \qquad j = 1, 2, 3 \tag{4.36}$$

Notice that these functions are continuously differentiable for all strictly positive prices, are homogeneous of degree zero, and obey Walras's law:

$$\begin{aligned} p_t x_1(p_t, p_{t+1}, p_{t+2}) &+ p_{t+1} x_2(p_t, p_{t+1}, p_{t+2}) \\ &+ p_{t+2} x_3(p_t, p_{t+1}\ p_{t+2}) = 0 \end{aligned} \tag{4.37}$$

In addition to these consumers, there are two others, an old consumer who lives only in period 1 and a middle-aged consumer who lives in periods 1 and 2. The old consumer, consumer -1, derives utility only from consumption of the single good in the first period, so we need not specify a utility function. We endow him with m_{-1} units of fiat money, which may be positive, negative, or zero. His excess demand function is

$$x_3^{-1}(p_1, m_{-1}) = m_{-1}/p_1 \tag{4.38}$$

The middle-aged consumer, consumer 0, has the utility function

$$u_0(c_2, c_3) = \frac{1}{b}(a_2 c_3^b + a_3 c_3^b) \tag{4.39}$$

an endowment stream (w_2^0, w_3^0) of goods, and an endowment m_0 of fiat money. His excess demand functions are

$$x_j^0(p_1, p_2, m_0) = \frac{a_j^\eta\left(\sum_{i=1}^{2} p_i w_{i+1}^0 + m_0\right)}{p_{j-1}^\eta \sum_{i=1}^{2} a_{i+1}^\eta p_i^{1-\eta}} - w_j^0 \qquad j = 2, 3 \quad (4.40)$$

The equilibrium conditions for this economy are

$$x_3^{-1}(p_1, m_{-1}) = x_2^0(p_1, p_2, m_0) + x_1(p_1, p_2, p_3) = 0 \quad (4.41)$$

for $t = 1$,

$$x_3^0(p_1, p_2, m_0) + x_2(p_1, p_2, p_3) + x_1(p_2, p_3, p_4) = 0 \quad (4.42)$$

for $t = 2$ and

$$x_3(p_{t-2}, p_{t-1}, p_t) + x_2(p_{t-1}, p_t, p_{t+1}) + x_1(p_t, p_{t+1}, p_{t+2}) = 0 \quad (4.43)$$

for $t > 1$.

By analogy with the two-period case, let $m = m_{-1} + m_0$. Multiplying the equilibrium conditions in the first two periods by the respective prices and adding yields an analog of (4.7):

$$m = -p_t x_1(p_1, p_2, p_3) - p_2 x_2(p_1, p_2, p_3) - p_2 x_1(p_2, p_3, p_4) \quad (4.44)$$

Repeated application of Walras's law (4.37) and the equilibrium condition (4.43) implies that

$$m = -p_t x_1(p_t, p_{t+1}, p_{t+2}) - p_{t+1} x_2(p_t, p_{t+1}, p_{t+2}) - p_{t+1} x_1(p_{t+1}, p_{t+2}, p_{t+3}) \quad (4.45)$$

for all t: just as in the two-period-lived model, the amount of fiat money stays constant over time.

Steady states also have the same structure as in the two-period-lived model. There are two types, real steady states in which $\beta \neq 1$ and

$$m = -x_1(1, \beta, \beta^2) - \beta x_2(1, \beta, \beta^2) - \beta x_1(1, \beta, \beta^2) = 0 \quad (4.46)$$

and nominal steady states in which, $\beta = 1$ and $m \neq 0$.

In the three-period case, the linearized equilibrium conditions are

$$\beta^2 D_1 x_3 p_{t-2} + (\beta^2 D_2 x_3 + \beta D_1 x_2) p_{t-1} + (\beta^2 D_3 x_3 + D_1 x_1) p_t + (\beta D_3 x_2 + D_2 x_1) p_{t+1} + D_3 x_1 p_{t+2} = 0 \quad (4.47)$$

Here all derivatives are evaluated at $(1, \beta, \beta^2)$. The eigenvalues are the roots of the corresponding fourth-order polynominal. These come from a 4×4 matrix like that in (4.29).

Consider the following parameter values:

Period	1	2	3
a_i	2	2	1
w_i	3	15	2

where $b = -4$. Notice that the representative consumer discounts consumption over time, has a hump in his life cycle earnings profile and has an elasticity of substitution in consumption over time of $1/(1 + 4) = 0.2$. This economy has one nominal steady state and three real steady states. They can be found by tracing out the graph of the function $m(\beta)$ given by (4.46). Real steady states occur where $m(\beta) = 1$. This is illustrated in figure 4.3.

To determine the values of the roots of the forth-order polynominal that corresponds to (4.47) we start by evaluating the partial derivatives of the excess demand functions (4.40) at $(p_t, p_{t+1}, p_{t+2}) = (1, \beta, \beta^2)$. At $\beta = 1$, for example, these derivatives are

$$\begin{bmatrix} D_1x_1 & D_2x_1 & D_3x_1 \\ D_1x_2 & D_2x_2 & D_3x_2 \\ D_1x_3 & D_2x_3 & D_3x_3 \end{bmatrix} = \begin{bmatrix} -2.29010 & 3.28375 & -0.99365 \\ -0.89664 & 1.89029 & -0.99365 \\ -0.78057 & 2.85867 & -2.07810 \end{bmatrix} \tag{4.48}$$

(Notice that, since this matrix has some negative off-diagonal elements, (x_1, x_2, x_3) violates gross substitutability.) The polynomial that we are interested in is

$$-0.78057 + 1.96230\lambda - 2.47791\lambda^2 + 2.29010\lambda^3 - 0.99365\lambda^4 = 0 \tag{4.49}$$

One of the roots is, of course, $\lambda = 1$. The other three are 0.93286, $0.18594 + 0.89862i$ and $0.18594 - 0.89862i$, as can easily be verified.

The roots at all four steady states are listed in table 4.1. The modulus of the pair of complex conjugates at the steady state where $\beta = 0.93295$ is 0.84513; where $\beta = 1$ it is 0.91766.

Let us first focus our attention on the steady state where $\beta = 0.93295$. Let $m_{-1} = m_0 = 0$ and let $w_2^0 = 10.84636$ and $w_3^0 = 2$. It is straightforward but tedious to check that $(p_1, p_2, p_3, p_4) = (1, 0.93295, (0.93295)^2, (0.93295)^3)$ satisfies the conditions for equilibrium in the first two periods. Since $\beta = 0.93295$ is a steady state, this is a legitimate equilibrium price path. Our earlier arguments imply that this is only one continuum. Since $m_{-1} = m_0 = 0$, the excess demands of generations -1 and 0 are homogeneous of degree zero and we can normalize prices by setting $p_1 = 1$. We can then choose

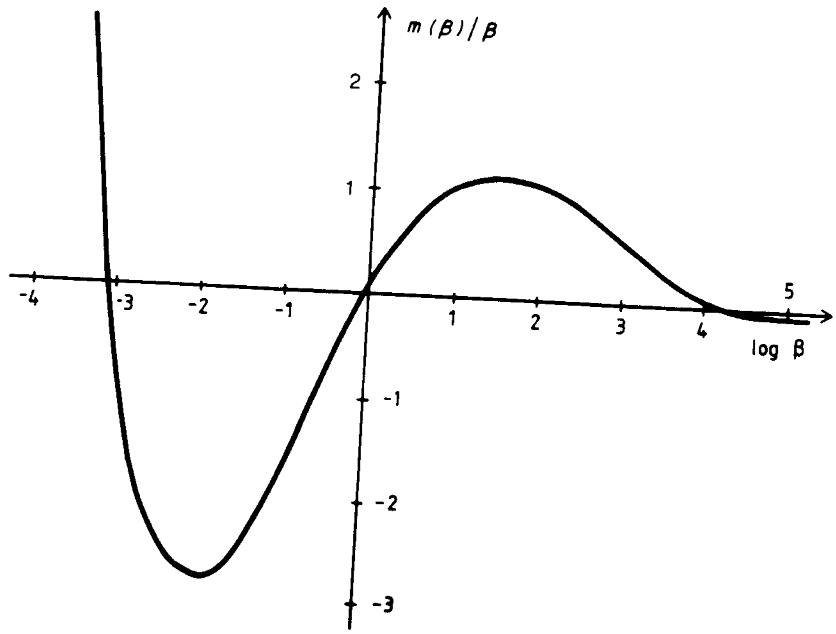


Figure 4.3

Table 4.1

	β		*Other roots*	
1	0.04239	1	0.47907	−0.01380
2	0.93295	1	0.17245 ± 0.82735i	
3	1	0.93286	0.18594 ± 0.89862i	
4	53.8056	1	2.04305	−121.45544

$p_2/p_1 = 0.93295 + \varepsilon$ for any small ε, positive or negative, and use the equilibrium conditions (4.41) and (4.42) to solve for p_3 and p_4. Using the equilibrium condition (4.43), we can then solve for an infinite price sequence. This price sequence must converge to unity where $p_{t+1}/p_t = 0.93295$ since the modulus of the root governing stability is less than 0.93295. The root $\lambda = 1$ is, as we have explained, irrelevant since $m = 0$ everywhere along this price path.

In fact, the value of ε need not be very small: every p_2/p_1 in the interval $0.26703 < p_2/p_1 \leqslant 16.67676$ determines a distinct equilibrium. Figure 4.4 illustrates some possibilities. Notice that

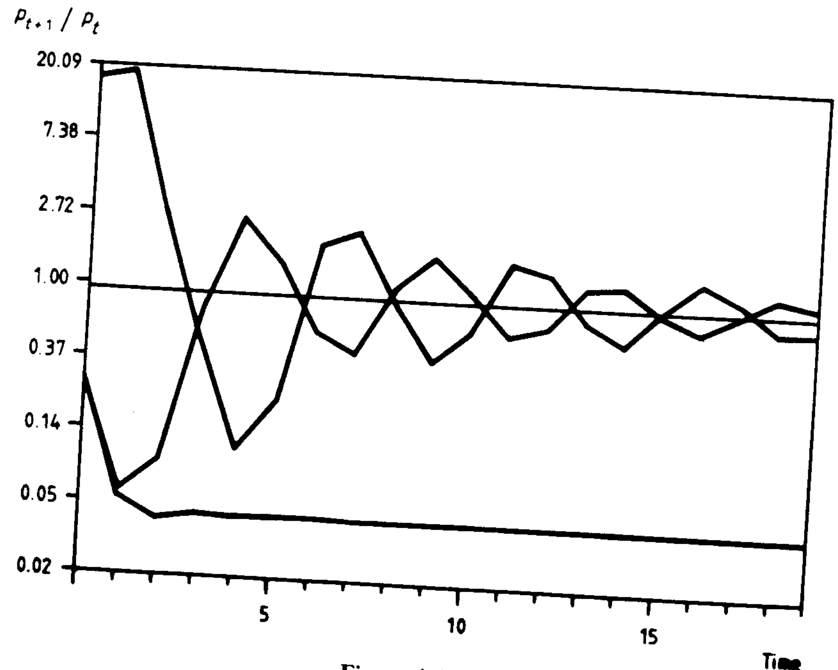


Figure 4.4

$p_2/p_1 = 0.26703$ determines an equilibrium that converges to the steady state where $\beta = 0.04239$. Otherwise, all values of p_2/p_1 outside this interval determine paths that eventually lead to a negative price.

Now consider the case where $m_{-1} = -0.08825$ and $m_0 = 0.18634$. Here it can be checked that $(p_1, p_2, p_3, p_4) = (1, 1, 1, 1)$ satisfies (4.41) and (4.42). In this case the excess demands of generations -1 and 0 are not homogeneous and we are not permitted a price normalization: money itself serves as numeraire. We can now choose $p_1 = 1 + \varepsilon_1$ and $p_2 = 1 + \varepsilon_2$ for any ε_1 and ε_2 small enough and use (4.41) and (4.42) to solve for p_3 and p_4. Again using (4.43), we can solve for an equilibrium price sequence that converges to unity where $p_{t+1}/p_t = 1$.

All these equilibria are Pareto efficient: those that converge to the steady state where $\beta = 0.93295$ all assign a finite value to the aggregate endowment, and so the standard proof of the first welfare theorem due to Debreu applies. Those that converge to the steady state where $\beta = 1$ satisfy the more general conditions for efficiency developed by Balasko and Shell (1980) and Burke (1987).

The relative price indeterminacy exhibited in this example does not need one of the consumers to come into the first year with negative fiat money. Suppose, for example, that $m_{-1} = 0$, that $m_0 = 0.09809$, and that $w_2^0 = 10.93461$ and $w_3^0 = 2$. Then $(p_1, p_2, p_3, p_4) = (1, 1, 1, 1)$ satisfies the equilibrium conditions in the first two periods. Again there is a two-dimensional indeterminacy. (Setting $m_{-1} = m_0 = 0$ does not, however, result in equilibrium conditions that are satisfied by $p_t = (0.93295)^{t-1}$.)

Notice that this example also has steady states of the more familiar sort: any equilibrium that converges to the steady state where $\beta = 0.04239$ is determinate. Any equilibrium that converges to $\beta = 53.80562$ and has no fiat money is also determinate. There is a one-dimensional manifold of paths that converge to the steady state if there is fiat money, however.

Choosing the parameters of this type of model suitably, we can illustrate other possibilities for behavior of equilibrium price paths near steady states. For example, the following parameter values correspond to an economy with four steady states with βs and other roots that are the reciprocals of those given above:

Period	1	2	3
a_i	1	2	2
w_i	2	15	3

where $b = -4$. Here the steady state where $\beta = 1$ is unstable: there are no paths that can approach it unless, by pure chance, $(1, 1, 1, 1)$ satisfies the equilibrium conditions in the first two periods. The steady state where $\beta = 1.07187 = (0.93295)^{-1}$ is also unstable for price paths with no nominal debt. There are, however, determinate price paths with nonzero nominal debt that converge to this steady state.

An essential feature of all the above examples is that they are robust: we can slightly perturb the parameters, and even the functional forms, of the demand functions of all the consumers, including the initial old consumers, and still have economy whose equilibria have the same qualitative features. We choose initial old consumers so that the steady state prices satisfy the equilibrium conditions in the first two periods only to make it easy to verify that there are prices that satisfy these equilibrium conditions and also converge to the steady state.

As we have explained, the examples of this section are special cases of a model with two two-period-lived consumers in each generation and two goods in each period. The main reason for using the three-period-lived model for examples is to keep the specifications as simple as possible. Suppose that consumer h, where $h = 1, 2$, in generation t solves the problem

$$\text{maximize} \frac{1}{\gamma_h} [\alpha_1^{1h}(c_t^1)^{\gamma_h} + \alpha_1^{2h}(c_t^2)^{\lambda_h} + \alpha_2^{1h}(c_{t+1}^1)^{\lambda_h} + \alpha_2^{2h}(c_{t+1}^2)^{\lambda_h}]$$

$$\text{subject to} \quad \sum_{j=1}^{2} \sum_{k=1}^{2} p_{t=K-1}^j c_{t+k-1}^j \leqslant \sum_{j=1}^{2} \sum_{k=1}^{2} p_{t+k-1}^j w_k^{jh} \tag{4.50}$$

If we set $\gamma_1 = \gamma_2 = b$, $\alpha_1^{11} = \alpha_1^{22} = a_1$, $\alpha_1^{21} = \alpha_2^{12} = a_2$, $\alpha_2^{11} \alpha_2^{22} = a_3$ and $\alpha_2^{21} = \alpha_1^{12} = 0$, and similarly set w_k^{jh}, then this model is formally the same as the three-period-lived model. While this model needs 18 parameters to specify it, of which 15 are not subject to normalization, the simple three-period-lived model needs only seven, of which five are not subject to normalization. It is still the case, however, that any small perturbation in the parameters of the two-period-lived model results in an economy whose equilibria have the same qualitative features as do the examples that we have presented.

5 Implications for Finite Horizon Models

There is a close relationship between models with infinite time horizons and models with long, but finite, time horizons. On the one hand, models with infinite horizons are only interesting in so far as they provide insights into the properties of finite horizon models. On the other hand, to approximate the equilibria of an infinite horizon model on a computer we would have to truncate the model after a finite number of periods. In this section we explore the relationship between equilibria of the finite horizon model and equilibria of the truncated model.

One way to truncate the model at period T would be to fix the expectations of what prices would be in period $T + 1$. Suppose that (p, β) is a steady state. We could require that $p_{T+1} = \beta p_T$ in the terminal equilibrium condition

$$z(p_{T-1}, p_T) + y(p_T, p_{T+1}) = 0 \tag{4.51}$$

or that $p_{T+1} = \|p_T\| \beta p$. (See Auerbach, Kotlikoff and Skinner (1983) for an example of this approach.) Another, sometimes equivalent, way to truncate the model would be to specify a terminal young generation y_T analogous to the initial old generation z_0.

This type of truncated model, which involves a finite number of variables in the same number of equilibrium conditions, generically has determinate equilibria. Furthermore, if the truncation date T is large enough, then an equilibrium of the truncated model serves as a reasonable approximation to an equilibrium of the actual model, at least in the early periods. In fact, the usual proof of the existence of equilibrium for the infinite horizon overlapping generations model depends on this property of the truncated model. We start by established the existence of equilibrium for the truncated model. We then construct a candidate for an equilibrium price path for the infinite horizon model by augmenting a vector of equilibrium prices $(p_1, p_2, \ldots, p_T)$ for the truncated model with prices $p_{T+1}, p_{T+2}, \ldots$ that are arbitrary. Letting the truncation date increase, we generate an infinite sequence of such price paths. Since we can use the boundary assumption on excess demand to prove that the space of possible price paths is compact in the product topology, this sequence has a convergent subsequence. Furthermore, since the equilibrium conditions are continuous in this topology, any limit point of the sequence is an equilibrium for the infinite horizon model (see, for example, Balasko, Cass and Shell 1980). There may, of course, be different limit points because of multiple equilibria, and that is why we need to consider subsequences converging.

Here convergence means convergence in the product topology: for any $\varepsilon > 0$ and any T there exists some $T' > T$ such that, if $(p_1, p_2, \ldots)$ is an infinite price path that satisfies the equilibrium conditions for the model truncated at T', then $(p_1, p_2 \ldots)$ is within ε of an equilibrium of the infinite horizon model for periods $1, 2, \ldots, T'$. Notice that we do not claim that this price path is within ε of an equilibrium of the infinite horizon model for periods $T + 1$, $T + 2, \ldots,$ T $'$. Nor, unfortunately, can we make any strong statements about the relationship between T, T'. We do not know how long a truncation date is needed to approximate an equilibrium for ten periods, for example.

Two questions about this approximation naturally arise. How does indeterminacy in the infinite horizon model, where $\bar{n}^s > n - 1$, manifest itself in the truncated model? How does instability, where $\bar{n}^s < n - 1$, manifest itself? To answer the first question, let us consider an infinite horizon model with a continuum of equilibria that converge to the same steady state $(p, \beta p)$. Choose two different price paths in that continuum, $(\bar{p}_1, \bar{p}_2, \ldots)$ and $(\hat{p}_1, \hat{p}_2, \ldots)$. For large enough T both $\bar{p}_{T+1}$ and $\hat{p}_{T+1}$ are very close to the steady state price vector p and, consequently, very close to each other. (We can think of normalizing the price sequence not by requiring that $p_1^1 = 1$ but rather by requiring that $p_{T+1}^1 = 1$.) By truncating the model at period T using the terminal condition (4.51) with $p_{T+1} = \bar{p}_{T+1}$, we generate $(\bar{p}_1, \bar{p}_2, \ldots, \bar{p}_T)$ as an equilibrium; with $p_{T+1} = \hat{p}_{T+1}$, we generate $(\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_T)$. No matter how large the difference between $\bar{p}_1$ and $\hat{p}_1$ there exists a T large enough so that $\bar{p}_{T+1}$ and $\hat{p}_{T+1}$ are arbitrarily close. Indeterminacy in the infinite horizon model can therefore be seen to manifest itself as sensitivity to terminal conditions in the truncated model, sensitivity that becomes more and more acute as the truncation date becomes larger and larger.

Another way to view this problem of indeterminacy credited by Calvo (1978) to Rolf Mantel is to consider the difference equation

$$z(p_{T-1-s}, p_{T-s}) + y(p_{T-s}, p_{T+1-s}) = 0 \qquad (4.52)$$

where $s = 1, 2, \ldots$, that runs backwards from the terminal conditions. It is trivial to show that the eigenvalues of the linearized versions of this difference equation are the reciprocals of the eigenvalues of the original system. If the original system has too many stable eigenvalues, the backwards system has too few and is therefore unstable: small changes in terminal conditions cause large changes in prices early in the price path.

Let us now turn our attention to the second question, the question of how instability manifests itself. Here we seem to be faced with a dilemma: We know that, if we truncate the model by requiring that $p_{T+1} = \|p_T\|\beta p$, we can compute an approximate equilibrium for the infinite horizon model. We also know, however, that it is extremely unlikely that for the infinite horizon model will have an equilibrium price path where p_{T+1} is close to these values. The solution to this dilemma lies in the nature of the approximation. We only know that the equilibria of the truncated model are close to the equilibria of the actual model in early periods; later they may diverge sharply. To approximate the equilibria of an infinite horizon model near an unstable steady state for any fixed number of periods we may have to choose a very large truncated date.

As we would expect, the problems of indeterminacy and instability represent two sides of the same coin. Indeterminacy manifests itself as sensitivity to terminal conditions. The larger the truncation date, the more sensitive prices early in the price path are to terminal conditions. Later prices, however, which all converge to the steady state, are relatively insensitive. Instability, in contrast, manifests itself as a need for a very large truncation date. The larger the truncation date is, the less sensitive prices early in the price path are to terminal conditions. Later prices, however, may diverge sharply from equilibrium prices for the actual model.

To see how indeterminacy manifests itself in a truncated model, let us consider again the simple three-period-lived model of the previous section. Figure 4.5 illustrates three equilibria that all converge to the steady state where $\beta = 0.93925$. The values of the price ratios p_{t+1}/p_t at these three equilibria are listed in table 4.2. Each of these three equilibria could be generated as an equilibrium of a truncated model with a suitable choice of a truncation date and terminal condition p_{T+1}/p_T. If, for example, we truncate the model at $T = 20$ and

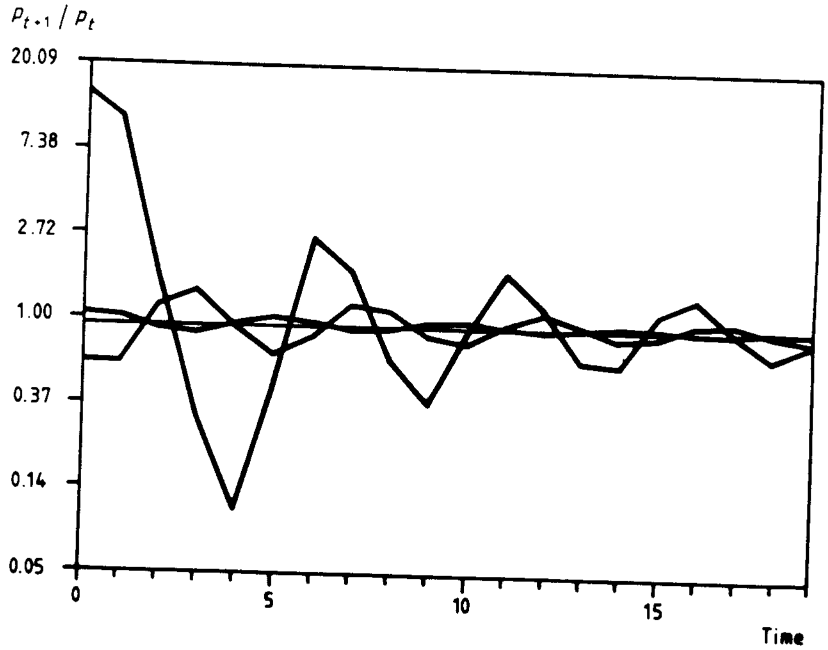


Figure 4.5

Table 4.2

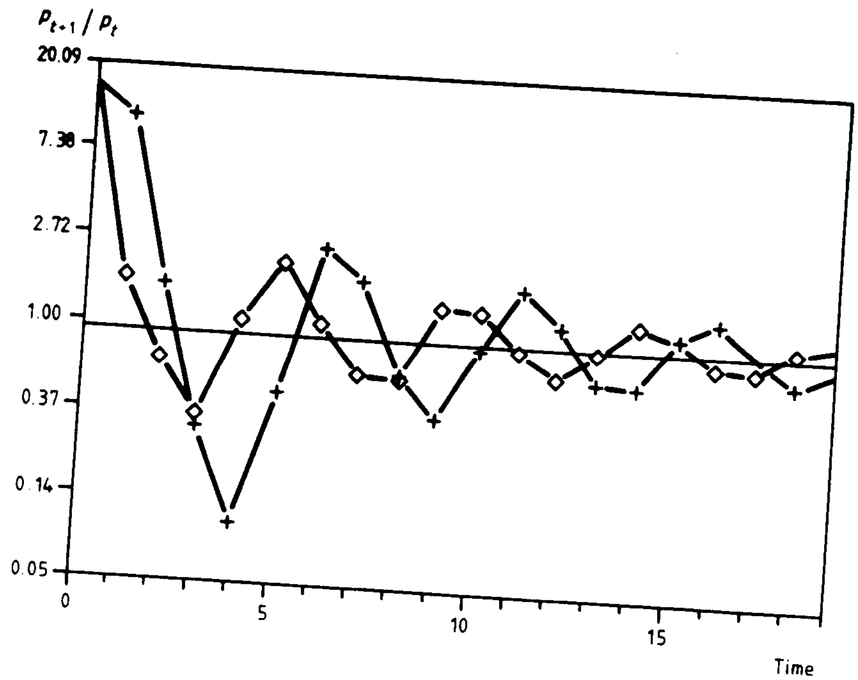
	P_{t+1}/P_t		
1	0.60000	1.05000	15.00000
2	0.59283	1.01905	10.76314
3	1.6288	0.88803	1.59627
4	1.38525	0.83980	0.30447
5	0.91629	0.94206	0.10258
6	0.66010	1.01934	0.47445
7	0.81814	0.95572	2.58841
8	1.18550	0.87554	1.77892
9	1.11307	0.89228	0.64069
10	0.82698	0.96736	0.38120
15	0.82304	0.96522	0.60597
20	0.85527	0.95375	0.81664
25	0.89286	0.94270	0.94927
30	0.92050	0.93544	0.99605

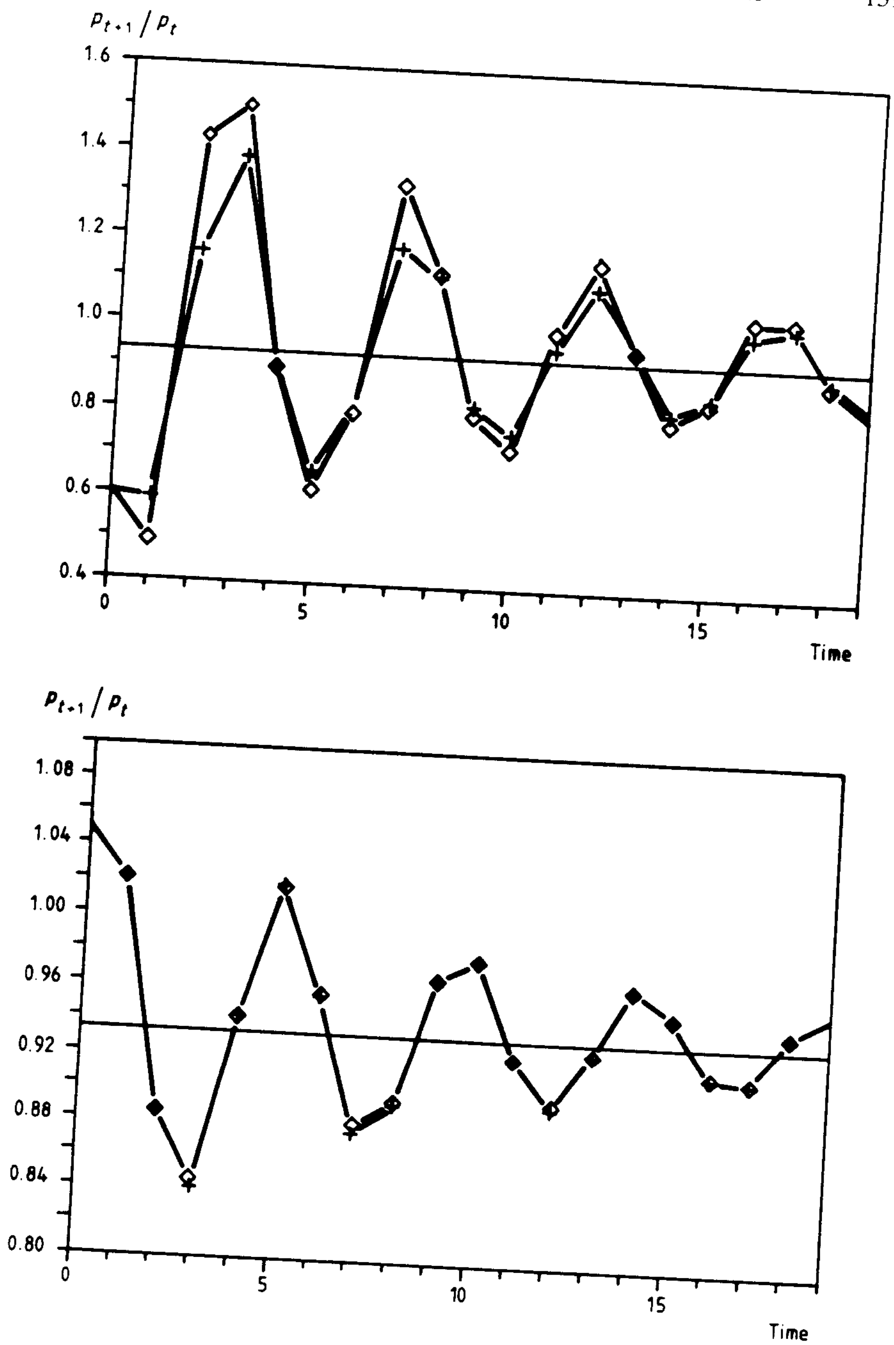
impose the terminal conditions $p_{21} = 0.85527p_{20}$, then we generate the first equilibrium. The terminal conditions $p_{21} = 0.95375p_{20}$ and

$p_{21} = 0.81664p_{20}$, however, result in the second and third equilibria respectively. Notice that very small differences in the terminal conditions cause large differences early in the price paths. Notice too that this sensitivity to terminal conditions is more acute if we truncate at $T = 30$ than it is if we truncate at $T = 20$.

One way to test a model for indeterminacy is to linearize the equilibrium conditions at the steady state of interest and to compute the eigenvalues of the matrix of the corresponding linear difference equation. A solution to the linearized equilibrium conditions provides an approximation to an equilibrium of the original model. Furthermore, this approximation becomes more and more accurate the closer the equilibrium is to the steady state. Figure 4.6 illustrates each of the three equilibria of figure 4.5 along with the corresponding solutions to the linearized system.

Another way to test for indeterminacy is to choose a large truncation date to vary the terminal conditions. Small changes in terminal conditions producing large changes in initial prices are symptoms of indeterminacy. Although this test is the simpler of the two to perform, evaluating its results is more difficult. What, for example, is a suitably larger truncated date? What works for one





p_{t+1}/p_t
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0
5
10
15
Time
p_{t+1}/p_t
1.08
1.04
1.00
0.96
0.92
0.88
0.84
0.80
0
5
10
15
Time

Figure 4.6

model may not work for another. This is because one model may have eigenvalues of its linearized system that are very close to the steady state values of β in modulus while the other may not.

6 Gross Substitutability and Determinacy

Although the overlapping generations model admits robust examples with indeterminate equilibria, such as that in the previous section, it also admits robust examples with determinate equilibria. The natural question is how to distinguish those parameter values for which the model has indeterminate equilibria from those for which it has determinate equilibria. The answer given in the previous section involves choosing numerical values for the parameters, linearizing the equilibrium conditions at the steady state of interest and computing eignvalues. In this section we explore one class of examples for which all this work does not have to be done: if the excess demand functions y and z exhibit gross substitutability, an easily checked condition, then the model has determinate equilibria if there is no nominal debt and a one-dimensional indeterminacy at most if there is nominal debt. See Kehoe et al. (1986) for a more detailed discussion.

In a static pure exchange model with n goods we say that an excess demand function $f(p)$ exhibits gross subtitutability if an increase in the price of goods i, all other prices remaining the same, causes the excess demand for all but good i to rise. If f is continuously differentiable, this condition holds in a neighborhood of price vector p if $(\partial f_j/\partial p_i)(p) > 0$ for $j \neq i$. In an overlapping generations model this condition holds near a steady state $(p, \beta p)$ if all the off-diagonal elements of the $2n \times 2n$ matrix

$$\begin{bmatrix} D_1 y & D_2 y \\ D_1 z & D_2 z \end{bmatrix}$$

are positive. The excess demand function in the example in the previous section does not exhibit gross substitutability since the matrix (4.48) of partial derivatives has negative off-diagonal elements. Beware that when the three-period-lived model is transformed into a two-period-lived model with two goods per period some of these off-diagonal elements become zero and strict gross subtitutability is lost. Since the analysis of this section can easily by extended to handle this case of weak gross substitutability with a mild indecomposability requirement, however, we shall ignore this minor technical detail (see, for example, Arrow and Hurwicz 1960).

The off-diagonal elements of the matrix of partial derivatives of the demand functions (4.36) used in the previous section take the form

$$D_k x_j(p_t, p_{t+1}, p_{t+2})$$
$$= \frac{a_j^\eta p_{t+j-1}^\eta \left[w_k \sum_{i=1}^{3} a_i^\eta p_{t+i-1}^{1-\eta} + (\eta - 1) a_k^\eta p_{t+k-1}^{-\eta} \sum_{i=1}^{3} p_{t+i-1} w_i \right]}{\left(p_{t+j-1}^\eta \sum_{i=1}^{3} a_i^\eta p_{t+i-1}^{1-\eta} \right)^2} \tag{4.53}$$

Notice that, if $w_k > 0$, this expression is always positive when $\eta > 1$, when $\eta < 1$, however, it may become negative. This is a well known feature of demand functions derived from CES utility functions, including functions that nest CES utility functions for consumption within a single period (such as those used by Auerbach, Kotlikoff and Skinner, 1983): to guarantee gross substitutability, the elasticity of substitution between any two goods must be greater than or equal to one. Unfortunately, a large number of empirical studies indicate that the elasticity of substitution in consumption over time is substantially lower: somewhere between 0.07 and 0.51 centered around approximately 0.2 (see Auerbach, Kotlikoff and Skinner, 1983; Mankiw, Rotemberg and Summers, 1985). An excess demand function arising from a CES utility function with $\eta < 1$ may, in fact, exhibit gross substitutability near a steady state $(p, \beta p)$, but this must be checked for by evaluating all the off-diagonal partial derivatives of y and z.

To clarify the role that gross substitutability plays in guaranteeing determinacy of equilibria in overlapping generations models, let us recall why it guarantees that a static pure exchange model has a unique equilibrium (see, for example, Arrow and Hahn, 1971, pp 221–7). Suppose, to the contrary, that an excess demand function f exhibits gross substitutability and has two equilibria, two strictly positive vectors $\bar{p}$ and $\hat{p}$, not proportional to each other, such that $f(\bar{p}) = f(\hat{p}) = 0$. Consider the ratios $\bar{p}_i/\hat{p}_i$, where $i = 1, \ldots, n$; let v be the largest such ratio, and suppose it is achieved for good k. Homogeneity implies that $f_k(v\hat{p}) = 0$. Now raise every price $v\hat{p}_i$ for which $v\hat{p}_i < \bar{p}_i$ until it equals $\bar{p}_i$. Since $\hat{p}$ is not proportional to $\bar{p}$, there is at least one such price. Gross substitutability implies that, as each of these prices is raised, f_k increases, which implies that $f_k(\bar{p}) > 0$. This contradicts the assumption that both $\bar{p}$ and $\hat{p}$ are equilibria. Consequently, f cannot have multiple equilibria.

Let us now apply this argument to the overlapping generations model. We first establish that, in the case with no nominal debt, the equilibria that converge to a steady state $(p, \beta p)$ are determinate. We start by proving a slightly different result: if z_0, as well as y and z, exhibits gross substitutability, then there is at most one equilibrium that converges to $(p, \beta p)$. Suppose, to the contrary, that z_0, y and z

all exhibit gross substitutability and that there are two such equilibria, two sequences of price vectors $(\bar{p}_1, \bar{p}_2, \ldots)$ and $(\hat{p}_1, \hat{p}_2, \ldots)$, not proportional to each other, that satisfy the equilibrium conditions and converge to the same steady state. Again consider the ratios $\bar{p}^i_t/\hat{p}^i_t$, where $i = 1, \ldots, n$ and $t = 1, 2, \ldots$. Since $\bar{p}_t$ and $\hat{p}_t$ both converge to the same steady state, $\lim_{t\to\infty}\bar{p}^i_t/\hat{p}^i_t = \lim_{t\to\infty}\|\bar{p}_t\|/\|\hat{p}_t\|$, for $i = 1, \ldots, n$. This limit may be the maximum ratio, it may be the minimum ratio, or it may be neither. It cannot, however, be both the maximum and the minimum since the two sequences are not proportional. Consequently, either the maximum or the minimum must be achieved for some $v = \bar{p}^k_t/\hat{p}^k_t$. If $t>1$, then we consider the equation

$$z^k(p_{t-1}, p_t) + y^k(p_t, p_{t+1}) = 0 \tag{4.54}$$

Homogeneity implies that $z^k(vp_{t-1}, vp_t) + y^k(vp_t) = 0$. If v is the maximum ratio, raises every price $vp^i_\tau < \bar{p}^i_\tau$ where $\tau = t-1, t, t+1$. If v is the maximum, then gross substitutability implies that $z^k(\bar{p}_{t-1}, \bar{p}_t) + y^k(\bar{p}_t, p_{t+1}) > 0$; if v is the minimum, then this expression is negative. Either situation contradicts the assumption that $(\bar{p}_1, \bar{p}_2, \ldots)$ is an equilibrium. If $t = 1$, then we apply the same argument to the equation

$$z^k_0(p_1, 0) + y^k(p_1, p_2) = 0 \tag{4.55}$$

Consequently, there cannot be multiple equilibria that converge to the same steady state.

This result implies that, if y and z exhibit gross substitutability, then equilibria that converge to the same steady state are determinate, regardless of z_0: our earlier discussion indicates that we can generically rule out the possibility of indeterminacy if we can generically rule out the possibility that the matrix G in the linearized equilibrium conditions satisfies $\bar{n}^s > n-1$. Suppose, to the contrary, that y and z are such that G has $\bar{n}^s > n-1$. Let us construct an excess demand function z_0 such that $z_0(p, 0) + y(p, \beta p) = 0$. Consider the Cobb–Douglas functions

$$z^j_0(p_1, 0) = a_j \frac{\sum_{i=1}^{n} p^i_1 w^i}{p^j_1} - w^j \qquad j = 1, \ldots, n \tag{4.56}$$

Choose $\zeta > 0$ large enough so that $\zeta > -y^j(p, \beta p)$, for $j = 1, \ldots, n$. Set $a_j = p^j/\Sigma^n_{i=1}p^i$, where $(p^1, \ldots, p^n)$ is the steady state price vector, and set $w^j = \zeta + y^j(p, \beta p)$. It is now trivial to check that, for all p_1, $(\partial z^i_0/\partial p^j_1)(p_1) > 0$ for $i \neq j$ and that $(p, \beta p, \beta^2 p, \ldots)$ is an equilibrium for this economy. Since $\bar{n}^s > n-1$, we know from our previous analysis that there is a continuum of equilibria. This is a contradiction, however, since we have already established that there can be at most one equilibrium.

Let us now argue that, in the case with nominal debt, there is at most a one-dimensional set of equilibria that converge to the same steady state. The argument is similar to the previous one. We start by proving that, if we fix the ratio between the price of that money and some index $\pi(p_1)$ of prices in the first period, $m = \pi(p_1)\mu$ where μ is a nonzero constant, then there is at most one equilibrium that converges to a given steady state. Let $\pi(p_1)$ be any price index, that is positive and homogeneous of degree one, e.g. $\pi(p_1) = p_1^1$. We now say that z_0 exhibits gross substitutability if there exsists such a price index such that an increase in p_1^i causes $z_0^j[p_1, \pi(p_1)m]$, $j \neq i$, to rise. An argument identical with that above establishes that, for any given m, there is at most one sequence $(p_1, p_2, \ldots)$ that converges to a given steady state and satisfies the equilibrium conditions. As we vary μ, however, we may allow additional equilibria. This implies that there is at most a one-dimensional set of equilibria and, again arguing as above, that it must be the case that $n^s \leqslant n$. We now use the excess demand function

$$z_0^j(p_1, m) = a_j \frac{\sum_{i=1}^{n} p_1^i w^i + m}{p_1^j} w^j \tag{4.57}$$

and again set $a_j = p^j / \Sigma p^i$ and $w^j = \zeta + y^j(p, \beta p)$.

7 Nonstationary Models

Thus far we have only considered stationary models and equilibrium price paths that converge to steady states of such models. In this section we indicate how our results can be extended to encompass models with some nonstationary structure and equilibrium price paths that converge to cycles of any finite length.

Let us first explain how our results can be extended to models with a constant rate of population growth. Suppose the demands of generations t are

$$\begin{aligned} y_t(p_t, p_{t+1}) &= \alpha^{t-1} y(p_t, p_{t+1}) \\ z_t(p_t, p_{t+1}) &= \alpha^{t-1} z_1(p_t, p_{t+1}) \end{aligned} \tag{4.58}$$

Here $\alpha - 1$ is the rate of population growth. With a suitable redefinition of prices and excess demand functions, this model can be transformed into one we have been working with. Let $\hat{p}_t = \alpha^{t-1} p_t$, $\hat{y}(p_t, p_{t+1}) = \alpha y_1(p_t, \alpha p_{t+1})$ and $\hat{z}(p_t, p_{t+1}) = z(p_t, \alpha p_{t+1})$. Notice that $\hat{y}$ and $\hat{z}$ are homogeneous of degree zero if y_1 and z_1 are.

Notice too that if y_1 and z_1 satisfy Walras's law

$$p_t'y_1(p_t, p_{t+1}) + p_{t+1}'z_1(p_t, p_{t+1}) = 0 \tag{4.59}$$

then so do $\hat{y}$ and $\hat{z}$:

$$\begin{aligned} 0 &= \alpha^{1-t}\hat{p}_t'y_1(\alpha^{1-t}\hat{p}_t, \alpha^{-t}\hat{p}_{t+1}) + \alpha^{-t}\hat{p}_{t+1}'z_1(\hat{p}_t, \alpha^{-t}\hat{p}_{t+1}) \\ &= \hat{p}_t'\hat{y}(\hat{p}_t, \hat{p}_{t+1}) + \hat{p}_{t+1}'\hat{z}(\hat{p}_t, \hat{p}_{t+1}) \end{aligned} \tag{4.60}$$

Finally, notice that if p_t, $t = 1, 2, \ldots$ satisfies the equilibrium conditions

$$0 = z_{t-1}(p_{t-1}, p_t) + y_t(p_t, p_{t+1}) \tag{4.61}$$

then $\hat{p}_t$, $t = 1, 2, \ldots$, satisfies the corresponding conditions

$$\begin{aligned} 0 &= \alpha^{t-2}z_1(\alpha^{2-t}\hat{p}_{t-1}, \alpha^{1-t}\hat{p}_t) + \alpha^{t-1}y_1(\alpha^{1-t}\hat{p}_t, \alpha^{-t}\hat{p}_{t+1}) \\ &\hat{z}(\hat{p}_{t-1}, \hat{p}_t) + \hat{y}(\hat{p}_t, \hat{p}_{t+1}) \end{aligned} \tag{4.62}$$

This transformation is obviously invertible: if we know $\hat{p}_t$, $\hat{y}$ and $\hat{z}$, and the growth factor α, we can recover p_t, y_t and z_t. Nominal steady states are those where $\hat{p}_t = \hat{p}_{t+1}$, which is equivalent to $p_t = \alpha p_{t+1}$. This implies Samuelson's result that the rate of interest at such a steady state is, in fact, the growth rate of the population.

Arbitrary forms of nonstationarity can be incorporated into our framework as long as the model is stationary for all generations after some generation T. In this case the equilibrium conditions for the first $T + 1$ periods serve the same role that the equilibrium conditions for the first period do in the stationary model. Generically, they determine all but one of the price vectors $p_1, p_2, \ldots, p_{T+1}$. The remaining price vector may, or may not, be determined by the conditions that p_T and p_{T+1} give rise to a price path that converges to a steady state when viewed as initial values for the difference equation corresponding to the remaining equilibrium conditions. The analysis of relative price indeterminacy remains the same.

Geanakoplos and Brown (1985) and Santos and Bona (forthcoming) have extended our results to more general nonstationary models. They find that, just as in stationary models, there are potentilly n dimensions of indeterminacy if there is fiat money and $n - 1$ dimensions if there is not. While these results are of considerable theoretical interest, they have little relevance for applied models. If nothing else, even to store all the parameters of a truly nonstationary model would require a computer with an infinitely large memory.

A restrictive aspect of our analysis is that we have only analyzed price paths near steady states. In fact, however, our analysis immediately extends to price paths near any cycle of finite length. Recall that, when we redefine generations, time periods and goods to convert the three-period-lived model into a two-period-lived model, two-period cycles become steady states. In general, suppose that a model has a cycle of length k, i.e. that $(p_{t+1}, p_{t+2}, \ldots, p_{t+k}) = \beta(p_{t-k+1}, p_{t-k+2}, \ldots, p_t)$ satisfies the equilibrium conditions. Suppose we redefine generations, time periods and goods so that generations $1, 2, \ldots, k$ are now generation 1 and so on. The cycle now corresponds to a steady state of the redefined model.

8 Concluding Remarks

Our results should be troubling to researchers interested in applications of the overlapping generations model. A model that does not give determinate results is not very useful for doing policy analysis. Unfortunately, the problem of indeterminacy of equilibria does not seem to be confined to pure exchange overlapping generations models. Muller and Woodford (1988) have extended the results presented here to stationary models that take into account production, including storage, infinitely lived assets and infinitely lived consumers. They find that, although the presence of infinitely lived assets, infinitely lived consumers or production may rule out nominal debt and inefficient equilibria, it does not rule out indeterminacy. They are able to identify a number of conditions that do rule our indeterminacy, however, just as we identify gross substitutability as such a condition in the pure exchange overlapping generations model. One obvious direction for future research is to find easily checked conditions that imply that a model has determinate equilibria. A warning should be given about the nature of our gross substitutability result. There is no presumption that gross substitutability implies determinacy of equilibrium in models with production. In static models with production, for example, gross substitutability in consumption does not imply uniqueness of equilibrium (see Kehoe, 1985).

It seems inevitable, however, that some very reasonable models have indeterminacy equilibria. Perhaps we are wrong to employ the hypothesis of perfect foresight. With adaptive expectations, for example, equilibrium price paths are generically determinate. Is there some general and economically meaningful way to choose a perfect foresight expectations mechanism that gives rise to determinate equilibria? If not, how far do we have to depart from the perfect foresight hypothesis to obtain determinacy?

That the overlapping generations model seems plagued by indeterminacy is not a satisfactory justification for completely abandoning it in favor of the model with a finite number of infinitely lived consumers. As Gale (1973) points out, "the reason for considering a population rather that a fixed set of agents is that the former is what in reality we have, the latter is what we have not". To build a useful intertemporal equilibrium model, however, it would be necessary to address the issues we have raised in this paper.

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