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Gross Substitutability in Large-Square Economies*

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This paper investigates the implications of gross substitutability of excess demand functions for the equilibria of exchange economies with countably many goods and arbitrarily many consumers. We provide uniqueness theorems for general models. We also show that the equilibrium trajectories of dynamic, stationary overlapping generations economies satisfy strong turnpike properties. In particular, we are able to extend Gale's characterization of the equilibrium dynamics of the model with one good per period, one consumer per generation, and two periods of life to models with many goods, many consumers, and many periods of life. *Journal of Economic Literature* Classification Numbers: 021, 111. © 1991 Academic Press, Inc.

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1. Introduction

In this paper, we study the implications of gross substitutability of excess demand for the equilibria of exchange economies with countably many goods and arbitrarily many consumers (large-square economies). We begin in Section 2 by providing two extensions of the classical uniqueness of equilibrium results to general (nonstationary) environments. Theorem A applies to economies admitting some equilibrium with finite aggregate wealth. Theorem B covers overlapping economies with a well-defined beginning and no uncertainty. Interestingly, a continuum of equilibria cannot be ruled out in all generality; we show this with a double-ended infinity example in Section 3. Sections 3 to 5 concentrate on the classical stationary overlapping generations model of Samuelson [25]. We are able to offer an exhaustive analysis of both the determinancy (the size of the equilibrium set) and the dynamic properties of equilibria.

Gale [11] provides a complete analysis of an overlapping generations model in which there is one good in each period and a single two-period-lived consumer in each generation whose demand function exhibits gross substitutability. He finds, in particular, that indeterminancy (the existence of a continuum of equilibra) is always associated with initial conditions that allow fiat money: if there is no fiat money, then there is a unique equilibrium. Even when there is fiat money, there is at most a one-dimensional set of equilibria; in other words, equilibria can be parameterized by a single number, for example, the level of real money balances. Balasko and Shell [4] extend this result to a model in which there are many goods in each period but a single two-period-lived consumer with a Cobb–Douglas utility function in each generation. Geanakoplos and Polemarchakis [12] further extend these results to a model in which the single two-period-lived consumer in each generation has an intertemporally separable untility function that generates an excess demand function exhibiting gross substitutability. (See also Kehoe and Levine [15].)

These results contrast with those of Kehoe and Levine [16, 17] who analyze the properties of a pure exchange overlapping generations model with an arbitrary number of consumers in each generation and an arbitrary finite number, n, of goods in each period. They find that there are robust examples of economies with an n dimensional manifold of equilibria if there is fiat money and an $n-1$ dimensional manifold of equilibria if there is no fiat money. There are, however, also robust examples of economies with unique equilibria.

We argue, in Sections 3 to 5, that Gale's analysis generalizes in its entirety to overlapping generations economies with n goods in which the excess demand functions exhibit gross substitutability. We show that his distinction between the classical and the Samuelson case continues to hold:

There are two types of asymptotic behavior possible. The first is a steady state, which we call the nominal steady state, in which the price level is constant and there is a nonzero amount of nominal debt passed from generation to generation. The second is a steady state, which we call the real steady state, in which the price level rises or falls exponentially and there is no nominal debt. The classical case is where the nominal steady state has a negative amount of nominal debt; in this case the real steady state has an exponentially falling price level. The Samuelson case is where the nominal steady state has a positive amount of nominal debt; in this case the real steady state has an exponentially rising price level. Moreover, the equilibria satisfy the same strong turnpike properties as in the one-good case. We emphasize that gross substitutability rules out cycles of the sort considered by Benhabib and Day [5], Benhabib and Nishimura [6], and Grandmont [13].

Section 3 presents the general stationary overlapping generations models and analyzes a one-good example. Section 4 analyzes the steady states of the general model. Theorem C shows that the monetary and the real steady state are unique and that economies can be classified as in Gale. Section 5 establishes the turnpike properties of equilibria. The basic arguent is in Theorem D. Theorems E and F offer then a complete characterization of the dynamics for, respectively, the double-ended and the single-ended model. We point out that, while our uniqueness results are the natural generalization of the classical static theorem for a finite number of goods, the results on dynamics have no finite counterpart. In Section 6, Theorem G generalizes the analysis to the overlapping generations economies with land considered by Muller and Woodford [22, 23]. Finally, Section 7 discusses the limitations of the gross substitutability hypothesis in intertemporal or production contexts.

2. General Nonstationary Economies

We first consider a general economy with a countable number of goods. The economy is specified by an excess demand function f: $R^{\infty}_{++} \to R^{\infty}$ that satisfies the following assumptions:

a.1. *Homogeneity of degree zero*: $f(\lambda p) = f(p)$ for all $\lambda > 0$ and $p \in R^{\infty}_{++}$.

a.2. *Boundedness below*: there exists $\omega \in R^{\infty}_{++}$ such that $f(p) \geqslant -\omega$ for all $p \in R^{\infty}_{++}$.

a.3. *Weak gross substitutability*: if $\hat{p} \geqslant \bar{p}$ and $\hat{p}_i = \bar{p}_i$ for some i, then $f_i(\hat{p}) \geqslant f_i(\bar{p})$.

a.4. *Indecomposability*: if $\hat{p} \geqslant \bar{p}$, $\hat{p}_i = \bar{p}_i$ for some i and $f(\hat{p}) = f(\bar{p})$, then $\hat{p} = \bar{p}$.

a.5. *Walras's law*: $p \cdot f(p) = 0$ whenever $p \cdot \omega < \infty$.

The vector ω is to be interpreted as the aggregate endowment. For any two vectors $x, y \in R^\infty$ we write $x \cdot y = \sum_{i=1}^{\infty} x_i \cdot y_j$ whenever the series converges absolutely.

An *equilibrium* is a price vector $\hat{p} \in R^\infty_{++}$ such that $f(\hat{p}) = 0$. Our first theorem says that, if there is an equilibrium that assigns finite value to the aggregate endowment, then it is the unique equilibrium of the economy.

THEOREM A. *Suppose that f satisfies assumptions* a.1–a.5 *and that there exists* $\hat{p} \in R^\infty_{++}$ *such that* $f(\hat{p}) = 0$ *and* $\hat{p} \cdot \omega < \infty$. *Then* $\hat{p}$ *is the unique equilibrium; that is,* $f(\bar{p}) = 0$ *implies* $\bar{p} = \lambda \hat{p}$ *for some* $\lambda > 0$.

Proof. Suppose that $\hat{p}$ and $\bar{p}$ are both equilibria and that $\hat{p} \cdot \omega < \infty$. Normalize prices so that $\hat{p}_1 = \bar{p}_1 = 1$. Set $p = \hat{p} \wedge \bar{p}$; that is, $p_i = \min(\hat{p}_i, \bar{p}_i)$ for all i. If $p_i = \hat{p}_i$, then $p \leqslant \hat{p}$ and weak gross substitutability imply that $f_i(p) \leqslant f_i(\hat{p}) = 0$. Similarly, if $p_i = \bar{p}_i$, $f_i(p) \leqslant f_i(\bar{p}) = 0$. Therefore $f(p) \leqslant 0$. Since p, $\hat{p}$, and ω are all positive, $p \leqslant \hat{p}$ implies that $p \cdot \omega \leqslant \hat{p} \cdot \omega < \infty$. Consequently, Walras's law implies that $f(p) = 0$. Since $\hat{p} \geqslant p$, $\hat{p}_1 = p_1$, and $f(\hat{p}) = f(p) = 0$, indecomposability implies that $\hat{p} = p$. This, in turn, implies that $\bar{p} \geqslant \hat{p}$. Now $\hat{p}_1 = \bar{p}_1$ and indecomposability imply that $\bar{p} = \hat{p}$. ∎

Note that Theorem A implies that, if there is an equilibrium that gives finite value to the initial endowment, then there cannot be another where the value is infinite. If one consumer (with nonsatiated preferences) owns a fraction of the aggregate endowment that is uniformly bounded away from zero for every good, then, since he must have finite wealth in equilibrium, it follows that total wealth is finite. In models where no consumer owns a fraction of the aggregate endowment, such as the standard overlapping generations model, the value of the aggregate endowment may be infinite, and Theorem A does not apply. It does apply, however, to the overlapping generations models with land or with infinitely lived consumers considered by Muller and Woodford [23]. Moreover, we can complicate the structure of that model, either by running time back to $-\infty$ or by allowing for uncertainty with contingent claims markets, and nevertheless conclude, when consumers' demands exhibit gross substitutability, that, if an equilibrium exists, it is unique. This is because the one-directional nature of time does not play any role in our argument. See Section 7 for more on economies with land. Theorem A also applies to the type of general large-square economy studied by Kehoe, Levine, Mas-Colell, and Zame [19].

There are many models of interest (including the standard overlapping generations model) where the finite wealth condition is not satisfied. We shall see in Section 3 that uniqueness may then well fail. It turns out, however, that under some additional regularity conditions uniqueness can be proved for a particularly important case: the classical overlapping generations model where the economy extends to infinity from a well-defined beginning at time one (the one-ended, as opposed to the two-ended, model). This we now proceed to show.

We consider general, nonstationary overlapping generations economies in which consumers live for two periods. Since we allow for many goods and consumers, the assumption of two periods of life is completely general: Balasko, Cass, and Shell [3] present a simple procedure for converting a model in which consumers live for any uniformly bounded number of periods into one in which they live for only two. This procedure, which redefines periods and generations, increases the number of goods in each period and the number of consumers in each generation.

The economy is now specified by a sequence of excess demand functions. Generation t, $t \geqslant 1$, has excess demand $y_t\colon R^{2n}_{++} \to R^n$ when young and $z_t\colon R^{2n}_{++} \to R^n$ when old. Generation 0 has excess demand $z_0\colon R^n_{++} \to R^n$. It is convenient for notation to let $y_0\colon R^n_{++} \to R^n$ be the function identically equal to zero. Given a price sequence p, the excess demand for goods in period t is

$$f_t(p) = f_t(p_{t-1}, p_t, p_{t+1}) = z_{t-1}(p_{t-1}, p_t) + y_t(p_t, p_{t+1}),$$

where we let p_0 be an empty symbol (this is a convention used repeatedly).

We assume:

A.1. Every y_t and z_t, $0 \leqslant t < \infty$, is *homogeneous of degree zero.*

A.2. The functions (y_t, z_t), $0 \leqslant t < \infty$, are *bounded below.* Namely, there exists a sequence of initial endowments $\omega_t \in R^{2n}_{++}$ such that $(y_t(q), z_t(q)) > -\omega_t$, for all $0 \leqslant t < \infty$ and $q \in R^{2n}_{++}$. Moreover, $\{\|\omega_t\|/\|\omega_{t-1}\| : 0 \leqslant t < \infty\}$ is bounded from above and away from zero.

A.3. Every (y_t, z_t), and therefore every f_t, $1 \leqslant t < \infty$, exhibits *weak gross substitutability*: if $(\hat{p}_{t-1}, \hat{p}_t, \hat{p}_{t+1}) \geqslant (\bar{p}_{t-1}, \bar{p}_t, \bar{p}_{t+1})$ and $\hat{p}^i_t = \bar{p}^i_t$ for some i then $f^i_t(\hat{p}_{t-1}, \hat{p}_t, \hat{p}_{t+1}) \geqslant f^i_t(\bar{p}_{t-1}, \bar{p}_t, \bar{p}_{t+1})$.

A.4. The functions f_t are *uniformly indecomposable*: for any $\varepsilon > 0$ there exists $\delta > 0$ such that if for any t we have $(\hat{p}_{t-1}, \hat{p}_t, \hat{p}_{t+1}) \geqslant (\bar{p}_{t-1}, \bar{p}_t, \bar{p}_{t-1}) \geqslant \varepsilon e$, $\hat{p}^i_t = \bar{p}^i_t$ for some i, $(\bar{p}_{t-1}, \bar{p}_t, \bar{p}_{t+1}) \leqslant (1/\varepsilon)e$ and $\|(1/\|(\hat{p}_{t-1}, \hat{p}_t, \hat{p}_{t+1})\|)\,(\hat{p}_{t-1}, \hat{p}_t, \hat{p}_{t+1}) - (1/\|(\bar{p}_{t-1}, \bar{p}_t, \bar{p}_{t+1})\|)(\bar{p}_{t-1}, \bar{p}_t, \bar{p}_{t+1})\| > \delta$, then $\|f_t(\hat{p}_{t-1}, \bar{p}_t, \bar{p}_{t+1}) - f_t(\bar{p}_{t-1}, \bar{p}_t, \bar{p}_{t+1})\|/\|\omega_t\| > \delta$. (Here e is the vector with all entries equal to one.)

A.5. The family of functions $\{(1/\|\omega_t\|)(y_t, z_t) : 0 \leqslant t < \infty\}$ is *uniformly continuous* on compact subsets of the domain.

A.6. (*Boundary*) For any sequences $t_k \geqslant 1$ and $q_k \in R^{2n}_{++}$ such that $q_k \to q \neq 0$ and $q^i = 0$ for some i, we have $\|(y_{t_k}(q_k), z_{t_k}(q_k))\| / \|\omega_t\| \to \infty$. Similarly, if $q_k \to q \neq 0$ and $q^i_k = 0$ for some i, then $\|z_0(q_k)\| \to \infty$.

Except for the uniformity restrictions, these assumptions are standard. Notice that Walras's law is not assumed. Assumption A.4 has been chosen, although it is strong, because it is weak enough to include situations where periods and generations have been redefined to convert a model in which consumers live for more than two generations into one in which they live for two. In a stationary economy (one where all generations are identical) the uniformities are automatically satisfied. Thus, they can be interpreted as putting some limit on how dissimilar generations can be. For example, the indecomposability assumption A.4 not only rules out situations in which some generation consists of two groups of consumers who have preferences for, and endowments of, disjoint sets of goods, but also situations in which the functions f_t converge to an excess demand function that is decomposable in this sense. If we derive the excess demand functions from utility maximization, we could show that A.6 is implied by the other hypothesis (see, for example, Arrow and Hahn [2, pp. 221–223]): indecomposability and homogeneity imply that the excess demand cannot be defined when some prices are zero.

An *equilibrium* is a price sequence p such that $f_t(p) = 0$ for all $t \geqslant 1$.

LEMMA 1. *Suppose that the* (y_t, z_t) *satisfy* A.1, A.2, A.5, *and* A.6 *and that* p *is an eventual equilibrium* (*in other words, there is* T *such that* $f_t(p) = 0$ *for* $t \geqslant T$). *Then there exists* $\gamma > 0$ *such that* $1/\gamma < p^i_t / p^j_\tau < \gamma$ *for all* $1 \leqslant i, j \leqslant n$, $1 \leqslant t < \infty$, *and* $\tau = t-1, t, t+1$. *In other words,* $\{(1/\|(p_{t-1}, p_t, p_{t+1})\|)(p_{t-1}, p_t, p_{t+1})) : 1 \leqslant t \leqslant \infty\}$ *has compact closure in* R^{3n}_{++}.

Proof. Suppose the contrary. Normalize the elements of the sequence $q_1, q_2, \ldots$, where $q_t = (p_t, p_{t+1})$, to all satisfy $\|q_t\| = 1$. Then q_t has a subsequence that converges to a vector $q \neq 0$ such that $q^i = 0$ for some i. Assumptions A.2 and A.6 imply that, as $t \to \infty$, there is no upper bound to $(1/\|\omega_t\|)(y_t(q_t), z_t(q_t))$. The assumption that $\|\omega_t\| / \|\omega_{t-1}\|$ is uniformly bounded now implies that q_t cannot satisfy the equilibrium conditions for t sufficiently large. ▌

We are now in a position to prove uniqueness:

THEOREM B. *Suppose that* (y_t, z_t), $0 \leqslant t < \infty$, *satisfy* A.1–A.6. *Then* (*up to normalization*) *there is at most one equilibrium.*

Proof. Given two arbitrary price sequences, $\hat{p}$ and $\bar{p}$, define

$\bar{r}_t = \max_i \hat{p}_t^i/\bar{p}_t^i$, $\underline{r}_t = \min_i \hat{p}_t^i/\bar{p}_t^i$. For later reference it is convenient to subdivide the proof into three steps.

Step 1. Here we assume only that $\hat{p}$, $\bar{p}$ are eventual equilibria and that $\bar{r}_t \to \bar{r}$ and $\underline{r}_t \to \underline{r}$ for some $\bar{r}$, $\underline{r}$. We show that $\bar{r} = \underline{r}$. Denote $\bar{v}_t = (\bar{p}_{t-1}, \bar{p}_t, \bar{p}_{t+1})$, $\bar{v}_t' = (\bar{r}_t \bar{p}_{t-1}, \bar{r}_t \bar{p}_t, \bar{r}_t p_{t+1})$, $\bar{v}_t'' = (\bar{r}_{t-1} \bar{p}_{t-1}, \bar{r}_t \bar{p}_t, \bar{r}_{t+1} \bar{p}_{t+1})$, and $\hat{v}_t = (\hat{p}_{t-1}, \hat{p}_t, \hat{p}_{t+1})$. By the uniform continuity hypothesis (and Lemma 1), we have $\|f_t(\bar{v}_t'') - f_t(\hat{v}_t)\| / \|\omega_t\| \to 0$. Because $\bar{v}_t'' \geqslant \hat{v}_t$ and $\bar{r}_t \bar{p}_t^i = \hat{p}_t^i$ for some i, the uniform indecomposability (and Lemma 1) implies $\|(1/\|\bar{v}_t''\|)\bar{v}_t'' - (1/\|\hat{v}_t\|)\hat{v}_t\| \to 0$. Now, $(1/\|\bar{v}_t''\|)\bar{v}_t'' \geqslant \|\hat{v}_t\|/\|\bar{v}_t''\|\ (1/\|\hat{v}_t\|)\hat{v}_t$ with equality for at least one component. Hence, without loss of generality, we can assume that $\|\hat{v}_t\|/\|\bar{v}_t''\| \to 1$. This implies $\|\hat{v}_t\|/\|\bar{v}_t\| \to \bar{r}$. Taking subsequences if necessary, we can use a similar argument to show that $\|\hat{v}_t\|/\|\bar{v}_t\| \to \underline{r}$. Hence $\bar{r} = \underline{r}$.

Step 2. Here we assume only that the sequences $\hat{p}$, $\bar{p}$ are equilibria for $t \geqslant T \geqslant 2$. We claim that $\bar{r}_{t-1} \leqslant \bar{r}_t$, $t \geqslant T$, implies $\bar{r}_t \leqslant \bar{r}_{t+1}$ and, similarly, that $\underline{r}_{t-1} \geqslant \underline{r}_t$, $t \geqslant T$, implies $\underline{r}_t \geqslant \underline{r}_{t+1}$. Indeed, suppose that $\bar{r}_{t-1} \leqslant \bar{r}_t$ and $\bar{r}_t > \bar{r}_{t+1}$. Then $(\bar{r}_t \bar{p}_{t-1}, \bar{r}_t \bar{p}_t, \bar{r}_t \bar{p}_{t+1}) \geqslant (\hat{p}_{t-1}, \hat{p}_t, \hat{p}_{t+1})$ and $\bar{r}_t \bar{p}_t^i = \hat{p}_t^i$ for some i. Again indecomposability and the equilibrium condition contradict $(\bar{r}_t \bar{p}_{t-1}, \bar{r}_t \bar{p}_t, \bar{r}_t \bar{p}_{t+1}) \neq (\hat{p}_{t-1}, \hat{p}_t, \hat{p}_{t+1})$. Consequently, $\bar{r}_t \leqslant \bar{r}_{t+1}$. Reversing the roles of $\hat{p}_t$ and $\bar{p}_t$, we can prove similarly that $\underline{r}_t \geqslant \underline{r}_{t+1}$.

Step 3. We now exploit the fact that both $\hat{p}$ and $\bar{p}$ are full equilibria. We establish first that $\bar{r}_2 \geqslant \bar{r}_1$ and $\underline{r}_2 \leqslant \underline{r}_1$. Suppose, to the contrary, that $\bar{r}_1 > \bar{r}_2$. Then $(\bar{r}_1 \bar{p}_1, \bar{r}_1 \bar{p}_2) \geqslant (\hat{p}_1, \hat{p}_2)$ and $\bar{r}_1 \bar{p}_1^i = \hat{p}_1^i$ for some i. Therefore, indecomposability and $f_1(\bar{r}_1 \bar{p}_1, \bar{r}_1 \bar{p}_2) = f_1(\hat{p}_1, \hat{p}_2) = 0$ contradict $(\bar{r}_1 \bar{p}_1, \bar{r}_1 \bar{p}_2) \neq (\hat{p}_1, \hat{p}_2)$. Hence, $\bar{r}_2 \geqslant \bar{r}_1$ and, similarly, $\underline{r}_1 \geqslant \underline{r}_2$. By Step 1, this implies $\bar{r}_1 \leqslant \cdots \leqslant \bar{r}_t \leqslant \cdots$ and $\underline{r}_1 \geqslant \cdots \geqslant \underline{r}_t \geqslant \cdots$. By Lemma 1, $\bar{r}_t - \underline{r}_t$ must be bounded. Therefore the two monotone sequences $\bar{r}_t$ and $\underline{r}_t$ are bounded. We can conclude that $\bar{r}_t \to \bar{r}$, $\underline{r}_t \to \underline{r}$ for some $\bar{r}$, $\underline{r}$.

Consequently, by Step 1, $\bar{r} = \underline{r}$ and $\underline{r}_t = \bar{r}_t = \bar{r}$ for all t; that is, $\hat{p} = \bar{r}\bar{p}$. ∎

The initial conditions, that is the excess demand of the old at time 1, are customarily described by means of a function $z_0' \colon J \to R^n$, where $J \subset R^n_{++} \times R$ is a cone of permissible combinations of price vectors p_1 and fiat money M. The function z_0' is homogeneous of degree zero on its arguments (p_1, M). To go from this to our previous specification we need a price index $\pi \colon R^n_{++} \to R_{++}$, that is, a continuous, monotonically increasing, and homogeneous of degree one function (for example, $\pi(p_1) = p_1^i$ for some i, or $\pi(p_1) = p_1 \cdot \omega_0, \ldots$). If then we let m stand for real balances we can put $z_0(p, m) = z_0'(p_1, \pi(p_1)m)$. For m fixed the function $z_0(p, m)$ is homogeneous of degree zero, and therefore, if A.1–A.6 are satisfied, we can conclude from Theorem B that for every level of real balances m there is at most one equilibrium.

The next proposition is a variation of Theorem B that does not involve the initial conditions z_0.

PROPOSITION 2. *Suppose that* (y_t, z_t), $1 \leqslant t < \infty$ *satisfy* A.1–A.6. *Let* $\pi(p_1)$ *be an arbitrary index function. Then for any* $q \in R^n_{++}$ *and* $m \in R$ *there is at most one price sequence* p *such that* $p_1 = q$, $f_t(p) = 0$ *for* $t > 1$, *and* $p_1 \cdot y_1(p_1, p_2) = \pi(p_1)m$.

Proof. Suppose, to the contrary, that there are two such price sequences $\bar{p}$ and $\hat{p}$. The proof proceeds exactly as that of Theorem B. The only change required is a modification of the argument in Step 3 showing that $\bar{r}_2 \geqslant \bar{r}_1$ and $\underline{r}_2 \leqslant \underline{r}_1$. Suppose that $\bar{r}_1 > \bar{r}_2$. In fact, $\bar{r}_1 = 1$ (because $\bar{p}_1 = \hat{p}_1 = q$). Hence, $\bar{r}_2 < 1$ means that $\hat{p}_2 < \bar{p}_1$. Because $(\hat{p}_1, \hat{p}_2) \leqslant (\bar{p}_1, \bar{p}_2)$ and $\hat{p}_1 = \bar{p}_1$ the weak gross substitute condition yields $y_1(\bar{p}_1, \bar{p}_2) \geqslant y_1(\hat{p}_1, \hat{p}_2)$. Since $q \cdot y_1(\bar{p}_1, \bar{p}_2) = q \cdot y_1(\hat{p}_1, \hat{p}_2) = \pi(q)m$, this implies $y_1(\bar{p}_1, \bar{p}_2) = y_2(\hat{p}_1, \hat{p}_2)$. Therefore, by indecomposability, $\bar{p}_2 = \hat{p}_2$, which is a contradiction. We conclude that $\bar{r}_2 \geqslant \bar{r}_1$. Similarly, $\underline{r}_2 \leqslant \underline{r}_1$, and the rest of the proof is as in Theorem B. ∎

Remark. It is interesting to observe that, if $\bar{p}$ and $\hat{p}$ are equilibria for $t \geqslant T = 2$ and $\bar{p}_1 = \hat{p}_1$, then $\bar{p}$, $\hat{p}$ are monotonically related; in other words, either $\bar{p} \leqslant \hat{p}$ or $\bar{p} \geqslant \hat{p}$. Indeed, the proof of Theorem B fails to apply only if either $\bar{r}_1 \geqslant \bar{r}_2 \geqslant \cdots \geqslant \bar{r}_t \geqslant \cdots$ or $\underline{r}_1 \leqslant \underline{r}_2 \leqslant \cdots \leqslant \underline{r}_t \leqslant \cdots$. Since $\bar{r}_1 = \underline{r}_1 = 1$, this means that either $\bar{r}_t \leqslant 1$ for all t, in which case $\hat{p} \leqslant \bar{p}$, or $\underline{r}_t \geqslant 1$ for all t, in which case $\hat{p} \geqslant \bar{p}$.

We discuss informally an important implication of Proposition 2. Whenever defined let $g(q, m) = y_1(q, p_2)$ where $p = (q, p_2, ...)$ is as in Proposition 2. Then $g(q, m)$ is a kind of reduced excess demand in period one; it is obtained under the condition that all other periods are in equilibrium. Bringing in the initial conditions, the overall equilibrium can then be found as the solution of the system of excess demand equations $z_0(q) + g(q, m) = 0$. Assume that z_0 satisfies Walras's law in the sense that $q \cdot z_0(q) = \pi(q)\bar{m}$ for some $\bar{m}$. Then we can set $m = -\bar{m}$ and, once prices have been normalized and one equation eliminated (courtesy of Walras's law), we are left with a system of $n-1$ equations in $n-1$ unknowns. Therefore, we would expect that typically the set of solutions is discrete. Observe that this heuristic argument (the rigorous treatment requires smoothness hypotheses) does not use any kind of gross substitute hypothesis on z_0. Consequently, remembering that we are free to redefine periods, we can conclude that economies that are eventually gross substitute typically do not have a continuum of equilibria.

3. Stationary Overlapping Generations Economies

We proceed to the study of stationary overlapping generations economies. The model is as in Section 2 except that now $(y_t, z_t) = (y, z)$ for all t. For the moment, we ignore initial conditions. Because their statement can be considerably simplified in the stationary case, we repeat assumptions A.1–A.6:

A.1. (y, z) are *homogeneous of degree zero.*

A.2. (y, z) are *bounded below.*

A.3. (y, z) exhibits *weak gross substitutability.*

A.4. (y, z) are *indecomposable*; that is, $\bar{q} \geqslant \hat{q}$, $\bar{q}^i = \hat{q}^i$ for some i and $y(\bar{q}) = y(\hat{q})$, $z(\bar{q}) = z(\hat{q})$ imply $\bar{q} = \hat{q}$.

A.5. (y, z) are *continuous.*

A.6. For any $q_k \in R^{2n}_{++}$ such that $q_k \to q \neq 0$, $q^i = 0$ for some i, we have $\|(y(q_k), z(q_k))\| \to \infty$.

To these assumptions we add now:

A.7. The excess demand function (y, z) obeys *Walras's law*: $p_t \cdot y(p_t, p_{t+1}) + p_{t+1} \cdot z(p_t, p_{t+1}) = 0$ for all $(p_t, p_{t+1}) \in R^{2n}_{++}$.

Of course, Walras's law is a natural assumption in a nonstationary environment as well; it plays no role in the results of the previous section, however.

There are two versions of this model. The version in which time begins at $t = 1$ is called the *single-ended infinity model.* The version in which time runs from $-\infty$ to $+\infty$ is called the *double-ended infinity model.* The double-ended model has an alternative interpretation: We can view it as a model with a fixed starting date and uncertainty over two states of nature in the initial period. If the first state occurs, the model runs to $+\infty$. If the second occurs, it runs to $-\infty$; that is, the preference and endowment patterns are reversed.

A *steady state* of this model is a price vector $p \in R^n_{++}$ and an inflation factor $\beta > 0$ such that $z(p, \beta p) + y(p, \beta p) = 0$, or, in other words, such that $p_t = \beta^t p$ is an equilibrium of the double-ended infinity model defined by (y, z). Steady states naturally divide themselves into two types. The steady state condition implies that $p \cdot z(p, \beta p) + p \cdot y(p, \beta p) = 0$; Walras's law implies that $\beta p \cdot z(p, \beta p) + p \cdot y(p, \beta p) = 0$. Consequently, $(\beta - 1)\, p \cdot z(p, \beta p) = 0$ where $p \cdot z(p, \beta, p)$ is the amount of nominal debt transferred from generation to generation in the steady state. A *nominal steady state* is one for which $\beta = 1$. A *real steady state* is one for which $p \cdot z(p, \beta p) = 0$. Kehoe and Levine [16] use the sort of fixed point argument used in static general equilibrium theory to prove the existence of a steady state of each type. Except in degenerate cases, $\beta = 1$ and $p \cdot z(p, \beta p) = 0$ do not occur simultaneously.

We first consider the case with a single good in each period and a single consumer in each generation with a Cobb–Douglas utility function over a two-period life. Since the Cobb–Douglas utility function gives rise to an excess demand function that exhibits gross substitutability, our model is a special case of that considered by Gale [11]. The consumer born in period t has a utility function of the form

$$u(c_t, c_{t+1}) = a_1 \log c_t + a_2 \log c_{t+1}$$

over consumption in periods t and $t+1$. Here $a_1, a_2 > 0$, $a_1 + a_2 = 1$. His endowment of goods is (ω_1, ω_2), which is strictly positive. If this consumer faces prices (p_t, p_{t+1}), then his demands are given by the familiar Cobb–Douglas demand functions

$$c_t = a_1(p_t\omega_1 + p_{t+1}\omega_2)/p_t$$

$$c_{t+1} = a_2(p_t\omega_1 + p_{t+1}\omega_2)/p_{t+1}.$$

An equilibrium of the double-ended infinity version of this model is an infinite price sequence that satisfies the requirement that demand equal supply in each period,

$$a_2(p_{t-1}\omega_1 + p_t\omega_2)/p_t + a_1(p_t\omega_1 + p_{t+1}\omega_2)/p_t = \omega_1 + \omega_2, \qquad -\infty < t < \infty,$$

which is easily converted into the second order linear difference equation

$$a_2\omega_1 p_{t-1} - (a_2\omega_1 + a_1\omega_2)p_t + a_1\omega_2 p_{t+1} = 0, \qquad -\infty < t < \infty.$$

There are two steady state equilibria of the form $p_t = \beta^t$. They can be found by solving the quadratic equation

$$a_2\omega_1 - (a_2\omega_1 + a_1\omega_2)\beta + a_1\omega_2\beta^2 = 0.$$

The solutions are $\beta = 1$ and $\beta = a_2\omega_1/(a_1\omega_2)$. Consequently, any solution to the difference equation has the form

$$p_t = k_1 + k_2\left[\frac{a_2\omega_1}{a_1\omega_2}\right]^t, \qquad -\infty < t < \infty,$$

where k_1 and k_2 are arbitrary constants. For a solution to make sense as an equilibrium of the economy, we require that it always be positive. This implies that k_1 and k_2 are nonnegative.

Since we do not consider two equilibria to be distinct if one is a scalar multiple of the other, we can impose a normalization such as $p_0 = k_1 + k_2 = 1$. Nevertheless, there is always a one-dimensional family of equilibria to this model except in the degenerate case where $a_2\omega_1 = a_1\omega_2$. Consider the excess demand function $f: R^{\infty}_{++} \to R^{\infty}$ given by

$$f_t(p_{t-1}, p_t, p_{t+1}) = (a_2\omega_1 p_{t-1} - (a_2\omega_1 + a_1\omega_2) p_t + a_1\omega_2 p_{t+1})/p_t, \qquad -\infty < t < \infty.$$

Then f satisfies the assumptions of Theorem A. The nonuniqueness of equilibrium does not contradict the theorem, however, because there is no equilibrium that satisfies the requirement $\sum_{t=-\infty}^{\infty} p_t(\omega_1 + \omega_2) < \infty$.

Consider now the single-ended infinity version of this model. There is an initial old consumer who may violate his budget constraint by demanding m of the initial young consumer's endowment. Here mp_1, which may be positive, negative, or zero, is the stock of fiat money. The equilibrium condition in the first period is, therefore,

$$m + a_1(p_1\omega_1 + p_2\omega_2)/p_1 = \omega_1,$$

which can be simplified to $a_2\omega_1 p_1 - a_1\omega_2 p_2 = mp_1$. This initial condition allows us to solve for k_1,

$$a_2\omega_1\left(k_1 + k_2\left[\frac{a_2\omega_1}{a_1\omega_2}\right]\right) - a_1\omega_2\left(k_1 + k_2\left[\frac{a_2\omega_1}{a_1\omega_2}\right]^2\right) = mp_1$$

or

$$(a_2\omega_1 - a_1\,\omega_2)k_1 = mp_1.$$

Hence, normalizing by $p_1 = 1$, we see that for fixed m we can have at most one equilibrium price sequence. This is in agreement with Theorem B. For some values of m an equilibrium may not exist, however.

There still is the requirement that p_t be positive for all $t > 0$ and therefore, if $a_2\omega_1 < a_1\omega_2$, then only nonpositive values of m are compatible with equilibrium. Note also that there is an equilibrium that satisfies $\sum_{t=1}^{\infty} p_t(\omega_1 + \omega_2) < \infty$ if and only if $m = 0$ and $a_2\omega_1 < a_1\omega_2$. In this case,

$$p_t = \left[\frac{a_2\omega_2}{a_1\omega_2}\right]^t, \qquad 1 \leqslant t < \infty.$$

The remaining analysis of the example constitutes a preview of the results to be established for considerably more general stationary models in Sections 4 and 5.

The equilibrium equations admit two steady states. In the nominal steady state $k_2 = 0$, and p_t is constant through time. This implies that

$m = a_2\omega_1 - a_1\omega_1$ in the single-ended model. In the real steady state $k_1 = 0$, and $p_t = \beta^{t-1}\ p_1$ for $\beta = a_2\omega_1/a_1\omega_2$. It implies that $m = 0$ in the single-ended model. The two steady states coincide if and only if $a_2\omega_1 = a_1\omega_2$.

The amount of nominal debt transferred from generation t to generation $t+1$ along an equilibrium path is $a_2\omega_1 p_{t-1} - a_1\omega_2 p_t$. The consumers' budget constraints and the equilibrium conditions imply that this amount remains constant over time. In the case where $a_2\omega_1 < a_1\omega_2$ this amount is negative at the nominal steady state. This is what Gale [11] calls the *classical case*. Note then that

(i) the real steady state necessarily involves exponential deflation since $a_2\omega_1/a_1\omega_2 < 1$,

(ii) in the double-ended model, every equilibrium path except the real steady state itself goes from the real to the nominal steady state,

(iii) in the one-ended model, equilibrium paths converge to the nominal steady state (except if $m = 0$).

The *Samuelson case* is where $a_2\omega_1 > a_1\omega_2$. Here nominal debt is positive at the nominal steady state. Also

(i) there is exponential inflation at the real steady state;

(ii) as in the classical case in the double-ended model, equilibrium paths go from the steady state with lower β to the steady state with higher β, but this now means that except for the nominal steady state itself equilibrium paths go from the nominal to the real steady state;

(iii) in the one-ended model, equilibrium paths converge to the real steady state (except if $m = a_2\omega_1 - a_1\omega_2$).

The amount of nominal debt passed from generation to generation can be interpreted as fiat money in the single-ended infinity model. It need not be interpreted that way, however, in the double-ended infinity model. As noted above, the model in which time runs from $-\infty$ to ∞ can be reinterpreted as a model with a fixed starting date and uncertainty over two states of nature. After the initial generation, in state 1 the representative consumer has endowments ω_1 when young and ω_2 when old and utility function $a_1 \log c_t + a_2 \log c_{t+1}$. In state 2 he has endowment stream (ω_2, ω_1) and utility function $a_2 \log c_t + a_1 \log c_{t+1}$. Assume that the random shock occurs before the second generation is born. This generation is not allowed to insure itself against what state it is born into and so faces two budget constraints. The initial old generation has endowment ω_2 in state 1 and endowment ω_1 in state 2. It faces one budget constraint. Furthermore, the representative consumer has von Neumann–Morgenstern utility function $\log c_1$ and assigns probability a_1 to state 1 occurring and a_2

to state 2 occurring. By construction, the equilibria of this example are isomorphic to those of the double-ended infinity model: Let p_t^i be the price of the good in period t and state i. Then $((p_1^1, p_1^2), (p_2^1, p_2^2), ...)$ is an equilibrium of the model with uncertainty if and only if $(..., p_2^2, p_1^2, p_1^1, p_2^1, ...)$ is an equilibrium of the double-ended infinity model. Yet another interpretation of this model is as a single-ended infinity model in which there are two goods in every period and two consumers in every generation but the first. Consumer 1 in each generation has preferences for, and endowment of, only good 1, and consumer 2 has preferences for, and endowment of, only good 2.

4. Stationary Overlapping Generations Economies: Steady States

We proceed now to the analysis of steady states in general stationary overlapping generations economies. Our next result says that the nominal and the real steady states are unique and that the classification of steady states given by Gale [11] for the model with one good in each period and one consumer in each generation holds more generally.

Theorem C. *Suppose that (y, z) satisfies* A.1–A.7. *Then the nominal and the real steady states exist and are unique. Moreover, (y, z) falls into one of three cases*:

(i) *the classical case in which $\beta < 1$ at the real steady state and $p \cdot z(p, p) < 0$ at the nominal steady state*;

(ii) *the Samuelson case in which $\beta > 1$ at the real steady state and $p \cdot z(p, p) > 0$ at the nominal steady state*;

(iii) *the coincidental case in which the real and the nominal steady states are the same.*

Before proving Theorem C we establish that there cannot be more than one real steady state. The proof is simple but, curiously, rather indirect.

Lemma 3. *Under hypotheses* A.1–A.7 *there is at most one real steady state for (y, z).*

Proof. Let $\underline{\beta} < \bar{\beta}$, with corresponding time one prices $\underline{q}$ and $\bar{q}$, be two steady states. Denote $\underline{p} = (\underline{q}, \underline{\beta}\underline{q}, \underline{\beta}^2\underline{q}, ...)$ and $\bar{p} = (\bar{q}, \bar{\beta}\bar{q}, \bar{\beta}^2\bar{q}, ...)$. We can assume that $\bar{p}_t \leqslant \underline{p}_t$ for $t \leqslant 2$.

Assume first that $\underline{\beta} < 1$.

Suppose that $(\bar{q}, \bar{\beta})$ is real; that is, $\bar{q} \cdot y(\bar{q}, \beta\bar{q}) = 0$. Let $z_0 \colon R^n_{++} \to R$ be an excess demand function satisfying A.1–A.7 and $z_0(\bar{q}) + y(\bar{q}, \beta\bar{q}) = 0$.

Obviously, such a z_0 exists. In particular, z_0 satisfies Walras's law; that is, $\bar{q} \cdot z_0(\bar{q}) = 0$. Observe then that $\bar{p}$ is an equilibrium of the one-ended infinity economy (z_0, y, z) and that this economy satisfies all the hypotheses of Theorem A.

The next step is similar to the proof of Theorem A. Define $p = \bar{p} \wedge \underline{p}$; that is, $p_t^i = \min\{\bar{p}_t^i, \underline{p}_t^i\}$. Since $p \leqslant \bar{p}$ and for any $t > 1$ and i, we have that either $p_t^i = \bar{p}_t^i$ or $p_t^i = \underline{p}_t^i$ and $f_t^i(\bar{p}) = f_t^i(\underline{p}) = 0$, we conclude that $f_t(p) \leqslant 0$ for $t > 1$. Because $p_1 = \bar{p}_1$, $p_2 = \bar{p}_2$, we also have $f_1(p) = f_1(\bar{p}) = 0$. Therefore, $f_t(p) \leqslant 0$ for all t.

Since $p \leqslant \underline{p}$ and $\underline{p}$ is exponentially decreasing ($\underline{\beta} < 1$), total wealth evaluated at p is finite. Hence, by Walras's law, $p \cdot f(p) = \sum_{t=1}^{\infty} p_t \cdot f_t(p) = 0$. This implies that $f_t(p) = 0$ for all $t \geqslant 1$. That is, p constitutes a finite wealth equilibrium of (z_0, y, z). By Theorem A, p is the unique equilibrium. Hence $\bar{p} = p$, or $\bar{p} \leqslant \underline{p}$. But this is impossible because $\underline{\beta} < \bar{\beta}$ implies that eventually $\underline{p}_t = \underline{\beta}^t \underline{q} < \bar{\beta}^t \bar{q} = \bar{p}_t$. This contradiction shows that $(\bar{q}, \bar{\beta})$ cannot be a real equilibrium.

Summarizing: any (y, z) that satisfies A.1–A.7 and admits a steady state (q, β) with $\beta < 1$ cannot have more than one real steady state.

Let now (y, z) be our given economy and let (q, β) be a steady state for it. If $\beta < 1$ we are done. If $\beta > 1$ consider the economy $(\tilde{y}, \tilde{z})$, where time is reversed in the obvious way: $\tilde{y}(p_t, p_{t+1}) = z(p_{t+1}, p_t)$ and $\tilde{z}(p_t, p_{t+1}) = y(p_{t+1}, p_t)$. (Note that $(\tilde{q}, \tilde{z})$ satisfies A.1–A.7 if (y, z) does.) Then $(q, 1/\beta)$ is a steady state for $(\tilde{y}, \tilde{z})$. Hence, $(\tilde{y}, \tilde{z})$, and therefore (y, z), has at most one real steady state. The only remaining case is where $\beta = 1$. But there is no more than one steady state with this value of β. (In other words, there is at most one monetary steady state.) This is fairly easy to see and, at any rate, is verified in the proof of Theorem C. ∎

Proof of Theorem C. Consider the function $h\colon R_{++}^n \times R_{++} \to R^n$ defined by the rule

$$h(p, \beta) = f(p, \beta p, \beta^2 p) - \frac{1}{e \cdot p}\, p \cdot f(p, \beta p, \beta^2 p) e,$$

where again $e = (1, ..., 1)$. Then h is continuous, homogeneous of degree zero in p, and satisfies $p \cdot h(p, \beta) \equiv 0$ for all $(p, \beta) \in R_{++}^n \times R_{++}$. Furthermore, h is bounded below and satisfies a boundary condition like A.4. Since, for fixed β, h has the same properties as an excess demand function of a static exchange economy, there exists $p(\beta) \in R_{++}^n$ such that $h(p(\beta), \beta) = 0$. Normalize $p(\beta)$ so that, for example, $e \cdot p(\beta) = 1$. We claim that $p\colon R_{++} \to R_{++}^n$ is a continuous, single-valued function. The continuity of h implies that it is an upper-hemi-continuous point-to-set correspondence. What we need to show is that $p(\beta)$ is unique. Suppose $\hat{p}$ and $\bar{p}$ both satisfy $h(p, \beta) = 0$. Let

$\bar{r} = \max_i \hat{p}^i/\bar{p}^i$, $\underline{r} = \min_i \hat{p}^i/\bar{p}^i$. Then for i such that $\hat{p}^i/\bar{p}^i = \bar{r}$, gross substitutability implies that

$$f_i(\bar{r}\bar{p}, \beta\bar{r}\bar{p}, \beta^2\bar{r}\bar{p}) \geqslant f_i(\hat{p}, \beta\hat{p}, \beta^2\hat{p}).$$

Since $h_i(\bar{p}, \beta) = h_i(\hat{p}, \beta)$, this implies that $\bar{p} \cdot f(\bar{p}, \beta\bar{p}, \beta^2\bar{p}) \geqslant \hat{p} \cdot f(\hat{p}, \beta\hat{p}, \beta^2\hat{p})$. For j such that $\hat{p}^j/\bar{p}^j = \underline{r}$, however, $f_j(\underline{r}\bar{p}, \beta\underline{r}\bar{p}, \beta^2\underline{r}\bar{p}) \leqslant f_j(\hat{p}, \beta\hat{p}, \beta^2\hat{p})$, which implies that $\bar{p} \cdot f(\bar{p}, \beta\bar{p}, \beta^2\bar{p}) \leqslant \hat{p} \cdot f(\hat{p}, \beta\hat{p}, \beta^2\hat{p})$. Consequently, $f(\bar{p}, \beta\bar{p}, \beta^2\bar{p}) = f(\hat{p}, \beta\hat{p}, \beta^2\hat{p})$, and indecomposability implies that $\bar{p} = \hat{p}$.

The pair (p, β) is a steady state if and only if $p = p(\beta)$ and $p \cdot f(p, \beta p, \beta^2 p) = 0$. By Walras's law $p \cdot f(p(\beta), \beta p(\beta), \beta^2(\beta)) = (1-\beta)\, p(\beta) \cdot z(p(\beta), \beta p(\beta))$. Denote $g(\beta) = (1-\beta) M(\beta)$ and $M(\beta) = p(\beta) \cdot z(p(\beta), \beta p(\beta))$. There are two distinct ways for $(p(\beta), \beta)$ to be a steady state: if $\beta = 1$ or if $M(\beta) = 0$. The function $M(\cdot)$ is continuous and the boundary condition A.4 implies that there exists $\underline{\beta} > 0$ such that $M(\beta) > 0$ for all $\beta < \underline{\beta}$ and $\bar{\beta} > 1$ such that $M(\beta) < 0$ for all $\beta > \bar{\beta}$; see Kehoe and Levine [16] for details. Since the graph of $M(\beta)$ intersects the line $\beta = 1$ only once and the line $M = 0$ only once, it must look like one of the three possibilities depicted in Fig. 1. (It is natural to conjecture that $M(\beta)$ is always downward sloping. We have not been able to prove this, however.) ∎

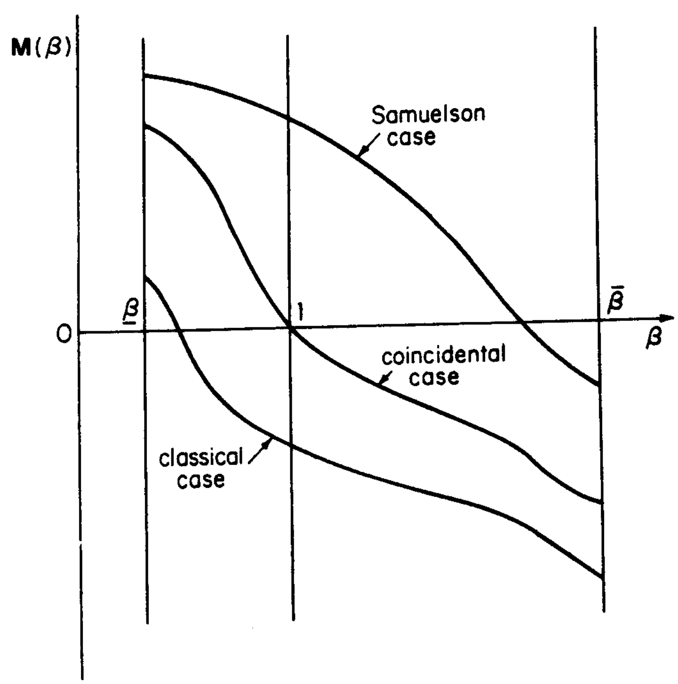

FIGURE 1

5. Stationary Overlapping Generations Economies: Turnpike Properties

In this section we study the equilibrium dynamics of the gross substitute economies described in the last section. We shall see that strong turnpike properties are satisfied. In contrast to the uniqueness theorems of Section 2, there is no analog of the turnpike property in the finite static model.

We say that a price sequence p converges (forward) to a steady state (q, β) if

$$\lim_{t \to \infty} \frac{1}{\|(p_t, p_{t+1})\|}(p_t, p_{t+1}) = \frac{1}{\|(q, \beta q)\|}(q, \beta q).$$

If the sequence p is double ended, then we can also speak of convergence backward to (q, β), namely,

$$\lim_{t \to -\infty} \frac{1}{\|(p_{t-1}, p_t)\|}(p_{t-1}, p_t) = \frac{1}{\|((1/\beta)q, q)\|}\left(\frac{1}{\beta}q, q\right).$$

The next theorem contains the basic turnpike argument.

THEOREM D. *Suppose that* (y, z) *satisfies* A.1–A.6. *Then every price sequence* p *that is an eventual equilibrium* (*there is* T *such that* $y(p_t, p_{t+1}) + z(p_{t-1}, p_t) = 0$ *for all* $t \geq T$) *converges to a steady state.*

Proof. Define the price sequences $\bar{p}$ and $\hat{p}$ by $\bar{p}_t = p_t$ and $\hat{p}_t = p_{t+1}$, in other words, $\hat{p}$ is a shift of $\bar{p} = p$. Because of stationarity both $\bar{p}$ and $\hat{p}$ are eventual equilibria. Define $\bar{r}_t = \max_i \hat{p}_t^i/\bar{p}_t^i$ $(= \max_i p_{t+1}^i/p_t^i)$, $\underline{r}_t = \min_i \hat{p}_t^i/\bar{p}_t^i$ $(= \min_i p_{t+1}^i/p_t^i)$. By Step 2 of the proof of Theorem B the sequences $\bar{r}_t$, $\underline{r}_t$ are such that, if $\bar{r}_t$ increases in one period $t > T$, then it cannot ever again decrease. Similarly, if $\underline{r}_t$ decreases, then it can never again increase. Hence, both sequences are eventually monotone. On the other hand, Lemma 1 applied to p implies that both sequences are bounded. Therefore, $\bar{r}_t \to \bar{r}$ and $\underline{r}_t \to \underline{r}$ for some $\bar{r}$, $\underline{r}$.

By Step 1 of the proof of Theorem B, we have $\bar{r} = \underline{r}$. Call this common value β. We can conclude that any accumulation point of the sequence $\{(1/\|(p_t, p_{t+1})\|)(p_t, p_{t+1})\}$ is of the form $(q, \beta q)$ and, by continuity, that it satisfies $y(q, \beta q) + z((1/\beta)q, q) = 0$. We saw in the previous section (proof of Theorem C) that given β this equation has at most one solution. Hence, $\{(1/\|(p_t, p_{t+1})\|)(p_t, p_{t+1})\}$ has at most one accumulation point, which implies that the price sequence converges to a steady state. ∎

We next put on record the basic conservation law implied by Walras's law:

PROPOSITION 4. *Suppose that* (y, z) *obeys Walras's law* A.7. *Then the amount of nominal debt transferred from generation to generation* $p_{t+1} \cdot z(p_t, p_{t+1})$ *is constant along an equilibrium price path of either the double-ended model or the single-ended model (whatever the initial conditions).*

Proof. Premultiplying the equilibrium conditions $z(p_{t-1}, p_t) + y(p_t, p_{t+1}) = 0$ by p_t and using Walras's law $p_t \cdot y(p_t, p_{t+1}) = -p_{t+1} \cdot z(p_t, p_{t+1})$, we get $p_t \cdot z(p_{t-1}, p_t) = p_{t+1} \cdot z(p_t, p_{t+1}) = \cdots$ ∎

We are now in a position to characterize the dynamics of the double-ended model.

THEOREM E. *Suppose that* (y, z) *satisfies* A.1–A.7. *Then every equilibrium of the double-ended infinity model goes from a steady state to a steady state with at least as large an inflation factor. More precisely, every equilibrium is characterized by one of the following three rules:*

(i) *in the classical case, every equilibrium, besides the two steady states, converges forward to the nominal steady state and backward to the real steady state,*

(ii) *in the Samuelson case, every equilibrium, besides the two steady states, converges forward to the real steady state and backward to the nominal steady state,*

(iii) *in the coincidental case, the single steady state is the unique equilibrium.*

Furthermore, in any of these cases the set of equilibria can be parameterized by at most n variables.

Proof. Let $p = (..., p_t, ...)$ be an equilibrium which is not a steady state. Put $\bar{r}_t = \max_i p^i_{t+1}/p^i_t$, $\underline{r}_t = \min_i p^i_{t+1}/p^i_t$. By Theorem D, $\bar{r}_t \to \beta$, $\underline{r}_t \to \beta$ as $t \to +\infty$ and, reversing time, $\bar{r}_t \to \alpha$, $\underline{r}_t \to \alpha$ as $t \to -\infty$. Because p is not a steady state we know that either $\bar{r}_t > \alpha$ or $\underline{r}_t < \alpha$ for some t. Say that $t = 0$. If $\bar{r}_0 > \beta$, then $\bar{r}_{t+1} > \bar{r}_t$ for some $t < 0$. But we saw in the proof of Theorem D that if $\bar{r}_{t+1} > \bar{r}_t$, then $\bar{r}_t > \bar{r}_t$ for all $t' > t$. Hence, $\beta > \alpha$. Similarly, if $\underline{r}_0 < \alpha$, then $\beta < \alpha$.

Therefore, p converges backwards and forwards to different steady states. In the coincidental case (iii), this implies that there cannot exist any non-steady state equilibrium. Suppose we are in the noncoincidental case, and call α and β the backward and forward limit rates of growth of p_t, respectively. By Proposition 3, $M = p_{t+1} \cdot z(p_t, p_{t+1})$ is a constant independent of t. Because either $\alpha = 1$ or $\beta = 1$ we get, taking the appropriate limit, that $M \neq 0$. In turn, this eliminates the possibility that $\beta < 1$ or $\alpha > 1$ (in either of these two cases we would get $M = 0$ by taking the appropriate limit). In summary, $\beta > \alpha$. Therefore, in the classical case we must have $\alpha < 1$ and $\beta = 1$ while in the Samuelson case we must $\alpha = 1$ and $\beta > 1$.

Finally, notice that, by Proposition 2, two different equilibria are not compatible with the same relative prices and real balances at $t = 0$. This provides the n parameters to index the equilibrium set. ∎

We do not know whether or not there are examples of classical case or Samuelson case economies in which the only equilibria are the two steady states.

We now turn to the single-ended model. Its equilibrium dynamics requires special study, because while it is true (and immediate to see) that every double-ended equilibrium can be made into an equilibrium of the single-ended model for appropriate initial conditions, this is not the case in the other direction: for some initial conditions the equilibria of the single-ended model cannot be viewed as restrictions of double-ended equilibria.

THEOREM F. *Suppose that* (z_0, y, z) *satisfies* A.1–A.7 *and that* z_0 *is of the form* $z_0(q, m)$, *where* $q \cdot z_0(q, m) = \pi(q)m$ *for* $\pi(q)$, *a price index. Then*

(i) *in the classical case, an equilibrium can exist only if* $m \leqslant 0$. *Moreover, except for* $m = 0$, *the (unique) equilibrium converges to the nominal steady state.*

(ii) *in the Samuelson case, suppose that* z_0 *is strictly increasing in* m *(normal demand). Then except for at most one value of* m *the (unique) equilibrium converges to the real steady state.*

Proof. (i) By Proposition 4, $M = \pi(p_1)m = p_{t+1} \cdot z(p_t, p_{t+1})$ for all t. By Theorem D, p_t converges to a steady state. If it converges to the real steady state, then $p_t \to 0$ and so $M = 0$. Therefore, if $m \neq 0$, then p_t converges to the monetary steady state $(q, 1)$. Since $q \cdot z(q, q) < 0$ (see Theorem C) it follows that $m < 0$.

(ii) Suppose, to the contrary, that there are two (distinct) equilibria $(\hat{p}_1, \hat{p}_2, ...)$, $(\bar{p}_1, \bar{p}_2, ...)$ for, respectively, $\hat{m}$ and $\bar{m}$, converging to the nominal steady state. Normalize prices so that $\lim_{t\to\infty} \hat{p}_t = \lim_{t\to\infty} \bar{p}_t$. By Proposition 4, $\pi(\hat{p}_1)\hat{m} = \pi(\bar{p}_1)\bar{m}$, in other words, nominal money balances are equal. On the other hand, Theorem B says that $\hat{m} \neq \bar{m}$—that is, real money balances cannot be equal—so $\pi(\hat{p}_1) \neq (\bar{p}_1)$. Assume, to be specific, that $\pi(\hat{p}_1) > \pi(\bar{p}_1)$, which implies $\hat{m} < \bar{m}$. Then, since π is monotonically increasing, $\hat{p}_1^i > \bar{p}_1^i$ for some i. As in the proof of Theorem B, consider the ratios $\bar{r}_t = \max_i \hat{p}_t^i/\bar{p}_t^i$. We know that $\bar{r}_1 > 1$. It must also be the case that $\bar{r}_1 > \bar{r}_2$; otherwise, by Step 1 of the proof of Theorem B, $1 < \bar{r}_1 \leqslant \bar{r}_2 \leqslant$

$\bar{r}_3 \leqslant \cdots$, which would contradict $\lim_{t\to\infty} \bar{r}_t = 1$. Since $(\bar{r}_1 \bar{p}_1, \bar{r}_1 \bar{p}_2) \geqslant (\hat{p}_1, \hat{p}_2)$ and $\bar{r}_1 \bar{p}_1^i = \hat{p}_1^i$ for some i, weak gross substitutability implies that

$$f_1^i(\bar{p}_1, \bar{p}_2, \hat{m}) = f_1^i(\bar{r}_1 \bar{p}_1, \bar{r}_1 \bar{p}_2, \hat{m}) \geqslant f_1^i(\hat{p}_1, \hat{p}_2, \hat{m}) = 0 = f_1^i(\bar{p}_1, \bar{p}_2, \bar{m}).$$

But this contradicts the normality hypothesis on z_0 since $\bar{m} > \hat{m}$. ▮

Remark. It is more natural to assume that z_0 is weakly increasing in m, that $\hat{m} \geqslant \bar{m}$ implies $z_0(p, \hat{m}) \geqslant z_0(p, \bar{m})$. To prove the result under this assumption would require that we strengthen our gross substitutability assumption enough to establish that $f_1^i(\bar{r}_1 \bar{p}_1, \bar{r}_1 \bar{p}_1 \hat{m}) > f_1^i(\hat{p}_1, \hat{p}_2, \hat{m})$.

Remark. We should emphasize that we do not concern ourselves with existence issues in this paper. Thus, while Theorem F provides conditions under which a single-ended infinity economy has at most one equilibrium with positive fiat money whose value does not converge to zero, we remain silent about the existence of such an equilibrium.

It is interesting to compare the turnpike results of this section with those familiar from the more traditional growth and equilibrium theory (see McKenzie [20] and Bewley [7]). In the latter, there is, so to speak, a single generation of infinitely lived agents. The key hypothesis in most of these results is that the rate of discount is low. This assumption cannot even be formulated in an overlapping generations context, and therefore our turnpike results are not particularly related to them. It would be worthwhile to explore, however, their relationship to the few turnpike theorems in the traditional literature which do not depend on patient consumers (notably Araujo and Scheinkman [1]).

6. Stationary Overlapping Generations Economies with Land

In this section we consider stationary overlapping generations economies with "land." As in Muller and Woodford [22, 23], land is an asset that yields a constant positive quantity of perishable consumption goods each period.

Let one unit of type i land yield one unit of good i each period, and let $d \in R_+^n$ be the vector giving the total quantity of each type of land. Note that land neither grows nor depreciates. For the single-ended infinity model land is owned by the initial old generation. If we let s denote the total initial value of all land, we can write the excess demand of the initial old as $z_0(p, s)$. We assume that it satisfies Walras's law. The excess demands of generations $t \geqslant 1$ are exactly as in the previous sections as these generations initially own no land. We continue to assume that A.1–A.2 and A.5–A.7 are satisfied.

Let q_t^i be the price of type i land at the beginning of period t, before the period t yield is collected. Then an equilibrium of the single-ended infinity model is a sequence of goods prices $(p_1, p_2, \ldots)$ and of land prices $(q_1, q_2, \ldots)$ such that

$$z_0(p_1, q_1 \cdot d) + y(p_1, p_2) = d$$

$$z(p_{t-1}, p_t) + y(p_t, p_{t+1}) = d, \qquad 2 \leqslant t \leqslant \infty,$$

$$q_t = p_t + q_{t+1}, \qquad 1 \leqslant t \leqslant \infty.$$

The first two conditions require that excess demand equal the yield on existing land. The final condition follows from arbitrage pricing: one unit of land in period t before the yield is collected is the same as one unit of land in period $t+1$ plus the period t yield.

The equilibrium conditions and Walras's law imply

$$\begin{aligned} p_{t+1} \cdot z(p_t, p_{t+1}) - q_{t+1} \cdot d &= -p_t \cdot y(p_t, p_{t+1}) - q_{t+1} \cdot d \\ &= p_t \cdot z(p_{t-1}, p_t) - p_t \cdot d - q_{t+1} \cdot d \\ &= p_t \cdot z(p_{t-1}, p_t) - q_t \cdot d. \end{aligned}$$

Since Walras's law for z_0 implies $p_1 \cdot z_0(p_1, q_1 \cdot d) = q_1 \cdot d$, it follows that $p_{t+1} \cdot z(p_t, p_{t+1}) = q_{t+1} \cdot d$ for all t; that is, the *aggregate transfer from generation to generation is always equal to the value of land.*

An equilibrium of the double-ended infinity model is a sequence of goods prices $(\ldots, p_{-1}, p_0, p_1, \ldots)$ and land prices $(\ldots, q_{-1}, q_0, q_1, \ldots)$ such that

$$z(p_{t-1}, p_t) + y(p_t, p_{t+1}) = d$$

$$q_t = p_t + q_{t+1}$$

$$p_t \cdot z(p_{t-1}, p_t) = q_t \cdot d.$$

The final condition requires that the value of aggregate savings equals the value of land. By the argument just given it will hold in all periods if it holds in any period.

If we assume free disposal of land, then equilibrium also requires $q_t \geqslant 0$ for all t. While this is the case of greatest interest, we have not assumed it in defining equilibrium. This is because it is useful to consider equilibria in which assets have negative values along with those in which they have positive values. For example, a steady state with positive land prices of a given double-ended infinity economy (y, z, d) corresponds to a steady state with negative land prices of the time-reversed economy just as with monetary steady states with positive and negative value of money. (Brock [8] has argued that equilibria with negative land prices are possible in an economy without limited liability for property owners.)

Our next goal is to argue that if $q_t \geqslant 0$ for all t, then

$$q_t = \sum_{\tau=t}^{\infty} p_\tau;$$

that is, there is no "speculative bubble" on the price of land. In particular, this is true if there is free disposal of land.

To see that there can be no bubble, observe that by induction on the arbitrage pricing condition

$$q_t = \sum_{\tau=t}^{T} p_\tau + q_{T+1}.$$

Since $q_t, q_{T+1} \geqslant 0$ and $p_\tau \geqslant 0$, it follows that $\sum_{\tau=t}^{\infty} p_\tau$ exists and is finite and that $\lim_{T\to\infty} q_T \geqslant 0$ exists. Moreover,

$$q_t = \sum_{\tau=t}^{\infty} p_\tau + \lim_{T\to\infty} q_T.$$

It remains to show that $\lim_{T\to\infty} q_T = 0$. If, in fact, $\lim_{T\to\infty} q^i_T > 0$, then young people in period T must spend $q^i_T d^i$ to purchase land from old people. On the other hand, $\sum_{t=1}^{\infty} p_t < \infty$ implies that eventually the value of the initial endowments of the young of some generation T is less than $q^i_T d^i$ which constitutes a contradiction. It follows that $\lim_{T\to\infty} q_T = 0$.

Note that if there is no free disposal, then $q_t < 0$ is possible; in other words, there can be "negative speculative bubbles" on the price of land in which $q_t < \sum_{\tau=t}^{\infty} p_\tau$.

In the case of an economy with land, we should adapt slightly our definitions of weak gross substitutability and of indecomposability. Let us define

$$f_1(p_1, p_2, s) = z_0(p_1, s) + y(p_1, p_2) - d$$

$$f_t(p_{t-1}, p_t, p_{t+1}) = z(p_{t-1}, p_t) + y(p_t, p_{t-1}) - d, \qquad 2 \leqslant t \leqslant \infty.$$

For $t \geqslant 2$ the weak gross substitute and indecomposability hypotheses (A.3 and A.4) are as before. The only change is in f_1 or, more precisely, in z_0. When we consider a price increase from $(\bar{p}_i, \bar{p}_2)$ to $(\hat{p}_1, \hat{p}_2)$ we should also increase $\bar{s}$ to some $\hat{s}$ so as to maintain some measure of real wealth for the initial old generation. If, heuristically, we think of s as the sum of the values of the yield of land, it is clear that in going from $(\bar{p}_1, \bar{p}_2)$ to $(\hat{p}_1, \hat{p}_2)$, and implicitly letting $\hat{p}_t \geqslant \bar{p}_t$ for all t, nominal wealth should increase by at least $(\hat{p}_1 - \bar{p}_1) \cdot d$; that is, $\hat{s} \geqslant \bar{s} + (\hat{p}_1 - \bar{p}_1) \cdot d$. With this motivation we assume

A.8 If $(\hat{p}_1, \hat{p}_2, \hat{s} - \hat{p}_1 \cdot d) \geqslant (\bar{p}_1, \bar{p}_2, \bar{s} - \bar{p}'_1 \cdot d)$ and $\hat{p}^i_1 = \bar{p}^i_1$ for some i, then $f^i_1(\hat{p}_1, \hat{p}_2, \hat{s}) \geqslant f^i_1(\bar{p}_1, \bar{p}_2, \bar{s})$. Moreover, equality holds only if $(\hat{p}_1, \hat{p}_2, \hat{s}) = (\bar{p}_1, \bar{p}_2, \bar{s})$.

Suppose that $(p_1, p_2, ...)$ is a price sequence with $\sum_{t=1}^{\infty} p_t < \infty$ and that

$$f_1\left(p_1, p_2, \left(\sum_{t=1}^{\infty} p_t\right) \cdot d\right) = f_t(p_{t-1}, p_t, p_{t+1}) = 0.$$

It is evident that if we define $q_t = \sum_{\tau=t}^{\infty} p_\tau$, then the arbitrage pricing condition is satisfied. Consequently, equilibria of the single-ended infinity economy consistent with free disposal coincide with the equilibria of an exchange economy in which the initial old have an endowment vector d in each period. Theorem B then applies directly.

PROPOSITION 5. *If the single-ended infinity economy* (z_0, y, z, d) *satisfies* A.1–A.8, *then there is at most one equilibrium in which* $q_t \geqslant 0$ *for all* t.

A steady state is an equilibrium of the double-ended infinity economy (y, z, d) such that, for some β, $p_t = \beta^t p_1$ and, correspondingly, $q_t = (\beta^t)/(1-\beta)\, p_1$. Observe that, necessarily, $0 < \beta$ and $\beta \neq 1$. A steady state is therefore consistent with free disposal if and only if $\beta < 1$. There is at least one steady state with $\beta < 1$ and at least one with $\beta > 1$. (Let $g(\beta)$ be defined from f as in the proof of Theorem C. As there, $\lim_{\beta \to 0} g(\beta) = \lim_{\beta \to \infty} g(\beta) = \infty$, but now $g(1) < 0$.)

The following characterization of the equilibrium dynamics is analogous to Theorems C–F for economies without land, and can be proven in a similar way:

THEOREM G. *Let the single infinity economy* (z_0, y, z, d) *satisfy* A.1–A.8. *Then*

(i) *There is a unique steady state with positive land prices and a unique steady state with negative land prices.*

(ii) *An equilibrium of the single-ended infinity model consistent with free disposal (which is unique) converges to the steady state with positive land prices as* $t \to \infty$. *Every other equilibrium converges to the steady state with negative land prices. The set of such equilibria can be parameterized by at most one parameter.*

(iii) *In the double-ended infinity model there is at most one equilibrium consistent with free disposal: it is the steady state with positive land prices. Every other equilibrium converges to the equilibrium with negative land prices as* $t \to \infty$ *and to the steady state with positive land prices as* $t \to -\infty$. *The set of such equilibria can be parameterized by at most* n *parameters.*

Examples of economies with continua of equilibria with negative land prices can easily be constructed using the demand functions of Section 3; see also the diagrammatic exposition of the one-good case by Brock [8].

It is also possible for there to exist equilibria with negative land prices and valued fiat money. This is mostly a matter of reinterpreting s; we shall not pursue it here.

Stationarity plays an important role in generating the uniqueness result for economies with free disposal of land. In a nonstationary economy, if endowments grow at a faster rate asymptotically than the aggregate dividends from land, then it is possible for there to be an equilibrium in which valued fiat money coexists with freely disposable land. See Wilson [27] for such an example. Tirole [26] interprets this sort of example as an economy with a speculative bubble on the price of land. In such cases there can exist continua of equilibria, in which all equilibria assign land a positive value forever.

7. Concluding Remarks

Unfortunately, our results do not apply to economies with production. Even in economies with a finite number of goods, gross substitutability in consumers' excess demand functions does not imply uniqueness if there is production, as an example due to Kehoe [14] demonstrates. Moreover, Calvo [9] presents a simple example of an overlapping generations economy with production in which the representative consumer in each generation can have gross substitutes excess demand but in which there can be a continuum of equilibria even though there is no fiat money. Furthermore, Reichlin [24] gives an example of a stationary, gross substitutes production economy in which the asymptotic behavior of the equilibria can be cyclical or chaotic.

In the case of decreasing returns production technologies, the equilibrium conditions still take the form $f(p_{t-1}, p_t, p_{t+1})=0$, where the function f represents excess demands of consumers minus net supplies of profit maximizing firms. All of our results still hold if the function f exhibits gross substitutability. Furthermore, there are meaningful economic conditions, albeit highly restrictive ones, under which this property holds. If firms are initially owned by the initial old generation, then the present discounted value of all profits enters the budget constraint of the initial old and no subsequent generation. Consequently, a pattern of preferences and endowments that produces a gross substitutes excess demand function in the pure exchange case produces the same excess demands for consumers in all generations but the initial old. Furthermore, there are known cases in which firm supplies exhibit gross substitutability. One case is that of land: net supply is a constant vector regardless of prices. Another case is production of a single output using a single input. In this case, net supply of both goods must be a function only of the price ratio of these two goods;

convexity of the profit function then guarantees that both net supplies are monotonic functions of this price ratio, which implies gross substitutability. This case applies to the first example of Calvo [9], in which a single perishable good is produced using labor and a fixed factor initially owned by the initial old generation. In this example he is only able to generate a continuum of equilibria by using consumer excess demand functions that violate gross substitutability. Calvo's second example, in contrast, has a continuum of equilibria without fiat money despite gross substitutability in the consumer excess demand functions; this depends on constant returns production without a fixed factor. (See the discussion of gross substitutability in production economies in Muller and Woodford [23]).

The case for gross substitutability in consumer excess demand functions is even weaker in intertemporal models than it is in static models. With constant-elasticity-of-substitution demand functions the elasticity of substitution must be greater than or equal to one for gross substitutability to hold at all price vectors; empirical studies have shown the intertemporal elasticity of substitution in consumption to be significantly less than one (see, for example, Mankiw, Rotemberg, and Summers [21]). See Fisher [10] for a general discussion of conditions on the utility functions that imply gross substitutability in the desired demand functions. Kehoe and Levine [18] present a simple example, with plausible parameters, of an overlapping generations model with one good in each period and a single three-period-lived consumer in each generation, that has a continuum of equilibria without fiat money and a two-dimensional set of equilibria with fiat money: the consumer's demand function violates gross substitutability.

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