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REGULAR PRODUCTION ECONOMIES*

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In this paper we extend the concept of regularity developed by Mas-Colell and Kehoe for constant-returns production economies to economies with primary and intermediate goods. To do so, we must deal with consumer demand functions that satisfy boundary conditions more general than any considered previously. We initially specify the production technology as a linear activity analysis model that allows free disposal of all commodities. Later, we indicate how our results can be extended to economies with more general production technologies.

1. Introduction

In recent years mathematical economists have made heavy use of the tools of differential topology in their study of economic equilibrium. The advantage of the differentiable approach is that the concept of regular economy, developed by Debreu (1970), distinguishes certain degenerate situations from generic ones. In an appropriately parameterized space of economies a regular economy is one whose equilibria are locally unique and vary continuously with the underlying data. Moreover, the concept of fixed point index, introduced into economics by Dierker (1972), provides a method for counting the equilibria that implies necessary and sufficient conditions for uniqueness of equilibrium in a regular economy. The appeal of the concept of regularity is enhanced by its genericity in the space of economies; almost all economies are regular in an appropriately defined topological or measure-theoretic sense.

With several notable exceptions [Fuchs (1974), Kehoe (1979, 1980), Mas-Colell (1975, 1976, 1978), and Smale (1974)], researchers in this area have

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focused their attention on the pure exchange model that allows no production. Both Mas-Colell (1978) and Kehoe (1980) have extended the concepts of regularity and fixed point index to economies with constant-returns-to-scale production technologies. Unfortunately, although the approaches employed by these two writers differ significantly, they share a common shortcoming: Neither allows for the existence of primary or intermediate goods.

To demonstrate the genericity of regular economies, researchers usually allow perturbations of the consumers' excess demand functions. This procedure is entirely natural for economies where all commodities enter into consumers' final demands. In economies where production plays an important role, however, there are likely to be commodities that are inelastically supplied as inputs into the production process and commodities that are only produced to serve as inputs in the production of other commodities. We call the first group of commodities primary goods and the second group intermediate goods. We call commodities that are neither primary goods nor intermediate goods final goods. Obviously, if we perturb the excess demand function of an economy with primary and intermediate goods, we may destroy the primary and intermediate characteristics of these commodities.

Actually, the approach used by Mas-Colell (1975) makes explicit provision for primary, but not intermediate, goods. His results, however, pertain only to equilibrium price vectors that are elements of some given compact set of strictly positive prices. To extend these results to all possible equilibrium prices requires assumptions about the behavior of the demand functions at vectors where some prices are zero that, in effect, rule out the possibility of primary goods. In fact, these assumptions are necessary to prove the existence of equilibrium for his model.

In this paper we extend the concept of regularity developed by Mas-Colell and Kehoe for constant-returns production economies to economies with primary and intermediate goods. To do so, we must deal with consumer demand functions that satisfy boundary conditions more general than any considered previously. We initially specify the production technology as an activity analysis model that allows free disposal of all commodities. Later, we indicate how our results can be extended to economies with more general production technologies.

2. The model

Let us begin by describing our economic model. We specify the consumption side of the model using the concept of excess demand directly, leaving the concepts of consumer preference orderings and initial endowments in the background. It may help the intuition, however, to picture the vector of market excess demand functions $\xi(\pi)=(\xi_1(\pi),\ldots,\xi_n(\pi))$ as generated by aggregating the responses of consumers to a vector of non-negative prices $\pi=(\pi_1,\ldots,\pi_n)$. Each consumer sells some initial endowment of resources to finance the purchase of a bundle of goods for consumption. Both the initial endowment and the consumption bundle are considered as vectors in R^n, the space of n perfectly divisible commodities. The excess demand for good i at prices π, $\xi_i(\pi)$, is the difference between aggregate consumption and aggregate initial endowment of the good. We take these functions to be completely arbitrary aside from the following assumptions:

(A.1) *Differentiability.* Let Z be a subset of the boundary of R^n_+ that includes the origin.

(a) ξ is a continuously differentiable function defined over the domain $R^n_+\backslash Z$.
(b) ξ is bounded from below on $R^n_+\backslash Z$.
(c) If $\pi^i\to\pi$ where $\pi^i\in R^n_+\backslash Z$ and $\pi\in Z\backslash\{0\}$, then $\|\xi(\pi^i)\|\to\infty$.

(A.2) *Homogeneity.* ξ is homogeneous of degree zero; that is, $\xi(t\pi)\equiv\xi(\pi)$ for all $t>0$.

(A.3) *Walras' law.* ξ obeys Walras' law; that is, $\pi'\xi(\pi)\equiv 0$.

Assumptions A.2 and A.3 are standard and require no comment. Several remarks may be necessary, however, to prevent the technical nature of assumption A.1 from obscuring its meaning. In Kehoe (1980) it is assumed that ξ is C^1 on all $R^n_+\backslash\{0\}$. Clearly, excess demand functions that satisfy this condition also satisfy assumption A.1 in the case where Z is the origin. Likewise, excess demand functions that satisfy the more familiar differentiability and boundary assumptions employed by both Debreu (1970) and Mas-Colell (1978) also satisfy assumption A.1 in the case where Z is the entire boundary of R^n_+. The generality of assumption A.1 is motivated by the situation that we are considering. A primary good is a commodity i for which $\xi_i(\pi)\equiv -w_i$ for some $w_i>0$; here w_i can be interpreted to be the aggregate initial endowment of commodity i. An intermediate good is a commodity i for which $\xi_i(\pi)\equiv 0$. Neither Kehoe's nor Debreu's and Mas-Colell's boundary assumptions are appropriate for an economy with primary and intermediate goods. In an economy with primary goods, for example, Walras's law is not compatible with the assumption that ξ is continuous at price vectors where the only price that is positive is that of some primary good. To illustrate this point, suppose that $\xi_i(\pi)\equiv -w_i<0$ is the demand for a primary good. Then at $\pi=(1,0,\ldots,0)$ $\pi'\xi(\pi)=-w_i\neq 0$ if ξ is, in fact, continuous. On the other hand, it is not possible to assume that $\|\xi(\pi)\|$ becomes unbounded at price vectors where the only price that is zero is that of a primary good. Walras's law and the assumption that ξ is bounded from below would then cause a contradiction.

One problem with assumption A.1 is that it requires ξ either to be continuous or to have unbounded norm at all price vectors in $R^n_+\backslash\{0\}$. Even this is too much to expect of price vectors whose only positive elements are the prices of some intermediate goods. It is natural to assume that consumers face the same situation at a price vector π^0 where the only positive price is that of an intermediate good as at the price vector where all prices are zero. Consequently, the homogeneity assumption makes it impossible to assign any value to $\xi(\pi^0)$ or even to claim that $\|\xi(\pi^i)\|$ becomes unbounded along all sequences $\pi^i \to \pi^0$. In our subsequent treatment of economies with intermediate goods we are able to avoid this problem by leaving the concept of excess demand undefined at all such price vectors.

Our initial characterization of the production technology is that of an activity analysis model. As is usual with constant-returns technologies, the delineation of individual producers or firms does not matter in the study of equilibria; all that matters is the aggregate technology. Such a technology is specified by an $n \times m$ activity analysis matrix A. Each column of A represents an activity, or known technological process, that transforms inputs taken from either the $n \times 1$ vector of initial endowments or the outputs of other activities into outputs, which are either consumed or further used as inputs. Positive entries in an activity vector denote quantities of outputs produced by the activity; negative entries denote quantities of inputs consumed. Aggregate production is denoted Ay where y is an $m \times 1$ vector of non-negative activity levels. We assume that A satisfies the following:

(A.4) *Free disposal.* A includes n free disposal activities, one for each commodity. In other words, the $n \times m$ matrix $-I$ is a submatrix of A.

(A.5) *Boundedness.* There are no outputs without any inputs: $\{x \in R^n \mid x = Ay \geqq 0,\ y \geqq 0\} = \{0\}$. Equivalently, there exists some vector $\bar{\pi} \in R^n$, strictly positive, such that $\bar{\pi}'A < 0$.

For our present purposes an economy is completely described by an excess demand function representing the consumption side and by an activity analysis matrix representing the production side. We define an equilibrium of (ξ, A) in the usual way:

Definition. An *equilibrium* of (ξ, A) is a price vector $\hat{\pi} \in R^n_+\backslash Z$ that satisfies the following conditions:

(a) $\hat{\pi}'A \leqq 0.$

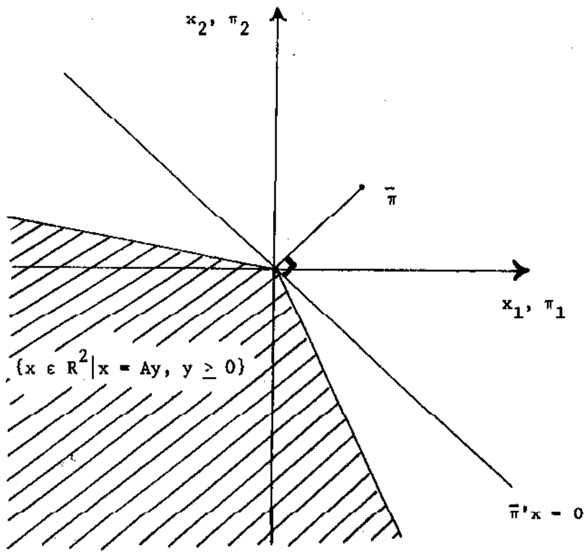

Fig. 1

(b) There exists $\hat{y} \geqq 0$ such that $\xi(\hat{\pi}) = A\hat{y}$.

(c) $\sum_{i=1}^{n} \hat{\pi}_i = 1$.

We shall find it useful to consider whole spaces of economies. To do so, we must specify some topological structure for the set of functions that satisfy assumptions A.1–A.3 and the set of activity analysis matrices that satisfy assumptions A.4–A.5. Let $\mathscr{A} \subset \{-I\} \times R^{n \times (m-n)}$ be the space of activity analysis matrices; here $-I$ is the submatrix of free disposal activities. We endow $\mathscr{A}$ with the standard topology on $R^{n \times (m-n)}$ by defining the metric

$$d(A^1, A^2) = \left(\sum_{i=1}^{n} \sum_{j=n+1}^{m} (a_{ij}^1 - a_{ij}^2)^2 \right)^{\frac{1}{2}} \quad \text{for any} \quad A^1, A^2 \in \mathscr{A}.$$

Let $\mathscr{D}^1$ be the space of excess demand functions. We endow $\mathscr{D}^1$ with the topology of uniform C^1 convergence on compacta. Letting M be some compact subset of $R_+^n \backslash Z$, we define the metric

$$d_M(\xi^1, \xi^2) = \sup_{i,\pi \in M} |\xi_i^1(\pi) - \xi_i^2(\pi)| + \sup_{i,j,\pi \in M} \left| \frac{\partial \xi_i^1}{\partial \pi_j}(\pi) - \frac{\partial \xi_i^2}{\partial \pi_j}(\pi) \right| \quad \text{for any} \quad \xi^1, \xi^2 \in \mathscr{D}^1.$$

The homogeneity assumption A.2 allows us to restrict our attention to compact subsets of $S\backslash Z$; here S is the unit simplex

$$\left\{\pi \in R^n \middle| \pi_i \geqq 0, \sum_{i=1}^{n} \pi_i = 1\right\}.$$

The space of economies $\mathscr{E}^1 = \mathscr{D}^1 \times \mathscr{A}$ has the induced product topology: For any $(\xi^1, A^1), (\xi^2, A^2) \in \mathscr{E}^1$ we define the metric

$$d_M[(\xi^1, A^1), (\xi^2, A^2)] = d_M(\xi^1, \xi^2) + d(A^1, A^2).$$

Definition. An *economy* (ξ, A) is an element of the topological space $\mathscr{E}^1 = \mathscr{D}^1 \times \mathscr{A}$.

3. Existence of equilibrium

To demonstrate the existence of equilibrium for an economy $(\xi, A) \in \mathscr{E}^1$, we construct a continuous, single-valued function $g: S \to S$ whose fixed points are equivalent to equilibria of (ξ, A). The generality of the boundary conditions embodied in assumption A.1 forces us to modify the approach used by Kehoe (1980).

We begin by noting that A.1 implies that there exists $w \in R^n_+$ such that $\xi(\pi) \geqq -w$ for all $\pi \in R^n_+ \backslash Z$. Observe that w could be interpreted as the vector of aggregate initial endowments, although, in the formal system that we have here, $-w$ need only be some lower bound on ξ. Assumption A.5 implies that the production possibility set $\{x \in R^n | x - Ay \geqq -w, y \geqq 0\}$ is bounded. Hence there exists $\bar{\mu} \in R^n_+$ such that $\bar{\mu} > x$ for all $x \in \{x \in R^n | x = Ay \geqq -w, y \geqq 0\}$. Letting $\bar{\pi} \in R^n$ be such that $\bar{\pi} > 0$ and $\bar{\pi}' A < 0$, we define the set

$$W = \{\pi \in R^n_+ \backslash \{0\} | \bar{\pi}' \xi(\pi) \leqq \bar{\pi}' \bar{\mu}\}.$$

Notice that we can choose the elements of $\bar{\mu}$ large enough so that W is non-empty and contains $\bar{\pi}$ in its interior. Assumption A.1 implies that $\bar{\pi}' \xi(\pi^l) \to \infty$ as $\pi^l \to \pi \in Z \backslash \{0\}$. Therefore $W \cap Z = \emptyset$. Note too that any equilibrium $\hat{\pi}$ is such that

$$\xi(\hat{\pi}) \in \{x \in R^n | x = Ay \geqq -w, y \geqq 0\}.$$

Consequently, $\bar{\pi}' \xi(\hat{\pi}) < \bar{\pi}' \bar{\mu}$ implies that any equilibrium $\hat{\pi}$ is contained in the interior of W relative to $R^n_+ \backslash \{0\}$. The motivation for these definitions should become apparent in the subsequent lemma.

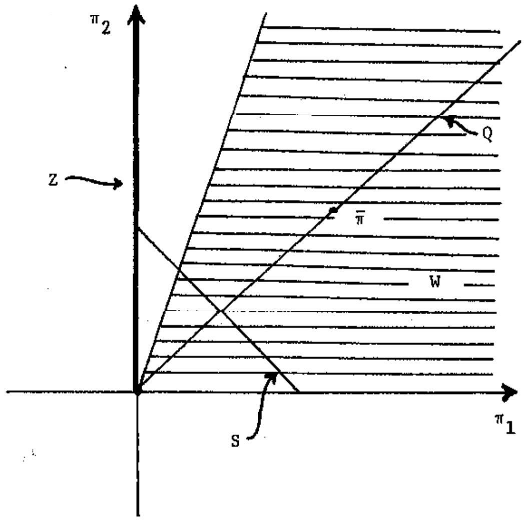

Fig. 2

We want to bound ξ away from infinity in such a way that it remains a smooth function and the equilibria of (ξ, A) are not affected.

Lemma 1. *Let* $\bar{\mu}$, $\bar{\pi} \in R^n_+$ *and* $W \subset R^n_+ \setminus \{0\}$ *be defined as above. For any* $(\xi, A) \in \mathscr{E}^1$ *there exists a* C^1 *function* $\xi^*: R^n_+ \setminus \{0\} \to R^n$ *such that:*

(a) ξ^* *satisfies assumptions A.2 and A.3.*

(b) $\xi^*(\pi) = \xi(\pi)$ *for all* $\pi \in W$.

(c) $\bar{\pi}' \xi^*(\pi) \geqq \bar{\pi}' \bar{\mu}$ *for all* $\pi \in R^n_+ \setminus \{0\}$, $\pi \notin W$.

Proof. Let

$$Q = \{\pi \in R^n_+ \mid \pi = t\bar{\pi},\ t \geqq 0\}.$$

We define the C^1 function $\zeta: R^n_+ \setminus Q \to R^n$ by the rule

$$\zeta_i(\pi) = \frac{\bar{\pi}_i \sum \pi_j^2 - \pi_i \sum \pi_j \bar{\pi}_j}{(\sum \pi_j^2)(\sum \bar{\pi}_j^2) - (\sum \pi_j \bar{\pi}_j)^2} \sum \bar{\pi}_j \bar{\mu}_j;$$

here all summations are over $j = 1, \ldots, n$. Observe that ζ satisfies A.2 and A.3 and is such that $\bar{\pi}' \zeta(\pi) = \bar{\pi}' \bar{\mu}$. By smoothing averaging ζ with ξ on $R^n_+ \setminus W$, we can construct the desired function ξ^*. Let $\theta: S \to R$ be a C^1 function with the following properties:

$$\theta(\pi)=1 \quad \text{if} \quad \bar{\pi}'\xi(\pi) \leqq \bar{\pi}'\bar{\mu},$$

$$\theta(\pi)=0 \quad \text{if} \quad \bar{\pi}'\xi(\pi) \geqq 2\bar{\pi}'\bar{\mu} \quad \text{or if} \quad \pi \in Z,$$

$$0<\theta(\pi)<1 \quad \text{if} \quad \bar{\pi}'\bar{\mu} < \bar{\pi}'\xi(\pi) < 2\bar{\pi}'\bar{\mu}.$$

Assumption A.1 implies that we can always find such a function. See Hirsch (1976, pp. 41–42) for an explicit construction. We let $\eta: R^n_+ \setminus \{0\} \to R$ be defined as $\eta(\pi)=\theta(\pi/e'\pi)$; here $e=(1,\dots,1)\in R^n$. To demonstrate our contention, we merely define

$$\begin{aligned} \xi^*(\pi) &= \eta(\pi)\xi(\pi)+(1-\eta(\pi))\zeta(\pi) && \text{if} \quad \pi \in R^n_+ \setminus (Z \cup Q) \\ &= \xi(\pi) && \text{if} \quad \pi \in Q \setminus \{0\}, \\ &= \zeta(\pi) && \text{if} \quad \pi \in Z \setminus \{0\}. \quad \square \end{aligned}$$

This lemma allows us to exploit the results contained in Kehoe (1980), where it is assumed the ξ is C^1 on all $R^n_+ \setminus \{0\}$. Letting N be any non-empty, closed, convex subset of R^n, we define $p^N: R^n \to N$ as the continuous map that projects any point in R^n into the point in N that is closest in terms of euclidean distance. We also define the set

$$S_A = \{\pi \in R^n \mid \pi' A \leqq 0, e'\pi = 1\}.$$

Note that, as a consequence of A.4 and A.5, S_A is a non-empty, closed, convex subset of S. In fact, it is a convex polyhedron.

Definition. For any economy $(\zeta, A) \in \mathscr{E}^1$ define the map $g: S \to S$ by the rule

$$g(\pi) = p^{S_A}(\pi + \xi^*(\pi)).$$

The definition of this map g is based on a similar construction due to Todd (1979). Kehoe (1980) has proven the equivalence of fixed points of g and equilibria of (ξ, A). We now present an alternative proof, which is more easily extended to general constant-returns production technologies.

Theorem 1. $g(\hat{\pi}) = \hat{\pi}$ *if and only if* $\hat{\pi}$ *is an equilibrium of* (ξ, A).

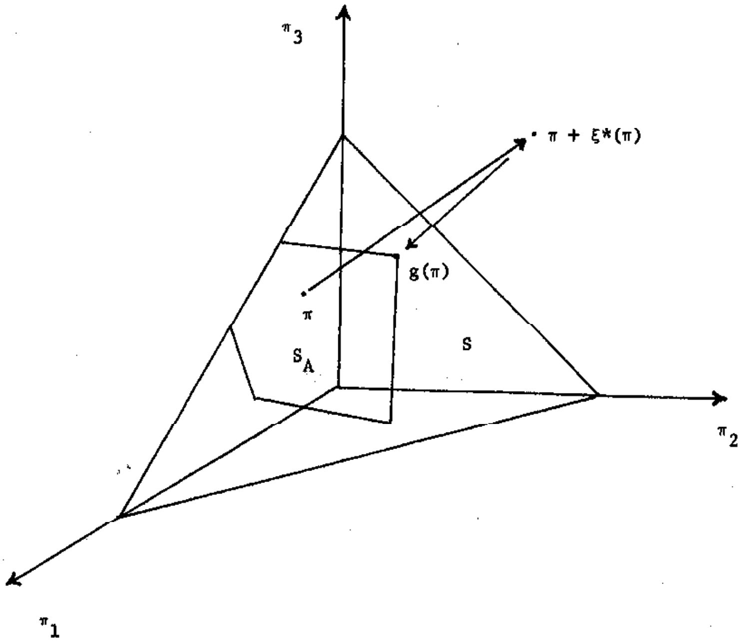

Fig. 3

Proof. $g=g(\pi)$ if and only if

$$(g-\pi-\xi^*(\pi))'(g-\pi-\xi^*(\pi))\leqq(p-\pi-\xi^*(\pi))'(p-\pi-\xi^*(\pi)),$$

for all $p\in S_A$. Because S_A is convex, this inequality holds if and only if

$$(g-\pi-\xi^*(\pi))'g\leqq(g-\pi-\xi^*(\pi))'p,$$

for all $p\in S_A$. For proof of this statement we refer to Berge (1963, p. 162). If $(\pi+\xi^*(\pi))\notin S_A$, then this inequality can be interpreted as stating that the hyperplane passing through g with normals $(g-\pi-\xi^*(\pi))$ separates g from S_A. If $(\pi+\xi^*(\pi))\in S_A$, then $g=\pi+\xi^*(\pi)$, and the inequality is trivial.

Suppose $\hat{\pi}=g(\hat{\pi})$. Then the above inequality implies that

$$p'\xi^*(\hat{\pi})\leqq\hat{\pi}\xi^*(\hat{\pi})=0,$$

for all $p\in S_A$. Since we can multiply this inequality by any positive constant without changing it, we obtain

$$p'\xi^*(\pi)\leqq 0 \quad \text{for all} \quad p\in Y_A^*=\{\pi\in R^n \mid \pi'A\leqq 0\}.$$

This inequality holds if and only if $\xi^*(\hat{\pi})$ is an element of the dual cone of Y_A^*, that is, if and only if $\xi^*(\hat{\pi})$ is an element of the production cone $Y_A = \{x \in R^n | x = Ay, y \geqq 0\}$. Now $\xi^*(\hat{\pi}) = A\hat{y}$ implies that $\hat{\pi}$ is in the relative interior of W. Otherwise, $\pi'\xi^*(\hat{\pi}) \geqq \bar{\pi}'\bar{\mu} > 0$ and $\bar{\pi}'A\hat{y} \leqq 0$ cause a contradiction.

Consequently, since $\hat{\pi} \in S_A$ and $\xi(\hat{\pi}) = \xi^*(\hat{\pi}) = A\hat{y}$, $\hat{\pi}$ is an equilibrium. Conversely, if $\hat{\pi}$ is an equilibrium then

$$p'\xi^*(\hat{\pi}) = p'\xi(\hat{\pi}) \leqq 0 \quad \text{for all} \quad p \in S_A,$$

which implies that $g(\hat{\pi}) = \hat{\pi}$. □

Observe that, because g is continuous and maps the non-empty, compact, convex set S into itself, it has a fixed point. Hence any economy $(\xi, A) \in \mathscr{E}^1$ has an equilibrium.

4. Regular economies

In the following analysis we focus our attention on derivatives of g. To simplify the presentation, we let $X \subset R^n$ be a smooth (that is, C^1) n-dimensional manifold with boundary that contains S in its interior, does not contain the origin, and is compact and convex. For example, it is easy to verify that

$$X = \left\{ x \in R^n \middle| \sum_{i=1}^{n} (x_i - \pi_i)^2 \leqq \varepsilon \quad \text{for some} \quad \pi \in S \right\}$$

satisfies these conditions if $0 < \varepsilon < 1/n$. We can easily extend ξ^* to a C^1 map on X. [See Lemma 1 in Kehoe (1980) where the approach follows closely that of Saigal and Simon (1973).] We are, therefore, justified in considering X as the domain of g.

We want to investigate the properties of economies (ξ, A) for which 0 is a regular value of the map $(g - I): X \to R^n$; here I is the identity map on R^n. Recall that for a C^1 map $f: M \to N$ from a smooth manifold of dimension m to a smooth manifold of dimension n the concept of regularity is usually defined as follows: A point $x \in M$ is a regular point if $\mathrm{D}f_x: T_x(M) \to T_{f(x)}(N)$ has rank n; in other words, is onto. A point $y \in N$ is a regular value if every point x for which $f(x) = y$ is a regular point. Points in M that are not regular points are critical points; points in N that are not regular values are critical values. By convention, any point y for which the set $f^{-1}(y)$ is empty is a regular value. Also, if $m < n$, then clearly every point $x \in M$ is a critical point. We extend these concepts to maps such as g that are not everywhere differentiable by requiring that the map $\mathrm{D}f_x$ exist at a point x for x to be a regular point.

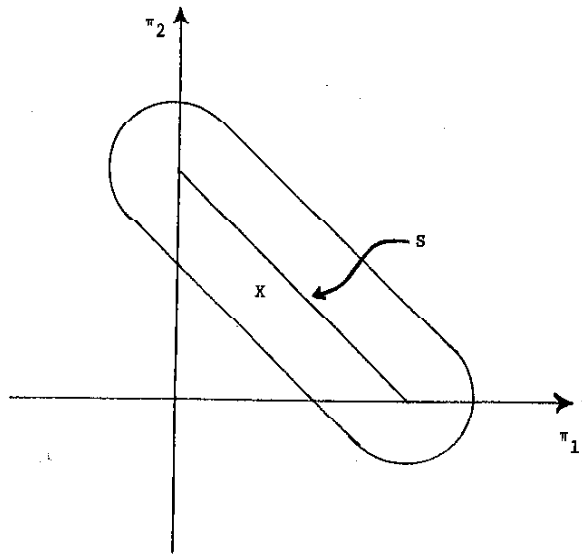

Fig. 4

To gain some intuitive understanding of this concept, consider the highly simplified equilibrium situations depicted in fig. 5. Although the diagrams here could be interpreted as representing the first coordinates, π_1 and $g_1(\pi) - \pi_1$, of a two-commodity economy where

$$\pi_2 = 1 - \pi_1 \quad \text{and} \quad g_2(\pi) - \pi_2 = \pi_1 - g_1(\pi),$$

we do not mean to be precise. The equilibria in (a) change character drastically with the slightest perturbation in the model's underlying parameters. In (b) the situation is even worse; there are an infinite number of equilibria. In contrast to (a) and (b), (c) depicts a situation where there exists a finite number of equilibria whose qualitative properties are stable under small perturbations. The distinguishing feature here is that 0 is a regular value of $g - I$ in (c), in the sense that $((\partial g/\partial \pi)(\hat{\pi}) - 1) \neq 0$ for $\hat{\pi} = g(\hat{\pi})$; in contrast, 0 is a critical value of $g - I$ in (a) and (b).

Unfortunately, the map g defined in the previous section is not everywhere differentiable. It is easy to verify, however, that, if (ξ, A) satisfies the following non-degeneracy assumptions, then g is differentiable in some neighborhood of every equilibrium price vector:

(A.6) No column of A can be expressed as a linear combination of fewer than n other columns.

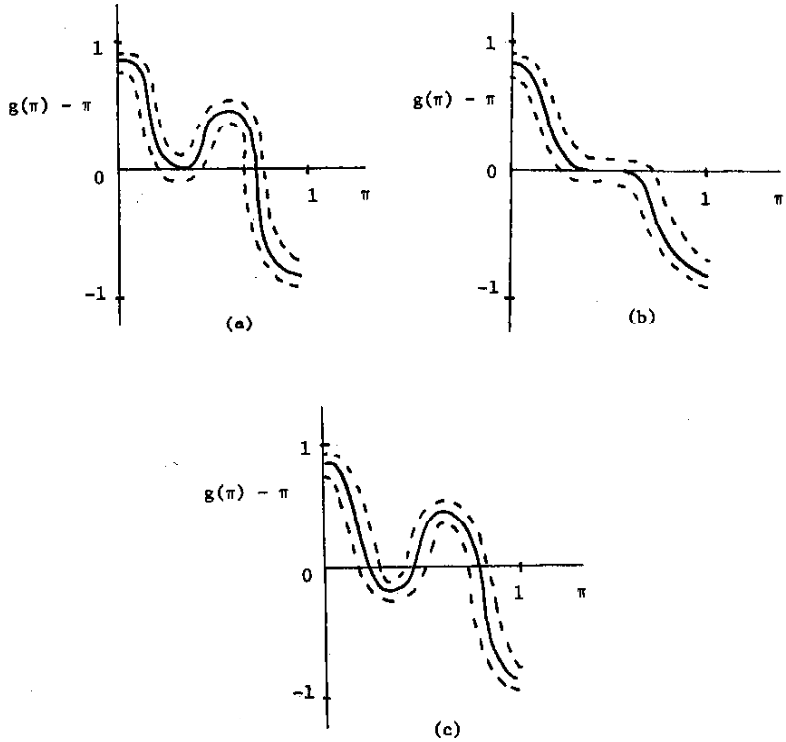

Fig. 5

(A.7) Let $B(\pi)$ denote the submatrix of A whose columns are all those activities (possibly none) that earn zero profit at π. At any equilibrium $\hat{\pi}$ all $\hat{y}_b$ are strictly positive in the equation $\xi(\hat{\pi}) = \sum_{b \in B(\hat{\pi})} \hat{y}_b b$.

Assumption A.7 is merely the requirement that all activities that earn zero profits are actually run at equilibrium. It is A.6 that deserves comment. Actually, it is somewhat stronger than needed. The requirement that $B(\hat{\pi})$ has full column rank suffices to prove that g is continuously differentiable at every equilibrium. Clearly A.6 implies that this condition holds; as we shall see it is also convenient for other reasons. Since it is trivial to prove the genericity of A.6, the alternative assumption on the column rank of $B(\hat{\pi})$ is also clearly generic.

Definition. An economy $(\xi, A) \in \mathscr{E}^1$ that satisfies assumptions A.6 and A.7

and is such that $(Dg_{\hat{\pi}}-I): R^n \to R^n$ is non-singular at every equilibrium is a *regular economy*. The set of regular economies is denoted as $\mathscr{R}^1 \subset \mathscr{E}^1$. Economies that are not regular are *critical economies*.

We justify this terminology by noting that 0 is a regular value of $g-I$ if (ξ, A) is a regular economy. In the next section we shall prove that the conditions that define a regular economy are satisfied by almost all economies in $\mathscr{E}^1$. The phrase 'almost all' is meant in the topological sense that $\mathscr{R}^1$ is open and dense in $\mathscr{E}^1$. These regularity conditions reduce to those given by Debreu (1970) in the special case of a pure exchange economy whose equilibria are all strictly positive.

Let us now briefly explore the significance of this definition of regular economy. See Kehoe (1980) for proofs of the results presented here.

Definition. The *equilibrium price correspondence* $\Pi: \mathscr{E}^1 \to S$ associates any economy $(\xi, A) \in \mathscr{E}^1$ with the set of its equilibrium price vectors.

Π is an upper-semi-continuous, point-to-set correspondence in the product topology that we have given to $\mathscr{E}^1$. Consequently for any $(\xi, A) \in \mathscr{E}^1$, the set $\Pi(\xi, A)$ is a closed subset of the compact set S, which implies that $\Pi(\xi, A)$ is compact. If $(\xi, A) \in \mathscr{R}^1$, the inverse function theorem applied to $g-I$ at any $\hat{\pi} \in \Pi(\xi, A)$ implies that the equilibria of (ξ, A) are isolated. Thus any economy $(\xi, A) \in \mathscr{R}^1$ has only a finite number of equilibria. Furthermore, the equilibrium price correspondence Π is continuous on $\mathscr{R}^1$. If $(\xi^0, A^0) \in \mathscr{R}^1$ with k equilibria, then there exists an open neighborhood $\mathscr{U} \subset \mathscr{E}^1$ of (ξ^0, A^0), such that all $(\xi, A) \in \mathscr{U}$ also have k equilibria. In fact, Π can be considered as the union of k continuous single-valued functions on $\mathscr{U}$. It can easily be demonstrated that we can choose $\mathscr{U}$ small enough so that all $(\xi, A) \in \mathscr{U}$ are regular. Thus the set $\mathscr{R}^1$ is open in $\mathscr{E}^1$. To prove that $\mathscr{R}^1$ is open and dense in $\mathscr{E}^1$, therefore, we need only prove that it is dense.

An important concept, which is closely related to regularity, is that of fixed point index. Whenever (ξ, A) is regular, we associate each fixed point of g with an index, which is either $+1$ or -1, depending on the local properties of the function at that point.

Definition. For any equilibrium $\hat{\pi}$ of a regular economy (ξ, A), *index* $(\hat{\pi})$ is defined as

$$(-1)^n \operatorname{sgn}(\det[Dg_{\hat{\pi}}-I]) = (-1)^n \operatorname{sgn}\left(\det \begin{bmatrix} 0 & e' & 0 \\ e & D\xi_{\hat{\pi}} & B(\hat{\pi}) \\ 0 & B'(\hat{\pi}) & 0 \end{bmatrix}\right).$$

Utilizing a version of the Lefschetz fixed point theorem given by Saigal and Simon (1973), we can demonstrate that $\sum_{\pi\in\Pi(\xi,A)}$ index $(\pi)=+1$ for all $(\xi,A)\in\mathscr{R}^1$. This index theorem provides us with a valuable tool for investigating the number of equilibria of an economy.

5. Genericity of regular economies

As we noted in the previous section, to prove that $\mathscr{R}^1$ is open and dense in $\mathscr{E}^1$, we need only prove that it is dense. Our approach to proving density can be motivated by an argument reminiscent of the counting of equations and unknowns by Walras. Any $\hat{\pi}\subset\Pi(\xi,A)$ satisfies the conditions

$$B'\hat{\pi}=0,\qquad \xi(\hat{\pi})=B\hat{y},$$

where B is the $n\times k$ matrix of activities $B(\hat{\pi})$ and y is an $k\times 1$ vector of non-negative activity levels. In this system of equations there are $n+k$ equations and $n+k$ variables, $\hat{\pi}$ and $\hat{y}$. Although Walras's law A.3 allows us to eliminate one of the equations, the homogeneity assumption A.2 allows us to add one more, $e'\hat{\pi}=1$. Thus these two assumptions offset each other, and we are left with a system with the same number of variables and equations. The regularity conditions are designed to ensure that the equilibrium pair $(\hat{\pi},\hat{y})$ is the locally unique solution to this system. If A.6 or the non-zero determinant condition do not hold, then the equations are not necessarily independent. If, on the other hand, A.7 does not hold, then we may have too many independent equations to expect a solution to exist. If the equilibrium conditions do not result in a system with as many independent equations as variables, then it is intuitively plausible that some very slight perturbation in the underlying parameters of the economy could make the equations independent. Similarly, if there are not as many independent variables as equations, the same slight perturbation could make a solution impossible. What we need is freedom to perturb the system in a sufficient number of directions. In the argument that follows we develop a method of perturbing the excess demand function that guarantees the density of economies satisfying A.7 and the non-zero determinant condition. The proof of the density of A.6 is relatively trivial.

Lemma 2. The set of activity analysis matrices that satisfy assumption A.6 is open and dense in $\mathscr{A}$.

Proof. Let $\mathscr{B}\subset\mathscr{A}$ be the subset of matrices in $\mathscr{A}$ that satisfy $\bar{\pi}'A<0$ for some $\bar{\pi}>0$. Obviously $\mathscr{B}$ is dense in $\mathscr{A}$ since $\mathscr{A}$ is contained in the closure of $\mathscr{B}$. Observe that the set of matrices that satisfy A.6 are a subset of $\mathscr{B}$. In other words, by imposing A.6 we are ruling out reversibility in the production process. Observe too that $\mathscr{B}$ is an open subset of both $\{-I\} \times R^{n\times(m-n)}$ and $\mathscr{A}$: For any $A^0 \in \mathscr{A}$, $\bar{\pi}'A<0$ for some $\bar{\pi}>0$ implies that there exists some $\varepsilon>0$ such that $\bar{\pi}'A<0$ for all $A\in\{-I\}\times R^{n\times(m-n)}$ that satisfy $d(A^0, A)<\varepsilon$. Since A.6 is a full rank condition, it is satisfied by some open dense set of $\{-I\}\times R^{n\times(m-n)}$. The observation that the intersection of two open dense sets is open and dense completes our argument. □

We can demonstrate that $\mathscr{R}^1$ is dense in $\mathscr{E}^1$ by proving that for any (ξ^0, A) that satisfies A.6, but not A.7, or not the non-zero determinant condition, there exists $(\xi^1, A)\in\mathscr{R}^1$ such that $d_M(\xi^0,\xi^1)<\varepsilon$ for any compact $M\subset S\backslash Z$ and any $\varepsilon>0$. The first step is to reduce the problem from one in the infinite-dimensional space $\mathscr{D}^1$ to the finite-dimensional space R^n. Define the perturbation function $\delta:(R^n_+\backslash\{0\})\times R^n\to R^n$ by the rule

$$\delta_i(\pi, v)=\left(\sum_{j=1}^{n}\pi_j v_j\Big/\sum_{j=1}^{n}\pi_j\right)-v_i, \qquad i=1,\ldots,n.$$

Since, for any fixed $v\in R^n$, δ is C^1 on $R^n_+\backslash\{0\}$ and satisfies A.2 and A.3, $(\xi+\delta)$ is an element of $\mathscr{D}^1$ if ξ is. For any $(\xi, A)\in\mathscr{E}^1$ consider the family $\{(\xi_v, A)\in\mathscr{E}^1\,|\,\xi_v(\pi)=\xi(\pi)+\delta(\pi,v),\ v\in R^n\}$. If we can demonstrate that the set of regular economies is dense in this n-parameter family, then the topology on $\mathscr{E}^1$ is fine enough so that we can conclude that $\mathscr{R}^1$ is dense in $\mathscr{E}^1$.

Our argument relies on a theorem in differential topology that is a direct consequence of Sard's theorem [Guillemin and Pollack (1974, pp. 67–79)].

Transversality Theorem. Let M, V, and N be smooth manifolds without boundary where $\dim M=m$, $\dim N=n$, and $m\leqq n$, and let $y\in N$. Suppose that $F: M\times V\to N$ is a C^1 map such that, for every (x,v) that satisfies $F(x,v)=y$, $\operatorname{rank} DF_{(x,v)}(x,v)=n$; then the set of $v\in V$ for which $F(x,v)=y$ implies that $\operatorname{rank} DF_x(x,v)=n$ has full Lebesgue measure. In other words, if y is a regular value of F, then for almost all $v\in V$ it is a regular value of $f_v(x)=F(x,v)$.

We now prove that (ξ_v, A) is regular for a dense set of perturbations $v\in R^n$. Letting B be any $n\times k$ submatrix at A, $0\leqq k\leqq n-1$, we define

$$K_B=\{x\in R^n\,|\,B'x=0,\ e'x=1\},$$

$$O_B=\{x\in R^n\,|\,B'x=0,\ e'x=0\}.$$

We also define the map $f^B:(K_B\cap \operatorname{int} X)\to O_B$ by the rule $f^B(\pi)=p^{O_B}(\xi^*(\pi))$. Recall that we have chosen X, a smooth n-dimensional manifold with boundary that contains the fixed points of g in its interior, to serve as the domain of ξ^*. Here, of course, p^{O_B} is the orthogonal projection of R^n into the

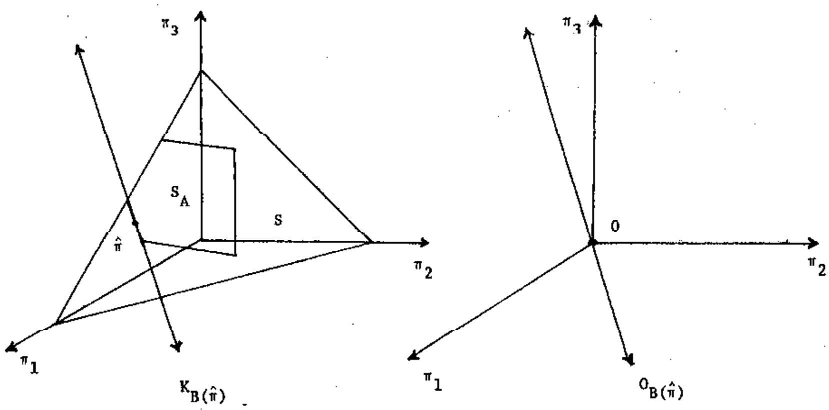

Fig. 6

null space of the $k+1$ columns of $C=[B\ e]$; that is, $p^{O_B}=I-C(C'C)^{-1}C'$. The advantage of this definition is that, if (ξ, A) satisfies A.6 and A.7 then $f^{B(\hat{\pi})}$ agrees with $g-I$ in some neighborhood of every $\hat{\pi}\in\Pi(\xi, A)$. Note, however, that although $\hat{\pi}=g(\hat{\pi})$ implies that $f^B(\hat{\pi})=0$ for some B, namely $B(\hat{\pi})$, the converse does not hold; in addition to $\hat{\pi}\in K_B$ and $f^B(\hat{\pi})=0$, the condition $\hat{\pi}\in S_A$ is needed for $\hat{\pi}=g(\hat{\pi})$. Defining $F^B(\pi, v)=p^{O_B}(\xi^*(\pi)+\delta(\pi, v))$, we focus our attention on the family of C^1 maps $F^B: (K_B\cap \operatorname{int} X)\times R^n\to O_B$ for which the matrices B are such that $K_B\cap S_A\neq\emptyset$.

The first step in proving the genericity of regularity is to demonstrate that we are able to perturb g in a sufficient number of directions. The exact requirement is given in the statement of transversality density theorem.

Lemma 3. For all $(\pi, v)\in(K_B\cap \operatorname{int} X)\times R^n$ *the derivative map* $DF_v^B: R^n\to O_B$ *has* rank $n-k-1$, *that is, is onto.*

Proof. $DF_v^B(\pi, v)=(I-C(C'C)^{-1}C')D\delta_v$. Now

$$D\delta_v(\pi, v)=\begin{bmatrix}\pi_1-1 & \dots & \pi_n\\ \vdots & & \vdots\\ \pi_1 & & \pi_n-1\end{bmatrix}$$

has rank $n-1$ since $D\delta_v(\pi, v)e=0$ while the $(n-1)\times(n-1)$ matrix formed by deleting any row and column j for which $\pi_j>0$ is non-singular. Letting $\rho=\operatorname{rank}(I-C(C'C)^{-1}C')D\delta_v$, we observe that

$$\operatorname{rank}\begin{bmatrix}(I-C(C'C)^{-1}C')D\delta_v & C\\ 0 & I\end{bmatrix}=\rho+k+1.$$

Taking the second column of this matrix, multiplying it by $(C'C)^{-1}C'D\delta_v$, and adding it to the first, we establish that this matrix has the same rank as

$$\operatorname{rank}\begin{bmatrix} D\delta_v & C \\ (C'C)^{-1}C'D\delta_v & I \end{bmatrix} = \operatorname{rank}\begin{bmatrix} D\delta_v & C \\ 0 & 0 \end{bmatrix} = \operatorname{rank}[D\delta_v \;\; C].$$

Clearly, however,

$$\operatorname{rank}[D\delta_v \;\; B \;\; e] \geqq n \quad \text{for all} \quad \pi \in K_B \cap \operatorname{int} X,$$

since $D\delta_v$ has rank $n-1$, $D\delta_v e = 0$, and $e'e = n$. Consequently, $\rho + k + 1 \geqq n$, which implies that $\rho \geqq n-k-1$. $(I - C(C'C)^{-1}C')D\delta_v$ maps into O_B, however, implying that $\rho = n-k-1$. □

If an economy does not satisfy A.7 at some equilibrium $\hat{\pi}$, then there are two distinct matrices B and B^*, where B is submatrix of B^*, such that $\hat{\pi}'B^* = 0$ and $\xi(\hat{\pi}) = B\hat{y}$. In other words, the matrix of activities that earn zero profits has more columns than the matrix of activities in use. In this case there are more equations than unknowns. By perturbing our equations we can get rid of the solution $(\hat{\pi}, \hat{y})$.

Lemma 4. If an economy $(\xi, A) \in \mathscr{E}^1$ satisfies A.6, then the set of $v \in V$ for which (ξ_v, A) satisfies A.7 has full Lebesgue measure.

Proof. Let B^* be some $n \times k^*$ submatrix of A, $1 \leqq k^* \leqq n-1$, and B be some $n \times k$ submatrix of B^*, $0 \leqq k < k^*$. Obviously there are only a finite number of such combinations for any $n \times m$ activity analysis matrix A. Notice that $K_{B^*} \cap \operatorname{int} X$ is an $n-k^*-1$ submanifold of the $n-k-1$ manifold $K_B \cap \operatorname{int} X$. For (ξ_v, A) to violate A.7 there would have to be some $\hat{\pi} \in S_A$ such that $\hat{\pi} \in K_{B^*} \cap \operatorname{int} X$ and $F^B(\hat{\pi}, v) = 0$ for some such pair B and B^*. Now, if we consider the map F^B with domain restricted to $(K_{B^*} \cap \operatorname{int} X) \times R^n \subset (K_B \cap \operatorname{int} X) \times R^n$, then we see that $DF^B_\pi(\pi, v)$ has rank at most $n-k^*-1$. Lemma 3 and the Transversality Theorem, however, imply that for almost all $v \in R^n$, if $F^B(\hat{\pi}, v) = 0$, then $DF^B_{\hat{\pi}}(\hat{\pi}, v)$ has full rank $n-k-1 > n-k^*-1$. Although F^B takes $(K_{B^*} \cap \operatorname{int} X) \times R^n$ into O_B, the image $F^B(K_{B^*} \cap \operatorname{int} X, v)$ is a very small subset of O_B for any fixed $v \in R^n$. Indeed, for some set $U \subset R^n$ of full Lebesgue measure the image $F^B(K_{B^*} \cap \operatorname{int} X, U)$ does not contain 0. We apply this same argument to all possible combinations B and B^*, noting that the intersection of a finite number of sets with full Lebesgue measure also has full Lebesgue measure. □

The third requirement for (ξ, A) to be regular is that $\det[Dg_{\hat{\pi}} - I] \neq 0$ for any $\hat{\pi} \in \Pi(\xi, A)$. If the requirement is not satisfied, then we do not have, at least infinitesimally, as many independent equations as unknowns. Letting $C=[B(\hat{\pi})\ e]$ we use the chain rule to establish that

$$Dg_{\hat{\pi}}=(I-C(C'C)^{-1}C')(I+D\xi_{\hat{\pi}}).$$

Lemma 5. Suppose that $F^B(\hat{\pi},v)=0$ *for some* $(\hat{\pi},v)\in(K_B\cap \operatorname{int} X)\times R^n$. *If* $DF^B_{\hat{\pi}}$ *maps* O_B *onto itself, that is, has* rank $n-k-1$, *then* $\det[(I-C(C'C)^{-1}C')(I+D\xi_{\hat{\pi}})-I]$ *does not vanish.*

Proof. $DF^B_{\hat{\pi}}(\hat{\pi},v)=(I-C(C'C)^{-1}C')D\xi_{\hat{\pi}}$ maps O_B onto itself only if $(I-C(C'C)^{-1}C')D\xi_{\hat{\pi}}$ maps R^n onto O_B. Let $x\in R^n$ be such that

$$(I-C(C'C)^{-1}C')D\xi_{\hat{\pi}}x-C(C'C)^{-1}C'x=0.$$

If we could demonstrate that $x=0$ is the only solution to this equation, then we would have established our contention. Pre-multiplying by $(I-C(C'C)^{-1}C')$, we have

$$(I-C(C'C)^{-1}C')D\xi_{\hat{\pi}}x=0.$$

On the other hand, pre-multiplying by $C(C'C)^{-1}C'$ produces

$$C(C'C)^{-1}C'x=0.$$

Since the columns of $(I-C(C'C)^{-1}C')D\xi_{\hat{\pi}}$ and those of $C(C'C)^{-1}C'$ together span all of R^n,

$$\{x\in R^n|(I-C(C'C)^{-1}C')D\xi_{\hat{\pi}}x=0\}\cap\{x\in R^n|C(C'C)^{-1}C'x=0\}=\{0\}.$$

Consequently,

$$[(I-C(C'C)^{-1}C')D\xi_{\hat{\pi}}-C(C'C)^{-1}C']$$
$$=[(I-C(C'C)^{-1}C')(I+D\xi_{\hat{\pi}})-I]$$

is non-singular. □

We combine this result with Lemmas 3 and 4 and the Transversality Theorem to produce the following result.

Theorem 2. The set of regular economies $\mathscr{R}^1$ *is open and dense in* $\mathscr{E}^1=\mathscr{D}^1\times A$.

Proof. As we have noted, it suffices to demonstrate that $\mathscr{R}^1$ is dense in $\mathscr{E}^1$. For any $(\xi, A), (\xi_v, A) \in \mathscr{E}^1$,

$$d_M[(\xi, A), (\xi_v, A)] = d_M(\xi, \xi_v).$$

Because $\xi_v(\pi) = \xi(\pi) + \delta(\pi, v)$,

$$d_M(\xi, \xi_v) = d_M(0, \delta) \quad \text{for fixed} \quad v \in R^n.$$

Now

$$d_M(0, \delta) = \sup_{i, \pi \in M} |\delta_i(\pi, v)| + \sup_{i, j, \pi \in M} \left| \frac{\partial \delta_i}{\partial \pi_j}(\pi, v) \right|.$$

For all π in any compact $M \subset S/Z$,

$$|\delta_i(\pi, v)| = \left| \frac{\partial \delta_j}{\partial \pi_i}(\pi, v) \right| = \left| \sum_{j=1}^{n} \pi_j v_j - v_i \right| \leqq \max_k |v_k|, \quad i = 1, \ldots, n.$$

Consequently, for any $\varepsilon > 0$ and any compact $M \subset S/Z$, there is some $\varepsilon' > 0$ such that $\|v\| < \varepsilon'$ implies that $d_M(0, \delta) < \varepsilon$, □

6. Economies with primary and intermediate goods

We can easily extend the results that we developed in the previous section to economies with primary goods. We modify assumption A.1 as follows:

(A.1′) The first l components of the excess demand function ξ, $l < n$, are such that $\xi_i(\pi) \equiv -w_i$ for some $w_i > 0$. ξ satisfies A.1 and the set Z includes the set $H = \{\pi \in R^n_+ \mid \pi_{l+1} = \cdots = \pi_n = 0\}$.

Denoting the space of excess demand functions satisfying assumptions A.1′, A.2 and A.3 as $\mathscr{D}^2$, we give the space of economies with primary goods, $\mathscr{E}^2 = \mathscr{D}^2 \times \mathscr{A}$, the same topological structure as that of $\mathscr{E}^1$. Note that $\mathscr{E}^2 \subset \mathscr{E}^1$, We define the map g and the concepts of regularity and index of an equilibrium as previously. The set of regular economies in $\mathscr{E}^2$, $\mathscr{R}^2$, is a subset of $\mathscr{R}^1$. Since $\mathscr{R}^1$ is open in $\mathscr{E}^1$, $\mathscr{R}^2$ is open in $\mathscr{E}^2$. Nevertheless, Theorem 2 gives us no reason to expect that $\mathscr{R}^2$ is dense in $\mathscr{E}^2$. In fact, for all we know, $\mathscr{R}^2$ may be empty. Fortunately, this is not the case.

Theorem 3. The set of regular economies $\mathscr{R}^2$ is open and dense in $\mathscr{E}^2 = \mathscr{D}^2 \times \mathscr{A}$.

Proof. Theorem 2 implies that $\mathscr{R}^2$ is open in $\mathscr{E}^2$ so all that we have to prove is that $\mathscr{R}^2$ is dense in $\mathscr{E}^2$.

We redefine the perturbation function $\delta(\pi, v)$ by the rule

$$\delta_i(\pi, v) = -v_i, \qquad i=1,\dots,l,$$

$$= \left(\sum_{j=1}^{n} \pi_j v_j \Big/ \sum_{j=l+1}^{n} \pi_j\right) - v_i, \qquad i=l+1,\dots,n.$$

Note that for any fixed $v \in R^n$, δ is C^1 on $R^n_+ \backslash H$ and satisfies A.2 and A.3. For any $(\xi, A) \in \mathscr{E}^2$, let

$$P_w = \{v \in R^n \mid v_i > -w_i,\ i=1,\dots,l\}.$$

P_w is obviously a smooth n-manifold without boundary. If $\xi(\pi)$ is an element of $\mathscr{D}^2$, then so is $\xi_v(\pi) = \xi(\pi) + \delta(\pi, v)$ whenever $v \in P_w$. We define the function $F^B: (K_B \cap \operatorname{int} X \backslash H) \times P_w \to O_B$ as before. Observe that

$$D\delta_v(\pi, v) = \begin{bmatrix} -1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & & \\ 0 & \dots & -1 & 0 & \dots & 0 \\ \dfrac{\pi_1}{\sum_{j=l+1}^{n} \pi_j} & \dots & \dfrac{\pi_l}{\sum \pi_j} & \dfrac{\pi_{l+1}}{\sum \pi_j} - 1 & \dots & \dfrac{\pi_n}{\sum \pi_j} \\ \dfrac{\pi_1}{\sum_{j=l+1}^{n} \pi_j} & \dots & \dfrac{\pi_l}{\sum \pi_j} & \dfrac{\pi_{l+1}}{\sum \pi_j} & \dots & \dfrac{\pi_n}{\sum \pi_j} - 1 \end{bmatrix}$$

has rank $n-1$ for all $(\pi, v) \in (S \backslash H) \times R^n$ because the $(n-l) \times (n-l)$ submatrix in the lower right-hand corner has rank $n-l-1$. Since, for any $(\xi, A) \in \mathscr{E}^2$, $(K_B \cap \operatorname{int} X \backslash H) \times P_w \subset (X \backslash H) \times R^n$, we can use the Transversality Theorem to prove that the set of regular economies is dense in the n-parameter family $\{(\xi_v, A)\} \subset \mathscr{E}^2$. We simply follow the same procedure as in Lemmas 3, 4, and 5.

Observe that $0 \in P_w$ and that, for any $v \in P_w$,

$$|\delta_i(\pi, v)| \leq \max_k \left| \frac{v_k}{\sum_{h=l+1}^{n} \pi_h} \right|,$$

$$\left| \frac{\partial \delta_j}{\partial \pi_i}(\pi, v) \right| \leq \max_k \left| \frac{v_k}{\left(\sum_{h=l+1}^{n} \pi_h\right)^2} \right|, \qquad i=1,\dots,n.$$

Consequently, for any $\varepsilon > 0$ and any compact subset M of $S \backslash Z$, we can choose $\varepsilon' > 0$ so that $\|v\| < \varepsilon'$ implies that

$$d_M[(\xi, A), (\xi_v, A)] = d_M[(0, \delta)] < \varepsilon.$$

Therefore $\mathscr{R}^2$ is dense in $\mathscr{E}^2$. □

Unfortunately, the situation with intermediate goods is not quite so simple. We could, of course, modify assumption A.1 again to allow for some $\xi_i(\pi) \equiv 0$ and then define the map g and the concepts of regularity and index of an equilibrium as previously. If we tried to mimic our proof of Theorem 3, however, we would find it impossible to prove that the set of regular economies is dense in a space of economies where the restriction $\xi_i(\pi) \equiv 0$ is placed on some components of the excess demand function. The Transversality Theorem requires that we be able to perturb our map g in a sufficient number of directions. The restriction $\xi_i(\pi) \equiv 0$ precludes us from doing this. This is no mere technicality; the set of regular economies is not dense in a space of economies with intermediate goods. Indeed, there are open sets of critical economies in such a space. We shall present a simple argument to illustrate this point, but let us first develop a few preliminary concepts.

We consider economies with n primary and final goods and h intermediate goods. For such an economy let $\xi(\pi) = (\xi_1(\pi), \ldots, \xi_n(\pi))$ be the aggregate excess demand function for the first n commodities where each ξ_i depends on only the first n prices. An economy with intermediate goods is specified as a pair (ξ, A) where ξ satisfies assumptions A.1′, A.2 and A.3 and A is an $(n+h) \times m$ activity analysis matrix satisfying A.4 and A.5. Thus the space of economies $\mathscr{E}^3$ is the product space of the space of excess demand functions for the first n commodities, $\mathscr{D}^3$, and the space of activity analysis matrices involving all $n+h$ commodities, $\mathscr{A}^3$. We endow $\mathscr{E}^3$ with a topological structure analogous to that of $\mathscr{E}^1$.

To see why regularity, as we have defined it, cannot be a generic property of $\mathscr{E}^3$, consider an economy $(\xi, A) \in \mathscr{E}^3$ that has an equilibrium at which no production takes place. It is easy to construct such an economy: Simply take some pure exchange economy $(\xi, -I) \in \mathscr{E}^1$ that has some equilibrium $\hat{\pi}$ strictly positive. Then choose a matrix $A \in \mathscr{A}^3$ such that the equilibrium $(\hat{\pi}_1, \ldots, \hat{\pi}_n)$ satisfies the condition

$$[\hat{\pi}' \;\; 0']A < 0 \quad \text{where} \quad \begin{bmatrix} \hat{\pi} \\ 0 \end{bmatrix} \in R^{n+h}.$$

Now, since $(\hat{\pi}, 0)$ is an equilibrium of $(\xi, A) \in \mathscr{E}^1$, so is any vector of prices $(\hat{\pi}, q)$ that satisfies $[\hat{\pi}' \;\; q']A < 0$. [That the elements of $(\hat{\pi}, q)$ may not sum to one is, of course, inconsequential; we merely divide by $1+\sum_{i=1}^{n} q_i > 0$.] Observe that $[\hat{\pi}' \; 0']A < 0$ implies that there is an infinite number of such prices. In other words, when no production takes place at equilibrium we have no way of determining the prices of intermediate goods. Furthermore, if the original economy $(\xi, -I) \in \mathscr{E}^1$ is regular, there is no way to eliminate this problem by perturbing our parameters (ξ, A). It is futile, therefore, to hope that we can prove a result analogous to Theorem 3 for $\mathscr{E}^3$.

We can get around this difficulty by altering several of our assumptions and definitions. It would hardly seem reasonable to expect that the prices of intermediate goods be uniquely determined at equilibria where no production takes place. We therefore want to develop a method for examining $\mathscr{E}^2$ that treats equilibria where no production takes place differently from those where it does.

To start with, let us examine the implications of our non-degeneracy assumption A.6 on A.

Lemma 6. Suppose $(\xi, A) \in \mathscr{E}^3$ satisfies assumption A.6. At any equilibrium $(\hat{\pi}, \hat{q})$, where some activities other than the first n disposal activities are in use, there are $k \geq h+1$ activities in use.

Proof. Assume not. Partition the matrix $B(\hat{\pi}, \hat{q})$ into $\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$ where B_1 is $n \times k$ and B_2 is $h \times k$. To simplify matters, assume that $B(\hat{\pi}, \hat{q})$ does not include any of the first n disposal activities. Now A.6 implies that the matrix $\begin{bmatrix} -I & B_1 \\ 0 & B_2 \end{bmatrix}$ has full column rank since $n+k \leq n+h$. Consequently, B_2 has full column rank. That $(\hat{\pi}, \hat{q})$ is an equilibrium, however, implies that

$$\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \hat{y}_B = \begin{bmatrix} \xi(\hat{\pi}) \\ 0 \end{bmatrix}, \qquad B_2 \hat{y}_B = 0,$$

which is contradiction. □

It is this result that makes assumption A.6 in its strong form attractive. If we had replaced A.6 with the assumption that the matrix of activities in use has full column rank at every equilibrium, then we would now want to prove that the conclusion of this lemma holds generically whether or not A.6 holds. Such a proof is, of course, easy, but it would complicate our exposition. The implication of Lemma 6 is that if production takes place then we have sufficient information to uniquely determine the prices of the intermediate goods. Since we always want to have at least $h+1$ activities in use at equilibrium, we must alter A.7 if we want to hold for an open dense subset of $\mathscr{E}^3$. That some activity makes zero profit at equilibrium $(\hat{\pi}, \hat{q})$ does not mean production is possible; something must be done to ensure that we have enough activities earning zero profits to come up with the required amounts of intermediate goods or get rid of the excess amounts produced.

(A.7′) Let $B(\pi, q)$ denote the submatrix of A whose columns are activities earning zero profit at $(\pi, q) \in R^{n+h}$. At an equilibrium $(\hat{\pi}, \hat{q})$, $B(\hat{\pi}, \hat{q})$ satisfies the following conditions:

(a) If some column b of $B(\hat{\pi}, \hat{q})$ is one of the first n disposal activities, then $\hat{y}_b$ is strictly positive in the equation

$$\begin{bmatrix} \xi(\hat{\pi}) \\ 0 \end{bmatrix} = \sum_{b \in B(\hat{\pi}, \hat{q})} \hat{y}_b \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

(b) If $B(\hat{\pi}, \hat{q})$ includes more than h columns, not counting any of the first n disposal activities, then all $\hat{y}_b$ are strictly positive in the above equation.

We modify our map g as follows: Partition A into $\left[\begin{smallmatrix} A_1 \\ A_2 \end{smallmatrix}\right]$ where A_1 is $n \times m$ and A_2 is $h \times m$. Let

$$S_A = \{\pi \in R^n \mid \pi' A_1 + q' A_2 \leqq 0 \text{ for some } q \in R^h, e'\pi = 1\}.$$

It is easy to see that S_A is a compact, non-empty, convex subset of S. Any q such that $\pi' A_1 + q' A_2 \leqq 0$ must be non-negative since A includes disposal activities for the intermediate goods. We alter slightly our definition of equilibrium to require that only the first n prices sum to one. Choosing $\bar{\pi} \in R^n$ such that $\bar{\pi} > 0$ and $\bar{\pi}' A_1 + q' A_2 \leqq 0$ for some $q \in R^h$, we define the function $\xi^*: R^n_+ \setminus \{0\} \to R^n$ as before. With these changes the definition of g as $p^{S_A}(\pi + \xi^*(\pi))$ should seem quite natural.

Theorem 4. Fixed points of the map g and equilibria of the economy $(\xi, A) \in \mathscr{E}^3$ are equivalent.

Proof. $\hat{\pi} = g(\hat{\pi})$ if and only if there is some $\hat{q} \in R^h$ such that $(\hat{\pi}, \hat{q})$ solves the quadratic programming problem

$$\min \tfrac{1}{2}(p - \hat{\pi} - \xi^*(\hat{\pi}))'(p - \hat{\pi} - \xi^*(\hat{\pi})),$$

$$\text{s.t. } p' A_1 + q' A_2 \leqq 0, \quad p' e = 1.$$

But $(\hat{\pi}, \hat{q})$ solves the problem if and only if

$$p' \xi^*(\hat{\pi}) + q' 0 \leqq \hat{\pi}' \xi^*(\hat{\pi}) + \hat{q}' 0 = 0,$$

for all

$$(p,q)\in Y_A^* = \{(p,q)\in R^{n+h} \mid p'A_1 + q'A_2 \leqq 0\}.$$

Once again, this is equivalent to the statement that, for some $y \geqq 0$,

$$\begin{bmatrix} \xi(\hat{\pi}) \\ 0 \end{bmatrix} = \begin{bmatrix} \xi^*(\hat{\pi}) \\ 0 \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \hat{y}. \quad \square$$

We define the concept of regularity in the obvious manner: A regular economy is one satisfying A.6 and A.7′ for which O is a regular value of $g-I$. If an economy $(\xi, A)\in\mathscr{E}^3$ satisfies the non-degeneracy assumptions A.6 and A.7′, we shall argue that the map g is differentiable in some neighborhood of every fixed point. Let us calculate an expression for $\det[Dg_{\hat{\pi}} - I]$. There are two cases worth distinguishing: The first is where no production, except for the possible use of some of the first n disposal activities, takes place. The second is where production does take place.

Suppose that, at some equilibrium $(\hat{\pi}, \hat{q})$, $B(\hat{\pi}, \hat{q})$ includes h or fewer activities, not counting the first n disposal activities. In this case A.6, A.7′, and Lemma 6 imply that, in some neighborhood of $\hat{\pi}$, $g(\pi)$ is equivalent to the vector p that solves the quadratic programming problem

$$\min \tfrac{1}{2}(p-\pi-\xi(\pi))'(p-\pi-\xi(\pi)),$$

$$\text{s.t.} \quad p'e = 1.$$

(There is, of course, an additional constraint $p_i = 0$ for each $\hat{\pi}_i = 0$.) In this case the computation of index $(\hat{\pi}) = (-1)^n \operatorname{sgn}(\det[Dg_{\hat{\pi}} - I])$ is the same as for the equilibrium $\hat{\pi}$ of the pure exchange economy $(\xi, -I)$ that ignores the h intermediate goods. Even if $\hat{\pi}$ is a regular point of $g-I$, however, there is no reason to expect that the prices of the intermediate goods are uniquely determined.

Let us turn our attention to the second possible case. Suppose that, at some equilibrium $\hat{\pi}$, $B(\hat{\pi}, \hat{q})$ includes more than h activities. In this case A.7′ implies that, in some neighborhood of $\hat{\pi}$, $g(\pi)$ is equivalent to the vector p that solves the quadratic programming problem

$$\min \tfrac{1}{2}(p-\pi-\xi(\pi))'(p-\pi-\xi(\pi)),$$

$$\text{s.t.} \quad p'[B_{11} \;\; B_{12}] + q'[B_{21} \;\; B_{22}] = 0, \quad p'e = 1.$$

Here we have partitioned the $(n+h)\times k$ matrix $B(\hat{\pi}, \hat{q})$ into B_{11} which is $n\times(k-h)$, B_{12} which is $n\times h$, B_{21} which is $h\times(k-h)$, and B_{22} which is $h\times h$.

Now A.6 implies that B_{22} is non-singular. We can therefore transform the expression

$$[p' \; q']\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}=0$$

into

$$[p' \; q']\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}\begin{bmatrix} I & 0 \\ -B_{22}^{-1}B_{21} & B_{22}^{-1} \end{bmatrix}=0,$$

$$[p' \; q']\begin{bmatrix} B_{11}-B_{12}B_{22}^{-1}B_{21} & B_{12}B_{22}^{-1} \\ 0 & I \end{bmatrix}=0.$$

This implies that $q=-(B_{12}B_{22}^{-1})'p$. Therefore, to find $g(\pi)$ in some neighborhood of $\hat{\pi}$, we can solve for p in the problem

$$\min \tfrac{1}{2}(p-\pi-\xi(\pi))'(p-\pi-\xi(\pi)),$$

$$\text{s.t.} \;\; p'(B_{11}-B_{12}B_{22}^{-1}B_{21})=0, \quad p'e=1.$$

Assumption A.6 implies that $B_{11}-B_{12}B_{22}^{-1}B_{21}$ has full column rank. Hence we can compute the expression

$$\text{index}\,(\hat{\pi})=(-1)^n \operatorname{sgn}\left(\det\begin{bmatrix} 0 & e' & 0 \\ e & \mathrm{D}\xi_\pi & B_{11}-B_{12}B_{22}^{-1}B_{21} \\ 0 & (B_{11}-B_{12}B_{22}^{-1}B_{21})' & 0 \end{bmatrix}\right).$$

If we rescale so that

$$\sum_{i=1}^{n}\hat{\pi}_i+\sum_{i=1}^{n}\hat{q}_i=1,$$

then it is easy to transform this expression into

$$(-1)^{n+h}\operatorname{sgn}\left(\det\begin{bmatrix} 0 & e' & e' & 0 & 0 \\ e & \mathrm{D}\xi_{\hat{\pi}} & 0 & B_{11} & B_{12} \\ e & 0 & 0 & B_{21} & B_{22} \\ 0 & B'_{11} & B'_{21} & 0 & 0 \\ 0 & B'_{12} & B'_{22} & 0 & 0 \end{bmatrix}\right).$$

This, of course, is the expression for index $(\hat{\pi}, \hat{q})$ that we compute when we ignore the intermediate character of the last h goods. Consequently, when production does take place we can expect the prices $\hat{q}$ of the h intermediate goods to be uniquely determined if $\hat{\pi}$ is a regular point of $g-I$.

One consequence of these arguments is:

Lemma 7. *Let* $(\xi, A) \in \mathscr{E}^3$ *satisfy assumptions A.6 and A.7′. The map g is differentiable in some open neighborhood of every fixed point* $\hat{\pi}$.

It is easy to demonstrate that regularity, as we now define it, is a generic property of $\mathscr{E}^3$. The alteration in the map g has reduced the problem from one in $n+h$ dimensions to one in n dimensions. The local properties of any equilibrium $(\hat{\pi}, \hat{q})$ of an economy $(\xi, A) \in \mathscr{E}^3$ that satisfies A.6 and A.7′ are the same as those of the equilibrium $\hat{\pi}$ of some economy with only n commodities and the same excess demand function ξ. The genericity of regular economies in $\mathscr{E}^3$ follows directly from Theorem 4.

Theorem 5. *The set of regular economies* $\mathscr{R}^3$ *is open and dense in* $\mathscr{E}^3 = \mathscr{D}^3 \times \mathscr{A}^3$.

Although the arguments presented in this section may have been laborious, the results are simple enough. The concepts of regularity and index of an equilibrium can be applied directly to economies with primary goods. The property of regularity is still generic in this case. The case with intermediate goods is different only in that we must make special provisions for equilibria where no production takes place. That intermediate goods can and should be ignored at such equilibria is hardly a surprising result.

7. Alternative production technologies

Let us consider a constant-returns production technology characterized by a set of feasible net-output combinations that is a closed convex cone $Y \subset R^n$. The assumption on Y that is analogous to A.4 is that the negative orthant $-R^n_+$ is a subset of Y. The assumption on Y that is analogous to A.5 is that $Y \cap R^n_+ = \{0\}$. If a production cone Y satisfies these assumptions, then the intersection of its dual cone,

$$Y^* = \{\pi \in R^n | \pi' x \leqq 0 \quad \text{for all} \quad x \in Y\},$$

and the set $\{\pi \in R^n | \pi' e = 1\}$ is a non-empty, closed, convex subset of the unit simplex. We denote this set as S_Y. An economy could be specified by an excess demand function satisfying assumptions A.1–A.3 and a production cone Y satisfying the above assumptions. An equilibrium for such an economy (ξ, Y) would be a price vector $\hat{\pi}$ such that $\hat{\pi} \in S_Y$ and $\xi(\hat{\pi}) \in Y$. This, of course, is simply a generalization of our previous definition where

$$Y = \{x \in R^n | x = Ay,\ y \geqq 0\}.$$

The proof that (ξ, Y) has an equilibrium is identical to that of Theorem 1.

To employ our differentiable approach we must place further restrictions on the production technology. We do this by employing the concept of a profit function. To gain some understanding of this concept, let us suppose that any vector x that satisfies the constraints

$$f(x) = 0 \quad \text{and} \quad x_i \leqq 0, \quad i = 1, \ldots, l,$$
$$\geqq 0, \quad i = l+1, \ldots, n,$$

is a feasible net-output combination. Here $f: R^n \to R$ is a constant-returns production function that employs the first l commodities as inputs and produces the final $n-l$ commodities as outputs. We assume that f is homogeneous of degree one and concave. For example

$$f(x_1, x_2, x_3) = \beta(-x_1)^{\alpha}(-x_2)^{1-\alpha} - x_3$$

is the familiar Cobb–Douglas production function when $1 \geqq \alpha \geqq 0$ and $\beta > 0$. To derive the profit function $a: R^n_+ \backslash \{0\} \to R$, we find a vector $x(\pi)$ that solves the problem

$$\max \pi' x,$$
$$\text{s.t.} \quad f(x) = 0, \quad \|x\| = 1, \quad x_i \leqq 0, \quad i = 1, \ldots, l,$$
$$\geqq 0, \quad i = l+1, \ldots, n.$$

We then set $a(\pi) = \pi' x(\pi)$. Thus the profit function tells us the maximum profit that can be earned at prices π when we are constrained by $\|x\| = 1$. Given our assumption of constant returns to scale, the profit that can be earned at π is unbounded if $a(\pi) > 0$. It is well known that $a(\pi)$ is homogeneous of degree one, convex, and continuous even if the vector $x(\pi)$ is not unique. If a is C^1, moreover, Hotelling's lemma provides us with a relationship between the profit function and the profit maximizing net-output vector $x(\pi)$. This relationship is $Da_{\pi}(\pi) = x(\pi)$ [see, for example, Diewert (1970)].

Suppose that Y is such that its dual cone can be defined by m C^2 profit functions $a_j(\pi), j=1,\dots,m$,

$$Y^* = \{\pi \in R^n \mid a_j(\pi) \leqq 0, \qquad j=1,\dots,m\}.$$

Clearly, this definition is a generalization of the activity analysis case where $a_j(\pi)$ is the linear function $\sum_{i=1}^n a_{ij}\pi_i$. If we impose assumptions on the functions a_j analogous to assumptions A.4–A.7, we can define the concepts of regularity and fixed point index as before.

Define $B(\hat{\pi})$ as the $n \times k$ matrix whose columns are the gradients of the k profit functions that satisfy $a_j(\hat{\pi})=0$. Hotelling's lemma and the assumption of constant returns allows us to interpret $B(\hat{\pi})$ as a matrix of activities. Further define $H(\hat{\pi})$ as the $n \times n$ weighted sum of the Hessian matrices of the same k functions; the weights are the appropriate activity levels. The calculations of index $(\hat{\pi})$ becomes

$$\operatorname{index}(\hat{\pi}) = (-1)^n \operatorname{sgn}\left(\det \begin{bmatrix} 0 & e' & 0 \\ e & D\xi_{\hat{\pi}} - H(\hat{\pi}) & B(\hat{\pi}) \\ 0 & B'(\hat{\pi}) & 0 \end{bmatrix} \right).$$

The activity analysis technology is, of course, the special case where $H(\hat{\pi}) \equiv 0$.

Utilizing the principle of duality, we can specify the production technology by a vector of profit functions $a(\pi) = (a_1(\pi), \dots, a_m(\pi))$ without having to mention the production cone Y. An economy would thus be specified as a pair (ξ, a). A more detailed analysis of this type of economy, including the calculation of the index, is given by Kehoe (1981).

An advantage to this more general approach is that it can easily be extended to economies with production technologies that exhibit decreasing returns to scale. In such an environment we have to specify production functions for individual firms and make provisions to distribute the profits of these firms to consumers. The situation can then be treated as a special case of constant returns production where we define an additional primary good to account for the profits of each firm; see McKenzie (1959) for details of the construction. If the profit function of each firm is C^2, then we can directly apply the results that we have derived here to such economies.

Another assumption about the production technology that can be weakened is A.4. Here we follow McKenzie (1959) in replacing free disposal with the assumption that the interior of the production cone Y is non-empty. To keep matters simple, let us assume that ξ is C^1 over the entire dual cone of Y except the origin, $Y^* \backslash \{0\}$. It is, of course, easy to weaken this condition to one analogous to A.1. Letting $\bar{x} \in \operatorname{int} Y$, we define

$$S_Y = Y^* \cap \{\pi \in R^n | \pi'\bar{x} = -1\}.$$

Clearly, this is a generalization of the previous case where the free disposal assumption allows us to set $\bar{x} = (-1, \ldots, -1)$.

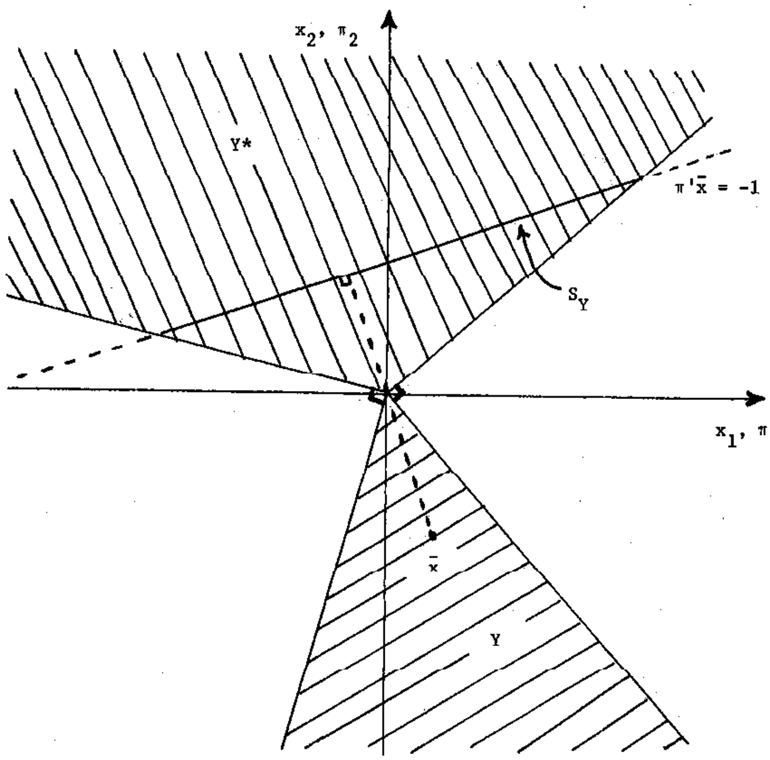

Fig. 7

The assumption that $Y \cap R^n_+ = \{0\}$ implies that S_Y is non-empty, compact, and convex. We again define X as a smooth n manifold that is a compact, convex subset of R^n, contains S_Y in its interior, and does not contain the origin. We define the function $g: X \to X$ by the rule $g(\pi) = p^{S_Y}(\pi + \xi(\pi))$. As before, it can easily be demonstrated that fixed points of g are equivalent to equilibrium of (ξ, Y). One problem that we face when we give the space of economies a topological structure is that we must somehow fix the domain of ξ. If we do this, and if we further specify the production technology using either the activity analysis specification or the profit function specification, we can prove the openness and density of the appropriately defined regularity conditions.

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