The comparative statics properties of tax models

TIMOTHY J. KEHOE Clare College, Cambridge

Abstract. Over the past decade economists have employed the mathematical tools of differential topology to investigate the comparative statics properties of general equilibrium models. In this paper we develop the appropriate concepts of regularity and fixed point index for economies that allow a wide variety of tax and subsidy schemes. As part of the analysis, we provide a proof of existence of equilibrium that is both simpler and more general than any given previously. Conditions that ensure local uniqueness and continuity of equilibria are not at all restrictive; they are satisfied by almost all economies. Unfortunately, it seems that conditions that ensure global uniqueness of equilibrium in these economies are even more elusive than in economies without distortions. This work should be of particular relevance to researchers who employ empirical general equilibrium models to do policy evaluation.

Les propriétés des modèles de fiscalité en statique comparative. Au cours de la dernière décennie, les économistes ont utilisé l'outillage mathématique de la topologie différentielle pour étudier les propriétés des modèles d'équilibre général en statique comparative. Dans ce mémoire, l'auteur développe les concepts de régularité et d'indice de point fixe pour des économies qui permettent l'existence de tout un éventail d'arrangements fiscaux. L'analyse produit, entre autres choses, une preuve de l'existence de l'équilibre qui est à la fois plus simple et plus générale que celles disponibles jusqu'ici. Les conditions qui assurent des équilibres localement uniques et continus ne s'avèrent pas tellement restrictives; elles sont satisfaites par presque toutes les économies. Malheureusement, il semble que les conditions qui assurent un équilibre global unique dans ces économies soient encore plus insaisissables que celles qu'on cherche dans des économies sans distorsions. Ce travail sera d'un intérêt particulier pour les chercheurs qui utilisent des modèles empiriques d'équilibre général pour faire l'évaluation de politiques.

INTRODUCTION

Over the past decade economists have employed the mathematical tools of differential topology to investigate the comparative statics properties of general equilibrium

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models. Debreu (1970) initiated this line of research with his definition of the concept of a regular economy, an economy whose equilibria are locally unique and vary continuously with its parameters. He proved that almost all economies are regular. (The phrase 'almost all' is, of course, given a precise mathematical meaning.) Dierker (1972) and Varian (1975) used the concept of a fixed point index to develop conditions that are both necessary and sufficient for a regular economy to have a unique equilibrium.

The work of these researchers was devoted to pure exchange models. More recently, Mas-Colell (1978) and Kehoe (1980, 1983) have extended the concepts of regularity and fixed point index to models with very general production technologies. In this paper we further extend these results to models that allow a wide variety of tax and subsidy schemes. As part of the analysis, we provide a proof of existence of equilibrium for such models that is both simpler and more general than those given previously by, for example, Shoven and Whalley (1973) and Todd (1979). A number of other researchers have studied the properties of general equilibrium models with taxes: Mantel (1975) and Shafer and Sonnenschein (1976) have provided very general existence theorems; Fuchs and Guesnerie (1983) have used differential topology to study the properties of a general equilibrium model with a decreasing returns production technology. Our approach differs from those of these researchers, however, and is more in line with that of Shoven and Whalley and Todd, in that it is intended for researchers who employ empirical general equilibrium models to do policy analysis.

To understand better the issues that we address in this paper, let us first consider a simple economic model specified by a system of n equations.

$$f_i(\pi_1, \ldots, \pi_n, t_1, \ldots, t_m) = 0, \qquad i = 1, \ldots, n. \tag{1}$$

Here π_i, $i = 1, \ldots, n$, are the endogenous variables, the prices of n goods, and t_i, $i = 1, \ldots, m$, are the exogenous variables, the tax parameters. The n equations can be thought of as requiring that demand minus supply equal zero. This system of equations can be written more compactly in vector notation as

$$f(\pi, t) = 0. \tag{2}$$

Suppose that, for a fixed vector of tax parameters t^0, the vector of prices π^0 solves (2), in other words, is an equilibrium price vector. There are a number of questions we could ask: for example, is π^0 locally unique? That is, would any small change in prices drive the system into disequilibrium? It is crucial for comparative statics analysis that the answer to this question be yes; otherwise, the specification of the model does not suffice to determine the values of the endogenous variables even locally. We could also ask whether the vector of equilibrium prices varies continuously with the tax parameters. Again an affirmative answer to this question is crucial. Continuity would mean that small errors in specifying the changes in taxes, or even in specifying the other parameters of the model, would not have a drastic effect on equilibrium prices.

Classical calculus techniques of the sort used by Hicks (1939) and Samuelson

(1947) provide answers to these questions. A sufficient condition for affirmative answers to both questions is that the $n \times n$ matrix of partial derivatives

$$\begin{bmatrix} (\partial f_1/\partial \pi_1)(\pi^0, t^0) & \dots & (\partial f_1/\partial \pi_n)(\pi^0, t^0) \\ \vdots & & \vdots \\ (\partial f_n/\partial \pi_1)(\pi^0, t^0) & \dots & (\partial f_n/\partial \pi_n)(\pi^0, t^0) \end{bmatrix}$$

be non-singular. In matrix notation we denote this matrix as $Df_\pi(\pi^0, t^0)$. If $Df_\pi(\pi^0, t^0)$ is non-singular, then the inverse function theorem implies that π^0 is a locally the only equilibrium. The non-singularity of this matrix intuitively means that there are locally enough independent equations f_i to determine the unknowns π_i. The implicit function theorem implies that for any small change in t there is a function $\pi(t)$ such that $f(\pi(t), t) \equiv 0$. In fact, we can compute the partial derivatives of $\pi(t)$, the comparative statics multipliers, by inverting $Df_\pi(\pi^0, t^0)$:

$$f(\pi(t), t) \equiv 0$$

$$Df_\pi(\pi(t), t)\, D\pi_t(t) + Df_t(\pi(t), t) \equiv 0$$

$$Df_\pi(\pi^0, t^0)\, D\pi_t(t^0) + Df_t(\pi^0, t^0) = 0$$

$$D\pi_t(t^0) = -(Df_\pi(\pi^0, t^0))^{-1}\, Df_t(\pi^0, t^0). \tag{3}$$

There are other interesting questions that calculus techniques by themselves cannot answer: For example, does a vector of equilibrium prices exist for a given vector of tax parameters? If it does exist, is it unique? Affirmative answers to these questions are no less crucial than they are to the earlier ones. Existence of equilibrium can often be established by an appeal to Brouwer's fixed point theorem, which states that any continuous function g that associates any point in a set with another point in the same set has a fixed point $\pi = g(\hat{\pi})$ if the set is non-empty, compact, and convex. Suppose that we can express the system of equations (2) as

$$\pi - g(\pi, t) = 0, \tag{4}$$

where, if π is an element of some compact convex set S, then $g(\pi, t)$ also is. Then Brouwer's fixed point theorem asserts the existence of an equilibrium. This conclusion can be motivated by a simple graph in the case where $n = 1$ and the price set is the closed interval [0, 1]. Here we are asserting that the graph of g must cross the diagonal, where $\pi = g(\pi, t)$. (See figure 1.) Furthermore, Scarf's (1973) fixed point algorithm, or one of its more recent variants, can be used to compute an equilibrium for the model.

More can be said: Suppose that neither 0 nor 1 is a fixed point of g and that the graph of g never becomes tangent to the diagonal. Then the graph of g must cross the diagonal at least once from above. After that it crosses once from below for every additional time it crosses from above. Let us associate an index +1 with a fixed point π if the graph of g crosses the diagonal from above and an index −1 if it crosses from

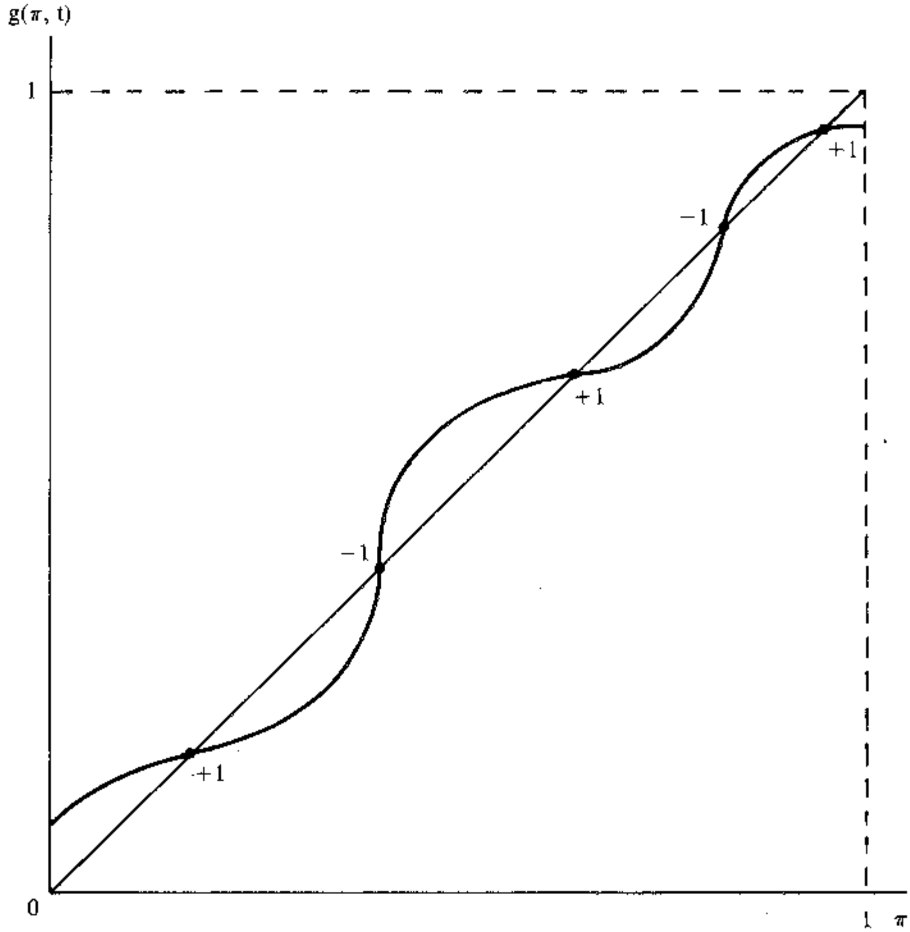

FIGURE 1

below. This index can easily be computed by finding the sign of the expression $1 - (\partial g/\partial \pi)(\hat{\pi}, t)$. The index theorem says that the sum of the indices of all equilibria is +1. Consequently, there is an odd number of equilibria, and, if index $(\hat{\pi}) = +1$ at every equilibrium, there is only one equilibrium. Furthermore, if index $(\hat{\pi}) = -1$ at any equilibrium, there must be multiple equilibria.

In the case with n goods, index $(\hat{\pi})$ can be computed by finding the sign of det $(I - Dg_\pi(\hat{\pi}, t))$, where I is the $n \times n$ identity matrix. We have already argued that, if this expression is non-zero, then the equilibrium $\hat{\pi}$ is locally unique and varies continuously with t. The index theorem tells us this expression is also crucial for conditions that guarantee the uniqueness of equilibrium.

THE MODEL

Let us begin by describing a simple economic model that allows for production and various tax and transfer schemes. The taxes may be taxes on consumption and production that are ad valorem or specific and income taxes that are linear or non-linear. The revenue generated by these taxes is distributed to consumers, one of whom may be a government. We initially describe a model in which the government only taxes and spends. We later extend our analysis to a model that allows a wide variety of subsidy schemes.

There are n goods in the model. The responses of consumers to an $n + 1$ vector of prices and tax revenue (π, r) are aggregated into an excess demand function $\xi : (R_+^n \setminus \{0\}) \times R_+ \to R^n$. Here $R_+^n \setminus \{0\}$ is the set of all non-negative price vectors except the origin. To keep our presentation as simple as possible, we assume that ξ is arbitrary, except for the following assumptions:

(A.1) ξ is C^1 (continuously differentiable).

(A.2) $\xi(\lambda\pi, \lambda r) \equiv \xi(\pi, r)$ for any $\lambda > 0$; that is, ξ is homogeneous of degree zero.

(A.3) ξ is bounded from below by some $-w$, $w \in R_+^n$.

(A.4) $\|\xi(\pi, r^i)\| \to \infty$ as $r^i \to \infty$ for any $\pi \in R_+^n \setminus \{0\}$.

The tax payments generated by consumption and income taxes are specified by a function $t : (R_+^n \setminus \{0\}) \times R_+ \to R_+$. Tax payments are expressed in the same units as expenditures, $\pi_i \xi_i(\pi, r)$. They are, or course, subject to whatever normalization we impose on prices. We assume that t satisfies

(A.5) t is C^1.

(A.6) $t(\lambda\pi, \lambda r) \equiv \lambda t(\pi, r)$ for any $\lambda > 0$; that is, t is homogeneous of degree one.

In addition, we assume that ξ and t satisfy a modified version of Walras's law:

(A.7) $\pi'\xi(\pi, r) + t(\pi, r) \equiv r$.

As in a model without taxes, Walras's law can be justified by adding up the budget constraints of all the individual consumers.

To get some intuition for the content of the above specification, consider the following example of consumption and income tax schemes in an economy with h consumers: Consumer j has an income that consists of the value of his initial endowments, $\sum_{i=1}^{n} \pi_i w_i^j$, and his share of tax revenue, $\theta_j r$. Here the share coefficients θ_j, $j = 1, \dots, h$, are non-negative and sum to one. If, for example, $\theta_h = 1$ while $\theta_1 = \theta_2 = \dots = \theta_{h-1} = 0$, then the hth consumer is the government. The endowment income of consumer j is taxed at a rate of $1 > \rho_j \geqslant 0$. The final demand for commodity i by consumer j is taxed at a rate of $\tau_{ij} \geqslant 0$ on its value. The problem that faces the consumer is

$$\max u_j(x_1{}^j, \ldots, x_n{}^j)$$

$$\text{s.t.} \quad \sum_{i=1}^{n} \pi_i(1 + \tau_{ij})x_i{}^j \leqslant (1 - \rho_j) \sum_{i=1}^{n} \pi_i w_i{}^j + \theta_j r, \qquad (5)$$

$$x_i{}^j \geqslant 0$$

where u_j is a strictly concave and monotonically increasing utility function.

The individual's excess demand functions $\xi_i{}^j(\pi, r) = x_i{}^j(\pi, r) - w_i{}^j$ that are derived by solving this problem are continuous, at least for strictly positive prices, homogeneous of degree zero in π and r, and bounded from below by $-w^j = -(w_1{}^j, \ldots, w_n{}^j)$. The aggregate excess demand function $\xi(\pi, r) = \sum_{j=1}^{h} \xi^j(\pi, r)$ has the property that for any $\pi \in R_+{}^n \setminus \{0\}$, $\|\xi(\pi, r^i)\| \to \infty$ as $r^i \to \infty$. This condition means that, everything else being equal, if tax revenue becomes arbitrarily large, then the income of at least one consumer (the government) becomes arbitrarily large, which then implies that excess demand for some good becomes arbitrarily large. Debreu (1972) and Mas-Colell (1974) have shown that assuming that the aggregate excess demand function is continuously differentiable rather than merely continuous is not very restrictive. Furthermore, Kehoe (1982b) has shown that assuming ξ to be continuous even when some, but not all, prices are zero is not a substantive assumption for our purposes.

In this model

$$t(\pi, r) = \sum_{j=1}^{h} \rho_j \sum_{i=1}^{n} \pi_i w_i{}^j + \sum_{j=1}^{h} \sum_{i=1}^{n} \sigma_{ij} \pi_i x_i{}^j(\pi, r). \qquad (6)$$

Notice that t is C^1 as long as the $x_i{}^j$ are, and homogeneous of degree one as long as the $x_i{}^j$ are homogenous of degree zero. Furthermore, since each individual demand function satisfies the budget constraint with equality, ξ and t satisfy Walras's law.

Specific taxes on consumption as well as income tax rates and revenue-shares that vary with income can also be included in this framework. Care must be taken, however, to ensure that ξ and t satisfy the relevant homogeneity properties. Consider for example, an economy with a specific tax rate τ_{ij} on consumption of commodity i by consumer j. If we set $t(\pi, r) = \sum_{j=1}^{h}\sum_{i=1}^{n}\tau_{ij}x_i{}^j(\pi, r)$, then t is not homogeneous of degree one. Instead we could set $t(\pi, r) = q(\pi)\sum_{j=1}^{h}\sum_{i=1}^{n}\tau_{ij}x_i{}^j(\pi, r)$, where $q(\pi)$ is some price index that is homogeneous of degree one in prices. Here $q(\pi)\tau_{ij}$ is the specific tax rate. For example, $q(\pi) = \pi_1$ requires that taxes be denominated in terms of the price of commodity 1, which serves as a numeraire; $q(\pi) = \sum_{i=1}^{n}\gamma_i \pi_i$ denominates taxes in some 'real' terms, that is, in terms of some price index. On the other hand, we do not want income tax rates or revenue shares to change if all prices are multiplied by a positive scaler. For example, if $\rho : R_+ \to [0, 1)$ corresponds to a progressive income tax schedule, we would want ρ to vary with $\left(\sum_{i=1}^{n}\pi_i w_i{}^j\right)/q(\pi)$ rather than $\sum_{i=1}^{n}\pi_i w_i{}^j$.

It should be stressed that we are not requiring that specific taxes or incomes taxes be fully indexed in any sense; only that they be denominated in terms of the price of some commodity or bundle of commodities. Subsequently, we normalize prices so that $\sum_{i=1}^{n}\pi_i = 1$. If we do not explicitly denominate taxes in terms of some prices, then we are implicitly denominating them in terms of the price index $\sum_{i=1}^{n}\pi_i$. If some good in the model is called money, then we can denominate taxes in terms of its price; it is meaningless to talk of taxes not being denominated in terms of some price index.

We initially specify the production technology by an $n \times m$ activity analysis matrix A. Later we indicate how our results can be extended to more general technologies. We assume that A satisfies

(A.8) A includes n free disposal activities, one for each commodity.

(A.9) No output is possible without any inputs; equivalently,

$$\{x \in R^n | x = Ay, y \geqslant 0\} \cap R_+^n = \{0\}.$$

Production taxes are specified by an $n \times m$ matrix A^* that satisfies

(A.10) $A^* \leqslant A$.

Let the output or input of commodity i in activity j be taxed at a rate of $\tau_{ij} \geqslant 0$ on its value. We construct A^* by setting $a_{ij}^* = a_{ij} - \tau_{ij}|a_{ij}|$. The revenue generated by production taxes at prices $\pi \in R_+^n \setminus \{0\}$ and activity levels $y \in R_+^m$ is $\pi'(A - A^*)y \geqslant 0$. We can specify specific production taxes similarly as long as the price index that denominates taxes is linear in π. We assume that there are no taxes, ad valorem or specific, on free disposal activities.

An *equilibrium* of an economy (ξ, t, A, A^*) is defined to be a vector $(\hat{\pi}, \hat{r})$ that satisfies the following conditions:

(E.1) $\hat{\pi}'A^* \leqslant 0$.

(E.2) $\xi(\hat{\pi}, \hat{r}) = A\hat{y}$ for some $\hat{y} \geqslant 0$.

(E.3) $\hat{r} = t(\hat{\pi}, \hat{r}) + \hat{\pi}'(A - A^*)\hat{y}$.

(E.4) $\hat{\pi}'e = 1$, where $e = (1, \dots, 1)$.

Notice that conditions (E.2) and (E.3), together with Walras's law, imply that $\hat{\pi}'A\hat{y} = \hat{\pi}'\xi(\hat{\pi}, \hat{r}) = \hat{r} - t(\hat{\pi}, \hat{r}) = \hat{\pi}'(A - A^*)\hat{y}$. Consequently, $\hat{\pi}'A^*\hat{y} = 0$, which, together with (E.1), implies that after-tax profits are maximized at equilibrium. (E.2) is the requirement that consumer excess demand can be supplied. (E.3) stipulates that disbursals of tax revenue equal tax receipts. This equilibrium condition can be thought of as the government's budget constraint. We need to use it as an equilibrium condition rather than as an identity, because the levels of expenditure, which depend on government income, need to be known before tax receipts can be calculated.

(E.4) is a normalization that we are permitted by the homogeneity of ξ and t. (E.1) and the free disposal assumption, which imply that equilibrium prices are non-negative, allow us to restrict our attention to the simplex $S = \{\pi \in R^n | \pi' e = 1, \pi \geqslant 0\}$. The advantage of using S as the price domain is that it is compact and convex. We want the set of all (π, r) that satisfy (E.1)–(E.3) to lie in a compact convex set. (A.4) says that $\|\xi(\pi, r^i)\| \to \infty$ as $r^i \to \infty$ for any $\pi \in S$. On the other hand, (A.3) and (A.9) imply that the production possibility set $\{x \in R^n | x = Ay \geqslant -w, y \geqslant 0\}$ is bounded and, therefore, that there exists some $\alpha > 0$ such that $\|x\| < \alpha$ for any vector in this set. Consequently, we can find some $\beta > 0$ such that $\|\xi(\pi, r)\| \geqslant \alpha$ if $\pi \in S$ and $r \geqslant \beta$. In searching for equilibria, we consider the set $S \times [0, \beta]$; it contains all the equilibria of (ξ, t, A, A^*).

An *economy* is defined to be a quadruple (ξ, t, A, A^*) that satisfies (A.1)–(A.10). We give the space of economies $\mathcal{T}$ the topological structure of a metric space by defining the concept of distance between two economies. We endow the space of excess demand functions with the uniform C^1 topology: ξ^1 and ξ^2 are close if their values and the values of their partial derivatives are close everywhere on the compact set $S \times [0, \beta]$. Similarly, we endow the space of tax functions with the uniform C^1 topology. We endow both spaces of activity analysis matrices with the standard topology of R^{nxm}: A^1 and A^2 are close if their columns are close in euclidean terms. $\mathcal{T}$ itself receives the induced product topology: two economies are close if all their components are close.

We give the space of economies a topological structure because we subsequently want to talk about properties that are satisfied by almost all economies. The phrase 'almost all' is taken to mean all economies in an open dense subset of the space of economies. That a subset is open means that any small perturbation in an economy in the subset produced another economy in the subset. That a subset is dense means that any economy in the space that is not in the subset can be approximated arbitrarily closely by an economy in the set. To make sense of phrases such as 'small perturbation' and 'arbitrarily close approximation,' we use the topological structure defined above.

EXISTENCE OF EQUILIBRIUM

To prove the existence of equilibrium for our model we define a continuous mapping of $S \times [0, \beta]$ into itself whose fixed points are equivalent to the equilibria of (ξ, t, A, A^*). For any (π, r) we define $g(\pi, r)$ to be the vector (p, q) that solves the problem.

$$\begin{aligned} &\min 1/2 \, [(p - \pi - \xi(\pi, r))'(p - \pi - \xi(\pi, r)) + (q - t(\pi, r))^2] \\ &\text{s.t. } p'A - (1 + q - r)\pi'(A - A^*) \leqslant 0 \\ &\quad p'e = 1 \\ &\quad 0 \leqslant q \leqslant \beta. \end{aligned} \tag{7}$$

Similar least-distance mappings have been used by Eaves (1971) and Todd (1979) to prove the existence of equilibrium for other models.

(A.9) implies that the constraint set is non-empty. Since we have assumed that there are no taxes on the disposal activities, (A.8) implies that it is a subset of $S \times [0, \beta]$. It is also closed and convex and varies continuously, as a point-to-set correspondence, with (π, r). Therefore, since the objective function is strictly concave, $g(\pi, r)$ is a continuous function.

THEORM 1. *$(\hat{\pi}, \hat{r}) \in S \times [0, \beta]$ is an equilibrium of (ξ, t, A, A^*) if and only if $(\hat{\pi}, \hat{r}) = g(\hat{\pi}, \hat{r})$.*

Proof. The Kuhn-Tucker theorem implies that $(p, q) = g(\pi, r)$, that is, (p, q) solves (7) if and only if there exists an $m \times 1$ vector $y \geqslant 0$ and scalers $\mu, \nu \geqslant 0$ and λ such that

$$p - \pi - \xi(\pi, r) + Ay + \lambda e = 0 \tag{8}$$

$$q - t(\pi, r) - \pi'(A - A^*)y - \mu + \nu = 0. \tag{9}$$

The complementary slackness conditions are

$$(p'A - (1 + q - r)\pi'(A - A^*))y = 0 \tag{10}$$

$$\mu q = 0 \tag{11}$$

$$\nu(\beta - q) = 0. \tag{12}$$

Suppose that $(\hat{\pi}, \hat{r}) = g(\hat{\pi}, \hat{r})$ and, for the moment, that $\hat{\mu} = \hat{\nu} = 0$. Since $(\hat{\pi}, \hat{r})$ satisfies the constraints of (7), $\hat{\pi}'A^* \leqslant 0$ and $\hat{\pi}'e = 1$. We therefore need prove only that $(\hat{\pi}, \hat{r})$ satisfies (E.2) and (E.3) to demonstrate that it is an equilibrium. We begin by pre-multiplying (8) by $\hat{\pi}'$ to produce

$$-\hat{\pi}'\xi(\hat{\pi}, \hat{r}) + \hat{\pi}'Ay + \hat{\lambda} = 0. \tag{13}$$

Notice that (10) becomes $\hat{\pi}'A^*\hat{y} = 0$. Using this condition and Walras's law, we obtain

$$-\hat{r} + t(\hat{\pi}, \hat{r}) + \hat{\pi}'(A - A^*)\hat{y} + \hat{\lambda} = 0. \tag{14}$$

Adding this equation to (9) yields

$$\hat{\lambda} = 0. \tag{15}$$

(8) and (14) now become the desired equilibrium conditions.

Since $t(\pi, r) + \pi'(A - A^*)y \geqslant 0$ for any $(\pi, r) \in S \times [0, \beta]$ and any $y \geqslant 0$, we are justified in ignoring the possibility that $\hat{\mu} \neq 0$. Consider, however, the case where $(\hat{\pi}, \hat{r})$ is a fixed point of g and $\hat{\nu} > 0$. We can use the above reasoning to demonstrate that $-\xi(\hat{\pi}, \beta) + A\hat{y} > 0$ at such a point, which would contradict our choice of β. Consequently, there are no fixed points where $\hat{\nu} > 0$.

The converse, that any equilibrium is a fixed point, is easily demonstrated by making the proper choice of $\hat{y}$ and setting $\hat{\lambda} = \hat{m} = \hat{\nu} = 0$ in (8) and (9).

Since $S \times [0, \beta]$ is non-empty, compact, and convex, and g is continuous, Brouwer's fixed point theorem implies the existence of a fixed point of g and, hence, an equilibrium of (ξ, t, A, A^*).

REGULAR ECONOMIES

Proving the existence of equilibrium demonstrates the logical consistency of our model. To do comparative statistics we would want to know more. When are the equilibria of the model locally unique? When do they vary continuously with the parameters of (ξ, t, A, A^*)? When is there a unique equilibrium? In this section we consider the first two questions; in the next we consider the third.

The traditional approach to comparative statics, as developed by Hicks (1939) and Samuelson (1947), involves application of the inverse function theorem and implicit function theorem of differential calculus to equations that determine an equilibrium. If B is the matrix of activities in use at an equilibrium (π, r), then these equations, given by (E.1)–(E.3), are

$$B^{*\prime}\pi = 0 \tag{16}$$

$$\xi(\pi, r) - By = 0 \tag{17}$$

$$t(\pi, r) - r + \pi'(B - B^*)y = 0. \tag{18}$$

The first step is to count equations and unknowns. If B is $n \times k$, there are $n + k + 1$ equations in the $n + k + 1$ unknowns π, r, and y. Walras's law implies that one of the equations in (17) or (18) is extraneous; homogeneity implies that one of the variables in (π, r) is extraneous. Suppose we eliminate (18) from the system and, if $\hat{r} > 0$, normalize nominal values by setting $r = \hat{r}$. The inverse function theorem then says that, if the Jacobean matrix

$$J = \begin{bmatrix} B^{*\prime} & 0 \\ D\xi_{\hat{\pi}} & -B \end{bmatrix} \tag{19}$$

is non-singular, then the remaining equations are locally independent and determine the values of the unknowns π and y. The implicit function says that if this same condition is satisfied, then π and y vary smoothly as we vary parameters of the economy. We are not explicit about what these parameters are. They can in fact be any parameters that describe an element of the infinite dimensional space of economies $\mathcal{T}$.

Theorem 1 gives us an alternative to (16)–(18) for determining equilibria:

$$(\pi, r) - g(\pi, r) = 0. \tag{20}$$

To apply our analysis to this system of equations we need to ensure that g is differentiable at all its fixed points. We can do this by imposing three additional assumptions on (ξ, t, A, A^*):

(R.1) $t(\pi, r) > 0$ for all $(\pi, r) \in S \times [0, \beta]$.

(R.2) No column of either A or A^* can be represented as a linear combination of fewer than n other columns.

(R.3) Every activity that earns zero profit at equilibrium is associated with a positive activity level.

(R.1) implies that $\hat{r} > 0$ at every equilibrium; (R.2) implies that both B and B^* have full column rank; and (R.3) implies that the $k \times 1$ vector $\hat{y}$ that satisfies $\xi(\hat{\pi}, \hat{r}) = B\hat{y}$ is strictly positive. (R.1) does not rule out subsidies; it merely requires that, for any values of prices and revenue, taxes outweigh subsidies. As we have mentioned, we later relax this assumption to allow more general subsidy schemes.

If (ξ, t, A, A^*) satisfies (R.1)–(R.3), then we can write out the conditions that determine $(p, q) = g(\pi, r)$ in some neighbourhood of an equlibrium as

$$p - \pi - \xi(\pi, r) + By + \lambda e = 0 \tag{21}$$

$$q - t(\pi, r) - \pi'(B - B^*)y = 0 \tag{22}$$

$$p'B - (1 + q - r)\pi'(B - B^*) = 0 \tag{23}$$

$$p'e = 1 \tag{24}$$

for some $y > 0$ and some λ.

The implicit function theorem says that if the Jacobean matrix of this system with respect to p, q, y, and λ is non-singular, then these variables vary smoothly as functions of π and r. Moreover, we can compute the partial derivatives of these functions at $(\hat{\pi}, \hat{r})$ by solving

$$\begin{bmatrix} I & 0 & B & e \\ 0 & 1 & \hat{\pi}'(B^* - B) & 0 \\ B' & (B^* - B)'\hat{\pi} & 0 & 0 \\ e' & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} Dp_{\hat{\pi}} & Dp_{\hat{r}} \\ Dq_{\hat{\pi}} & Dq_{\hat{r}} \\ Dy_{\hat{\pi}} & D\lambda_{\hat{r}} \\ D\lambda_{\hat{\pi}} & D\lambda_{\hat{r}} \end{bmatrix}$$

$$= \begin{bmatrix} I + D\xi_{\hat{\pi}} & D\xi_{\hat{r}} \\ Dt_{\hat{\pi}} \_\_ y'(B - B^*)' & Dt_{\hat{r}} \\ (1 + \hat{q} - \hat{r})(B - B^*)' & (B^* - B)'\hat{\pi} \\ 0 & 0 \end{bmatrix}. \tag{25}$$

Let

$$C = \begin{bmatrix} B & e \\ -\hat{\pi}'B & 0 \end{bmatrix}, \quad G = \begin{bmatrix} I + D\xi_{\hat{\pi}} & D\xi_{\hat{r}} \\ Dt_{\hat{\pi}} + \hat{y}'(B - B^*)' & Dt_{\hat{r}} \end{bmatrix},$$

$$F = \begin{bmatrix} (B - B^*)' & -B'\hat{\pi} \\ 0 & 0 \end{bmatrix}.$$

Further let z denote the $(k + 1) \times 1$ vector (y, λ). (25) can be rewritten as

$$\begin{bmatrix} I & C \\ C' & 0 \end{bmatrix} \begin{bmatrix} Dg_{(\hat{\pi}, \hat{r})} \\ Dz_{(\hat{\pi}, \hat{r})} \end{bmatrix} = \begin{bmatrix} G \\ F \end{bmatrix}. \tag{26}$$

(R.2) implies that C has full column rank, since B has full column rank and $\hat{\pi}'B - \hat{\pi}'B = 0$, while $\hat{\pi}'e - 0 = 1$. Consequently, we can solve (26) to obtain

$$Dg_{(\hat{\pi}, \hat{r})} = (I - C(C'C)^{-1}C')G + C(C'C)^{-1}F. \tag{27}$$

To compare this approach with the previous one, we compare det $(I - Dg_{(\hat{\pi}, \hat{r})})$ with det (J). Notice that, since $C'C$ is positive definite and hence non-singular, we can write

$$\det (I - Dg_{(\hat{\pi}, \hat{r})}) = \det (C'C)^{-1} \det \begin{bmatrix} I - (I - C(C'C)^{-1}C')G - C(C'C)^{-1}F & C \\ 0 & C'C \end{bmatrix}. \qquad (28)$$

We can perform elementary row and column operations on the matrix on the far right of this expression without changing its determinant. First, subtract the second column post-multiplied by $(C'C)^{-1}C'G$ from the first; then subtract the first row pre-multiplied by C' from the second. The result is

$$\det (I - Dg_{\hat{\pi}, \hat{r})}) = \det (C'C)^{-1} \det \begin{bmatrix} I - G & C \\ F - C' & 0 \end{bmatrix}$$

$$= \det (C'C)^{-1} \det \begin{bmatrix} -D\xi_{\hat{\pi}} & -D\xi_{\hat{r}} & B & e \\ -Dt_{\hat{\pi}} - \hat{y}'(B - B^*)' & 1 - Dt_{\hat{r}} & -\hat{\pi}'B & 0 \\ -B^{*\prime} & 0 & 0 & 0 \\ -e' & 0 & 0 & 0 \end{bmatrix}. \qquad (29)$$

Differentiating Walras's law, we can establish that $\hat{\pi}'D\xi_{\hat{\pi}} + Dt_{\hat{\pi}} = -\xi(\hat{\pi})' = -\hat{y}'B$ and $\hat{\pi}'D\xi_{\hat{r}} + Dt_{\hat{r}} = 1$. Consequently, adding the first row of the above matrix pre-multiplied by $\hat{\pi}'$ to the second and adding the third row pre-multiplied by $\hat{y}'$ to the second yields

$$\det (I - Dg_{(\hat{\pi}, \hat{r})}) = \det (C'C)^{-1} \det \begin{bmatrix} 0 & -e' & 0 \\ D\xi_{\hat{r}} & -D\xi_{\hat{\pi}} & B \\ 0 & -B^{*\prime} & 0 \end{bmatrix}. \qquad (30)$$

Similarly differentiating (A.2), we can subtract the second column of this matrix post-multiplied by $\hat{\pi}$ from the first multiplied by $\hat{r}$ to produce

$$\det (I - Dg_{(\hat{\pi}, \hat{r})}) = 1/\hat{r} \det (C'C)^{-1} \det \begin{bmatrix} -D\xi_{\hat{\pi}} & B \\ -B^{*\prime} & 0 \end{bmatrix}. \qquad (31)$$

Notice that this expression is non-vanishing if and only if the Jacobian matrix in (19) is non-singular.

A *regular economy* is defined to be one that satisfies (R.1)–(R.3) and one additional restriction:

(R.4) $I - Dg_{(\hat{\pi}, \hat{r})}$ is non-singular at every equilibrium $(\hat{\pi}, \hat{r})$.

THEOREM 2. *If* $(\xi, t, A, A^*) \in \mathcal{S}$ *is a regular economy, then it has a finite number of equilibria that vary continuously in the topology we have defined on* $\mathcal{T}$.

For a proof of a similar theorem see Kehoe (1980).

The appeal of the concept of regularity is enhanced by its genericity in the space of economies.

THEOREM 3. *Regular economies form an open dense subset of $\mathcal{T}$.*

In other words, if an economy is regular, then any sufficiently small perturbation produces another regular economy, but, if an economy is not regular, then an arbitrarily small perturbation produces a regular economy.

The proof of this theorem follows the same lines as that in Kehoe (1980). The idea of the proof is to demonstrate that we can perturb the system of equations (20) or alternatively (16)–(18) in a suitable number of directions. If (R.1) or (R.3) is not satisfied, then there are more equations than unknowns in the system. If we could perturb the system in a suitable number of directions, we would not, in general, expect a solution to exist where these assumptions are violated. If (R.2) or (R.4) is not satisfied, then there are more unknowns than independent equations. Perturbing the system would, in general, make these equations independent. Figure 2 depicts an economy that is not regular and two different perturbations that make it regular.

Demonstrating that almost all functions t satisfy (R.1) and that almost all matrices A and A^* satisfy (R.2) is trivial. Actually, both restrictions are stronger than needed. (R.1), for example, can be replaced with the assumption that $t(\pi, r) = 0$ on some neighbourhood of $(\hat{\pi}, \hat{r})$ if $t(\hat{\pi}, \hat{r}) = 0$. To demonstrate that almost all economies satisfy (R.3) and (R.4), we perturb ξ and t using a vector (u, v) that lies in an open subset of R^{n+l} that contains the origin

$$\xi_{(u,v)}(\pi, r) = \xi(\pi, r) + [(\pi' u - v)/\pi' e]\, e - u \tag{32}$$

$$t_{(u,v)}(\pi, r) = \xi(\pi, r) + v. \tag{33}$$

It is easy to verify that $(\xi_{(u,v)}, t_{(u,v)}, A, A^*) \in \mathcal{T}$ if (ξ, t, A, A^*) is. It can also be proved that, for any fixed (ξ, t, A, A^*) that satisfies (R.1) and (R.2), almost all (u, v) are such that $(\xi_{(u,v)}, t_{(u,v)}, A, A^*)$ satisfies (R.3) and (R.4).

THE INDEX THEOREM

We define the *index* of an equilibrium $(\hat{\pi}, \hat{r})$ of a regular economy to be sgn [det $(I - Dg_{(\hat{\pi}, \hat{r})})$]. Our discussion in the previous section indicates that

$$\text{index}(\hat{\pi}, \hat{r}) = \text{sgn}\left(\det \begin{bmatrix} -D\xi_{\hat{\pi}} & B \\ -B^{*\prime} & 0 \end{bmatrix}\right). \tag{34}$$

Relying on the approach used by Saigal and Simon (1973), we can prove the following theorem:

THEOREM 4. *If (ξ, t, A, A^*) is a regular economy, then $\sum$index$(\hat{\pi}, \hat{r}) = +1$ where the sum is over all equilibria.*

Notice that this result implies that there are an odd number of equilibria and, in particular, that there is at least one equilibrium.

The concepts of regularity and fixed-point index that we have discussed here can easily be extended to economies with more general smooth production technologies,

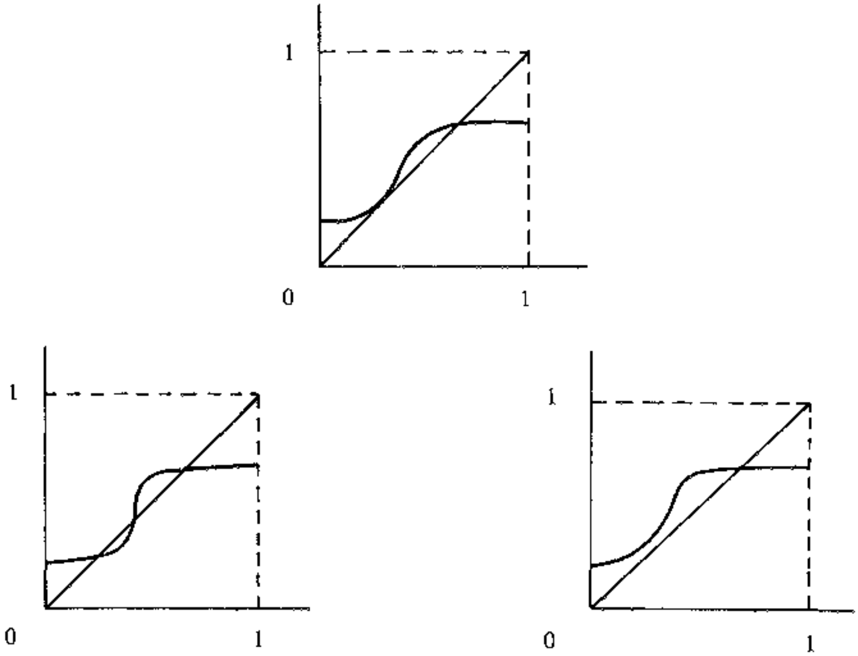

FIGURE 2

involving both constant and decreasing returns. In situations where efficient production techniques vary with prices we can compute the index of an equilibrium as

$$\text{index}(\hat{\pi}, \hat{r}) = \text{sgn}\left(\det\begin{bmatrix} -D\xi_{\hat{\pi}} + H(\hat{\pi}) & B(\hat{\pi}) \\ B^{*\prime}(\hat{\pi}) & 0 \end{bmatrix}\right). \tag{35}$$

Here $H(\pi)$ is the matrix of partial derivatives of the vector $B(\pi)y$ with respect to the vector π where y is held fixed (see Kehoe (1983)).

The most significant consequence of the index theorem is that it permits us to establish conditions sufficient for uniqueness of equilibrium. If index $(\hat{\pi}, \hat{r}) = +1$ at every equilibrium, then (π, t, A, A^*) has a unique equilibrium. Moreover, these conditions are necessary for uniqueness in almost all cases: If $(\hat{\pi}, \hat{r})$ is an equilibrium where index$(\hat{\pi}, \hat{r}) = -1$, then (π, t, A, A^*) has multiple equilibria.

Kehoe (1982a) studies the question of when an economy with production, but no distortions, has a unique equilibrium. He discusses two conditions, which correspond to easily interpretable economy assumptions, that imply uniqueness of equilibrium. The first is that ξ satisfies the weak axiom of revealed preference. An easily interpretable assumption that implies that this condition holds is that there is a representative consumer who generates ξ. The second condition is that there are always $n - 1$ activities in use at equilibrium. An assumption that implies that this condition holds is that the economy has an input-output structure: There is one non-produced good; there are no initial endowments of produced goods; and positive production of produced goods is feasible.

Unfortunately, the distortions present in the model with taxes make even these assumptions, which are extremely restrictive, too weak to ensure uniqueness. This point has been made by Foster and Sonnenschein (1970) and Hatta (1977), who present graphical examples of single-consumer economies and input-output production technologies with multiple equilibria. As these authors have pointed out, such examples depend on inferiority of at least one good.

Suppose that there are two produced goods and one factor of production, which is supplied inelastically. The production possibility frontier in the space of the two produced goods is a straight line. In a model without taxes the slope of this line would be the negative of the relative price ratio for the two goods. Using this relative price ratio and the zero profit condition, we could easily compute the unique equilibrium prices without regard to demand. In a model with taxes there are two crucial differences. First, the prices faced by consumers do not necessarily correspond to those given by the slope of the production possibility frontier. Second, the level of demand is crucial for determining the level of tax revenue, which is an essential part of the definition of an equilibrium. Figure 3 depicts a model with three equilibria, each of which has the same price vector but a different level of revenue.

Using our index theorem, we are able to generalize a condition due to Hatta that does imply uniqueness of equilibrium in economies with input-output production technologies with one non-produced good: Let $\bar{\pi}$ be the vector of efficiency prices associated with the production matrix B in the sense that $\bar{\pi}'B = 0$. If $\bar{\pi}'D\xi_r(\pi, r) > 0$ for any choice of π and r, then the economy has a unique equilibrium. In contrast to this result, which holds for economies with many consumers, Hatta's condition is developed for the one consumer model and is expressed in terms of that consumer's compensated demand function.

To see why this condition works, we partition (34) into

$$\text{index}\ (\hat{\pi}, \hat{r}) = \text{sign}\left(\det \begin{bmatrix} -d_{11} & -d_{12} & b_1 \\ -d_{21} & -D_{22} & B_2 \\ -b_1{}^{*\prime} & -B_2{}^{*\prime} & 0 \end{bmatrix}\right). \tag{36}$$

Here d_{11} is 1×1, d_{12} is $1 \times (n-1)$, d_{21} is $(n-1) \times 1$, D_{22} is $(n-1) \times (n-1)$, and so on. We have numbered goods so that the first is the non-produced factor of production. For production to be feasible, B_2 must be a productive Leontief matrix. Since $B^* \leqslant B$ but $[\hat{\pi}_2 \ldots \hat{\pi}_n]\, B_2{}^* = -\hat{\pi}_1 b^{*\prime}{}_1 \geqslant 0$ for some $\hat{\pi} \in S$, $B_2{}^*$ must also be a productive Leontief matrix. Performing elementary row and column operations, and using (A.2) to establish that $D\xi_{\hat{\pi}}\hat{\pi} + D\xi_{\hat{r}}\hat{r} = 0$, we can obtain

$$\begin{aligned} \text{index}\ (\hat{\pi}, \hat{r}) &= \text{sgn}\left((\hat{r}/\hat{\pi}_1\bar{\pi}_1) \det \begin{bmatrix} \bar{\pi}'D\xi_{\hat{r}} & 0 & 0 \\ 0 & 0 & B_2 \\ 0 & -B_2{}^{*\prime} & 0 \end{bmatrix}\right) \\ &= \text{sgn}\ (\bar{\pi}'D\xi_{\hat{r}} \det (B_2{}^*) \det (B_2)) \\ &= \text{sgn}\ (\bar{\pi}'D\xi_{\hat{r}}) \end{aligned} \tag{37}$$

since $\det (B_2{}^*) > 0$ and $\det (B_2) > 0$.

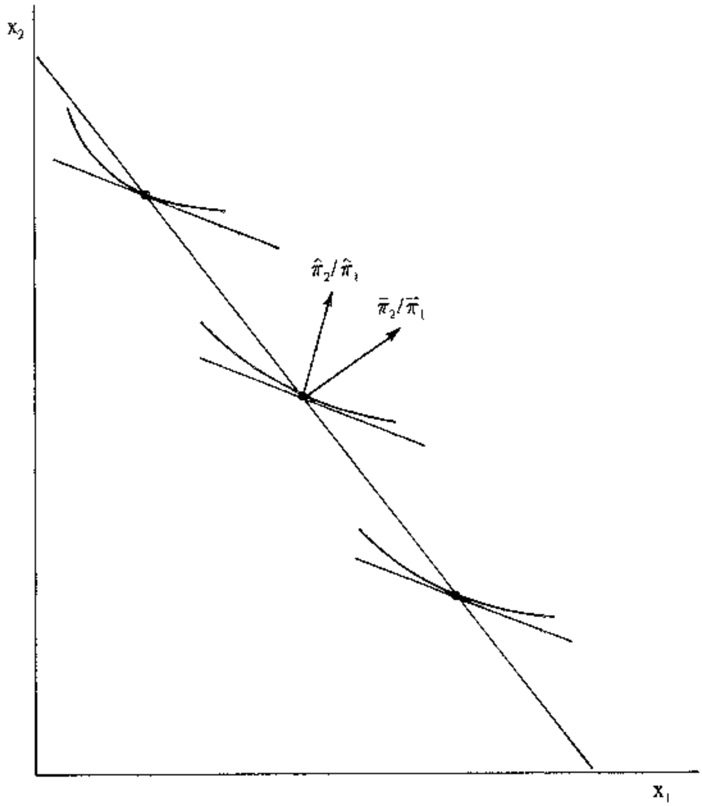

FIGURE 3

SUBSIDIES

The model analysed in the previous sections has very limited scope for analysing subsidies. We have imposed conditions on $t(\pi, r)$ and A^* that ensure that $\hat{r} \geqslant 0$ at every equilibrium. If this is not the case, then the government may find it impossible to balance its budget. This can cause serious problems in existence proofs (see Shafer and Sonnenschein, 1976). In this section we extend our analysis to a model that allows a wide variety of subsidy schemes.

We allow the same sorts of subsidies as we do taxes: ad valorem and specific subsidies on production and consumption and linear and non-linear subsidies on income. We need to be able to guarantee, however, that the government can pay these subsidies out of its tax revenues. To do this, we introduce another variable, s, that is equal to the fraction of the subsidy payments the government can afford to make: If $s = 1$, the government has enough tax revenues to make all the subsidy payments. If $s = 0$, the government cannot afford to make any subsidy payments. Define $\xi(\pi, r, s)$ as the aggregate excess demand function and $t(\pi, r, s)$ as the tax payments by consumers net of subsidies to consumers. Both ξ and t are assumed to be C^1. For fixed s, $0 \leqslant s \leqslant 1$, ξ is assumed to satisfy assumptions (A.2)–(A.4), and t is assumed to satisfy (A.6). ξ and t are assumed to satisfy (A.7), Walras's law. We assume that $t(\pi, r, 0) > 0$ for every $(\pi, r) \in (R_+^n \setminus \{0\}) \times R_+$.

To clarify the role of s, let us modify our example of an economy with taxes to allow subsidies: The problem that faces the consumer becomes

$$\max u_j(x_1^j, \ldots, x_n^j)$$

$$\text{s.t.} \sum_{i=1}^{n} \pi_i(1 + \tau_{ij} - s\sigma_{ij})x_i^j > (1 - \rho_j) \sum_{i=1}^{n} \pi_i w_i^j + sq(\pi)\eta_j + \theta_j r. \qquad (38)$$

Here $1 > \sigma_{ij} \geqslant 0$ is the ad valorem subsidy rate on the final demand for commodity i by consumer j, $\eta_j \geqslant 0$ is a fixed income transfer, and $q(\pi)$ is some homogeneous of degree one price index.

To model subsidies to producers we construct a matrix A** by setting $a_{ij}^{**} = \sigma_{ij}|a_{ij}|$ where now $\sigma_{ij} \geqslant 0$ is the ad valorem subsidy rate on the output of input of commodity i in activity j. We assume that there are no subsidies on free disposal activities.

The concept of equilibrium is modified to be a vector $(\hat{\pi}, \hat{r}, \hat{s})$ that satisfies the following conditions:

(E.1) $\hat{\pi}'(A^* + \hat{s}A^{**}) \leqslant 0$.

(E.2) $\xi(\hat{\pi}, \hat{r}, \hat{s}) = A\hat{y}$ for some $\hat{y} \geqslant 0$.

(E.3) $\hat{r} = t(\hat{\pi}, \hat{r}, \hat{s}) + \hat{\pi}'(A - A^* - \hat{s}A^{**})\hat{y}$.

(E.4) $\hat{\pi}'e = 1$.

(E.5) If $\hat{s} < 1$, then $\hat{r} = 0$, and if $\hat{r} > 0$, then $\hat{s} = 1$.

Conditions (E.1)–(E.4) require no comment, since they are analogous to the conditions of the previous definition. Condition (E.5) says that the first commitment of the government tax receipts is to subsidies, and only after making all the subsidy payments can it transfer revenue to consumers. Furthermore, (E.3) and (E.5) together imply that, if $\hat{s} = 0$, then $t(\hat{\pi}, \hat{r}, 0) + \hat{\pi}'(A - A^*)\hat{y} = 0$.

To prove the existence of equilibrium for this model we define a continuous mapping of $S \times [0, \beta] \times [0, 1]$ into itself, whose fixed points are equivalent to the equilibria of (ξ, t, A, A^*, A^{**}). For any (π, r, s) we define $g(\pi, r, s)$ to be the vector (p, q, u) that solves the problem

$$\min \tfrac{1}{2}[(p - \pi - \xi(\pi, r, s))'(p - \pi - \xi(\pi, r, s)) + (q - t(\pi, r, s))^2 + (u - s - t(\pi, r, s))^2]$$

$$\text{s.t. } p'A - (1 + q - r + u - s)\pi'(A - A^* - A^{**}) - (1 + u - s)(1 - s)\pi'A^{**} \leqslant 0$$

$$p'e = 1 \tag{39}$$

$$0 \leqslant q \leqslant \beta$$

$$0 \leqslant u \leqslant 1.$$

The Kuhn-Tucker conditions that characterize the unique solution to this problem are analogous to (8)–(12):

$$p - \pi - \xi(\pi, r, s) + Ay + \lambda e = 0 \tag{40}$$

$$q - t(\pi, r, s) - \pi'(A - A^* - A^{**})y - \mu + \nu = 0 \tag{41}$$

$$(p'A - (1 + q - r + u - s)\pi'(A - A^* - A^{**}) - (1 - u - s)(1 - s)\pi'A^{**})y = 0 \tag{42}$$

$$\mu q = 0 \tag{43}$$

$$\nu(\beta - q) = 0. \tag{44}$$

In addition, there is a condition associated with the new variable s:

$$u - s - t(\pi, r, s) - \pi'(A - A^* - sA^{**})y - \phi + \psi = 0. \tag{45}$$

Here ϕ and ψ are non-negative scalars such that

$$\phi u = 0 \tag{46}$$

$$\psi(1 - u) = 0. \tag{47}$$

The proof that a vector $(\hat{\pi}, \hat{r}, \hat{s})$ is an equilibrium of (ξ, t, A, A^*, A^{**}) is analogous to the proof of theorem 1: since $(\hat{\pi}, \hat{r}, \hat{s})$ satisfies the constraints of (39), the conditions (E.1) and (E.4) are satisfied. Suppose for the moment that $\hat{\nu} = \hat{\phi} = 0$. Pre-multiplying (40) by $\hat{\pi}'$ produces

$$-\hat{\pi}'\xi(\hat{\pi}, \hat{r}, \hat{s}) + \hat{\pi}'A\hat{y} + \hat{\lambda} = 0. \tag{48}$$

There are two cases to be considered. First, suppose that $\hat{r} > 0$. Then $\hat{\mu} = 0$. Using Walras's law, (41), (42), and (48), we can argue, as in the proof of theorem 1, that $\hat{\lambda} = 0$. Consequently, (40) becomes (E.2) and (41) becomes (E.3). Second, suppose that $\hat{r} = 0$. Then $t(\hat{\pi}, \hat{r}, \hat{s}) + \hat{\pi}'(A - A^* - A^{**})\hat{y} < 0$. This implies that $\hat{\psi} = 0$. Now we can use Walras's law, (42), (45), and (48) to establish that $\hat{\lambda} = 0$. In either case $\hat{\psi} = \hat{r}$, and (47) becomes (E.5). We can again rule out the possibility that $\hat{\nu} > 0$ at a fixed point by arguing that this would imply that $-\xi(\hat{\pi}, \beta, \hat{s}) + A\hat{y} > 0$ at such a point, which would contradict our choice of β. The assumption that $t(\hat{\pi}, \hat{r}, 0) > 0$ precludes the possibility that $\hat{\phi} > 0$ at a fixed point.

The next step of the analysis is to define the concept of a regular economy and to analyse its properties. We modify (R.1) to require that $\iota(\pi, r, 0) > 0$ for any $(\pi, r) \in S \times [0, \beta]$. It is too much to assume that no column of $A^* + sA^{**}$ can be expressed as a linear combination of fewer than n other columns for any $s \in [0, 1]$. Instead, we assume that this condition holds at any equilibrium, and we add this condition to (R.2). (R.3) remains the same. (R.4) becomes the requirement that $I - Dg_{(\hat{\pi}, \hat{r}, \hat{s})}$ is non-singular at every equilibrium $(\hat{\pi}, \hat{r}, \hat{s})$. A regular economy is defined as one that satisfies (R.1) and (R.4) and a new restriction.

(R.5) If $\hat{s} = 1$, then $\hat{r} > 0$.

This rules out the case where $\hat{s} = 1$ and $\hat{r} = 0$. As before, a regular economy has a finite number of equilibria that vary continuously with its parameters. Also, as before, almost all economies are regular.

(R.5) says that in some neighbourhood of an equilibrium either $u = 1$ or $q = 0$ in the calculations of (p, q, u) in (39). In the case where $u = 1$ we can use the equations

$$p - \pi - \xi(\pi, r, s) + By + e = 0 \tag{49}$$

$$q - \iota(\pi, r, s) - \pi'(B - B^*)y = 0 \tag{50}$$

$$p'B - (1 + q - r)\pi'(B - B^*) = 0 \tag{51}$$

$$p'e = 1 \tag{52}$$

$$u = 1 \tag{53}$$

and the implicit function theorem to compute $Dg_{(\hat{\pi}, \hat{r}, \hat{s})}$. The formula for the index remains the same:

$$\text{index } (\hat{\pi}, \hat{r}, \hat{s}) = \text{sgn} \left(\det \begin{bmatrix} -D\xi_{\hat{\pi}} & B \\ -B^{*\prime} - \hat{s}B^{**\prime} & 0 \end{bmatrix} \right). \tag{54}$$

In the case where $\hat{q} = 0$ we can replace (50) with

$$q = 0 \tag{55}$$

and (53) with

$$u - s - \iota(\pi, r, s) - \pi'(A - A^* - sA^{**})y = 0. \tag{56}$$

The formula for the index is again the same. In both cases the derivation is virtually identical to steps (25)–(31).

CONCLUDING REMARKS

Assumptions that imply local uniqueness and continuity of equilibria are not at all restrictive; they are satisfied by almost all economies. Unfortunately, global uniqueness of equilibrium is much more elusive. Not even the representative consumer and input-output assumptions are enough to ensure it.

It would seem necessary, therefore, to develop a comparative statics methodology that does not depend on uniqueness of equilibrium. Hatta (1977) attempts to do something of this sort, utilizing a Marshallian concept of local stability. His analysis, however, pertains to economies with a representative consumer and an input-output structure. In such a model it can be shown that equilibria with index $+1$ are locally stable, while those with index -1 are unstable. Unfortunately, although his analysis can be generalized to include more than one consumer, it does not seem possible to extend it to more general production technologies.

An alternative is to perform comparative statics only in some neighbourhood of a regular economy. Although this approach may be defensible, a warning should be given with regard to the continuity of equilibria at regular economies. Although almost all economies are regular, discrete changes in parameters may necessarily pass through critical economies where mathematical catastrophes occur. It does not seem that any sort of comparative statics methodology is applicable in such circumstances.

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