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Uniqueness and Stability

TIMOTHY J. KEHOE

1 Introduction

Consider an economist working with an applied general equilibrium model. This economist starts by calibrating, or statistically fitting, the parameters of the model so that it has an equilibrium that replicates transactions observed in the data. He or she then changes some of the parameters to simulate a change in policy, and computes an equilibrium to the perturbed model. The economist then uses the changes in the values of variables from the initial equilibrium to the new one as an indication of the changes that he or she would expect to see in the corresponding variables in the economy if the simulated policy change were to occur. This is the comparative statics method. If there is more than one possible equilibrium after the parameter change, the method becomes problematic.

Since the time of Wald (1936) economists have searched for conditions that ensure uniqueness of equilibrium in general equilibrium models. Arrow and Hahn (1971: ch. 9), Kehoe (1985b, 1991), and Mas-Colell (1991) provide surveys of the results obtained. This chapter presents an overview of different approaches, summarizes general results, and provides examples of economies with multiple equilibria. Conditions that guarantee uniqueness of equilibrium are those that rule out the features of these examples that allow multiple equilibrium. We begin by considering exchange economies, and then successfully generalize the analysis to allow for production and distortionary taxes. The focus here is on economies with a finite number of goods. The conclusion briefly indicates how the analysis can be extended to economies with an infinite number of goods, an essential step for studying economies with time and uncertainty.

As we shall see, useful conditions that guarantee the uniqueness of equilibrium are very restrictive. In this chapter we consider a general set of mathematical conditions that are sufficient and, in general, necessary for uniqueness. The problem is that in translating these mathematical conditions into easy-to-check and interpretable economic conditions, they lose their necessity. It may be the case that most applied models have unique equilibria. Unfortunately, however, these models seldom satisfy analytical conditions that are known to guarantee uniqueness, and are often too large and complex to allow exhaustive searches to numerically verify uniqueness. More research is obviously needed.

2 Exchange Economies

Consider a model economy with m consumers who trade their endowments of n goods among themselves. Each consumer is specified by a utility function $u_i(x_1, \ldots, x_n)$ that is defined on a consumption set that is the nonnegative orthant of R^n, denoted R^n_+, and an endowment vector $w^i = (w^i_1, \ldots, w^i_n)$ that is strictly positive, $w^i \in R^n_{++}$. An equilibrium is a price vector $\hat{p}$ and an allocation $(\hat{x}^1, \ldots, \hat{x}^m)$, where $\hat{x}^i = (\hat{x}^i_1, \ldots, \hat{x}^i_n)$, such that

- given $\hat{p}$, each $\hat{x}^i$ solves
  maximize $u_i(x)$
  subject to $\hat{p} \cdot x \leq \hat{p} \cdot w^i$
  $x \geq 0$;
- $\sum_{i=1}^m \hat{x}^i \leq \sum_{i=1}^m w^i$.

(In the consumer's budget constraint, $\hat{p} \cdot x$ is, of course, the inner product $\sum_{j=1}^n \hat{p}_j x_j$.)

In the case where $m = n = 2$ the equilibrium can be depicted in an Edgeworth box. Unfortunately, even in this simple case there can be multiple equilibria, as the following example demonstrates.

2.1 EXAMPLE 1

Consumer i, $i = 1, 2$, has the utility function

$$u_i(x_1, x_2) = a^i_1(x_1^{b_i} - 1)/b_i + a^i_2(x_2^{b_i} - 1)/b_i$$

where $a^i_j > 0$ and $b_i < 1$. This is the familiar constant elasticity of substitution (C.E.S.) utility function with elasticity of substitution $\eta_i = 1/(1 - b_i)$; in the limit where $b_i = 0$, l'Hôpital's rule says that the utility function is $a^i_1 \log x_1 + a^i_2 \log x_2$. Suppose that $b_1 = b_2 = -4$, so that $\eta_1 = \eta_2 = 0.2$, and that the two consumers have the symmetric parameters $a^1_1 = a^2_2 = 1024$, $a^1_2 = a^2_1 = 1$, $w^1_1 = w^2_2 = 60$, and $w^2_1 = w^1_2 = 5$.

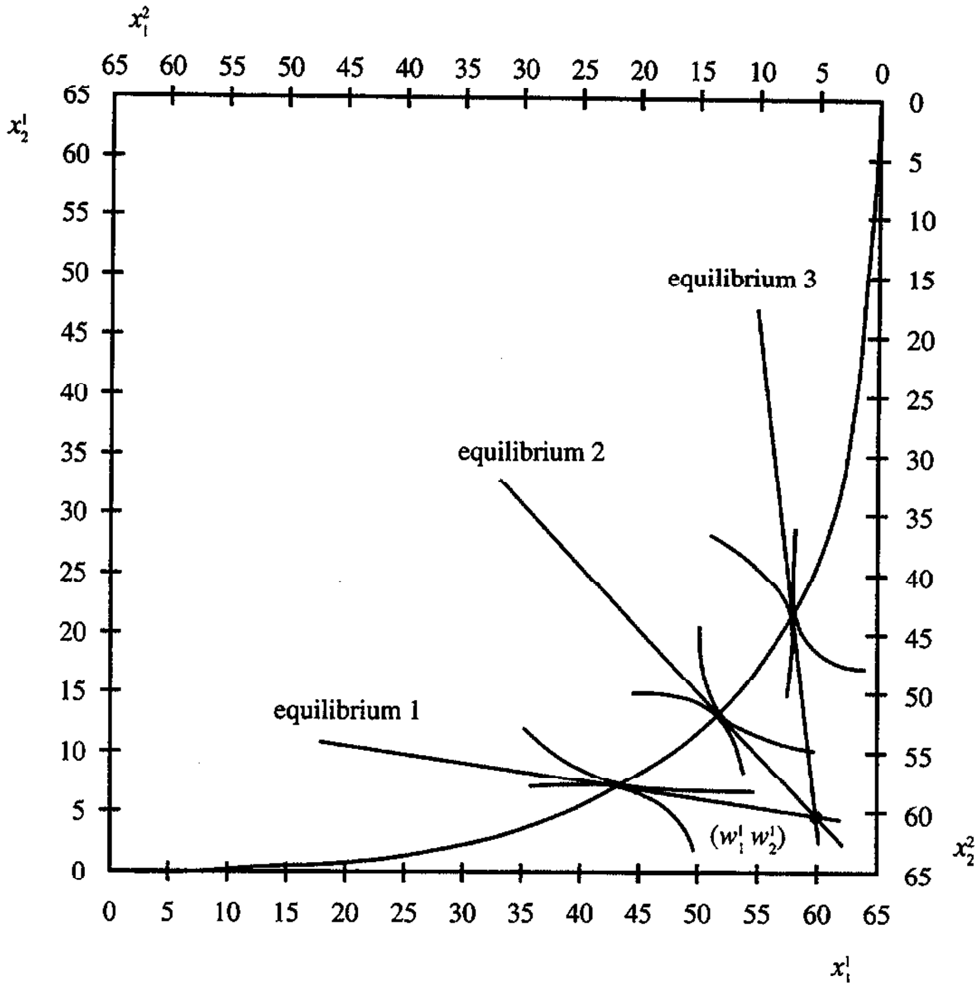

Figure 3.1 Nonuniqueness example in an Edgeworth box

This economy has three equilibria, which are depicted in the Edgeworth box in figure 3.1 and whose corresponding prices and allocations are listed below (The utility indices reported are 100,000 $(u_i - 1025/4)$; in other words, the constant terms $-a_j^i/b_i$ have been eliminated and the numbers scaled up.)

Equilibrium 1

	$\hat{x}_1^i$	$\hat{x}_2^i$	u_i
$\hat{x}_j^1$	43.1565	7.1442	−16.9770
$\hat{x}_j^2$	21.8435	57.8558	−2.3946
$\hat{p}_j$	0.1273	1.0	

Equilibrium 2

	$\hat{x}_1^i$	$\hat{x}_2^i$	u_i
$\hat{x}_j^1$	52.0	13.0	−4.3766
$\hat{x}_j^2$	13.0	52.0	−4.3766
$\hat{p}_j$	1.0	1.0	

Equilibrium 3

	$\hat{x}_1^i$	$\hat{x}_2^i$	u_i
$\hat{x}_j^1$	57.8558	21.8435	−2.3946
$\hat{x}_j^2$	7.1442	43.1565	−16.9770
$\hat{p}_j$	7.8555	1.0	

Notice that if all prices are multiplied by the same positive constant, the consumer's budget set does not change, and the price vector remains an equilibrium. Since we do not want to think of this indeterminacy of the absolute price level as multiplicity of equilibria, we need to somehow normalize prices. We have done so here by making the second good the numeraire and setting $\hat{p}_2 = 1$.

2.2 EXCESS DEMAND FUNCTIONS

The usual approach to analyzing the possibility of multiplicity of equilibria is to use the economy's aggregate excess demand function. We solve the maximization problem of consumer i to derive the demand function $x^i(p) = (x_1^i(p), \ldots, x_n^i(p))$. Our assumptions on $u_i(x)$ and w^i imply that $x^i(p)$ is continuous, at least for strictly positive price vectors; that it is homogeneous of degree zero, $x^i(\theta p) \equiv x^i(p)$ for all $\theta > 0$ and all price vectors p; and that it satisfies the budget identity $p^i \cdot x^i(p) \equiv p \cdot w^i$. The aggregate excess demand function

$$f(p) = \sum_{i=1}^{m} (x^i(p) - w^i),$$

therefore, is continuous, at least for all $p \in R^n_{++}$, is homogeneous of degree zero, and obeys Walras's law,

$$p \cdot f(p) \equiv 0.$$

An equilibrium is now specified as a price vector $\hat{p}$ for which

- $f_j(\hat{p}) \leq 0, j = 1, \ldots, n.$

(Walras's law implies that $f_j(\hat{p}) = 0$ if $\hat{p}_j > 0$, but we allow for free goods.) Obviously, $\hat{p}, x^1(\hat{p}), \ldots, x^m(\hat{p})$ is the corresponding equilibrium in the previous specification.

Unfortunately, utility maximization does not imply that f is continuous on the set of all nonnegative prices except $p = 0$, denoted $R^n_+\backslash\{0\}$. Excess demand may become unbounded from above on some sequence $p^k \to p^0$ where $p^k \in R^n_{++}$ and $p^0 \in R^n_+\backslash\{0\}$, $p^0_i = 0$ for some i. One approach to handling this minor technical problem is to bound the consumers' budget sets with a constraint like $x^i \leq 2\sum_{j=1}^m w^j$ (see Debreu 1959: ch. 5). With such a constraint $x^i(p)$, and therefore $f(p)$, is continuous on all $R^n_+\backslash\{0\}$. Moreover, the constraint $x^i \leq 2\sum_{j=1}^m w^j$ cannot bind in any equilibrium, since we know that $\sum_{i=1}^m \hat{x}^i \leq \sum_{i=1}^m w^i$ and $\hat{x}^i \geq 0$. Alternatively, we could modify the aggregate excess demand function $f(p)$ itself so that it is continuous on all $R^n_+\backslash\{0\}$, but is unaffected on a large open subset of prices that includes any possible equilibrium (see Kehoe 1982). In any case, the potential unboundedness of excess demand as some prices tend to zero plays little substantive role in the matters discussed here.

The excess demand function allows us to reduce our search for equilibria to a search for price vectors that satisfy $f(\hat{p}) \leq 0$ and $f_j(\hat{p}) = 0$ if $\hat{p}_j > 0$. Homogeneity allows us to normalize prices. Rather than choosing a numeraire by setting, say, $p_n = 1$, we normalize $\sum_{j=1}^n p_j = 1$. This has the advantages of restricting prices to a compact set and of allowing any good to potentially be a free good. Walras's law allows us to neglect one of the conditions $f_j(\hat{p}) = 0$ unless $\hat{p}_j = 0$. In general, therefore, we look for price vectors $\hat{p}$ that satisfy $\sum_{j=1}^n \hat{p}_j = 1$, $\hat{p}_j = 0$ and $f_j(\hat{p}) \leq 0, j = 1, \ldots, n-1$, with $f_j(\hat{p}) = 0$ if $\hat{p}_j > 0$.

2.3 EXAMPLE 1 (CONTINUED)

The demand function for good j by consumer i with C.E.S. utility is

$$x^i_j(p_1, p_2) = \frac{(a^i_j)^{\eta_i}(p_1 w^i_1 + p_2 w^i_2)}{p_j^{\eta_i}((a^i_1)^{\eta_i} p_1^{1-\eta_i} + (a^i_2)^{\eta_i} p_2^{1-\eta_i})}, \; i, j = 1, 2.$$

(Once again, $\eta_i = 1/(1 - b_i)$ is the elasticity of substitution.) For the parameters of our example,

$$f_1(p_1, p_2) = \frac{4(60p_1 + 5p_2)}{p_1^{0.2}(4p_1^{0.8} + p_2^{0.8})} + \frac{(5p_1 + 60p_2)}{p_1^{0.2}(p_1^{0.8} + 4p_2^{0.8})} - 65.$$

The equation $f_1(p_1, 1 - p_1) = 0$ has three solutions: $\hat{p}_1 = 0.1129$, which is equilibrium 1; $\hat{p}_1 = 0.5$, which is equilibrium 2; and $\hat{p}_1 = 0.8871$, which is equilibrium 3. Figure 3.2 depicts the graph of the function $f_1(p_1, 1 - p_1)$. An equilibrium is either a zero of this function, or $\hat{p}_1 = 0$ if $f_1(0, 1) \leq 0$, or $\hat{p}_1 = 1$ if $f_1(1, 0) = 0$ and where we need to check that $f_2(1, 0) \leq 0$.

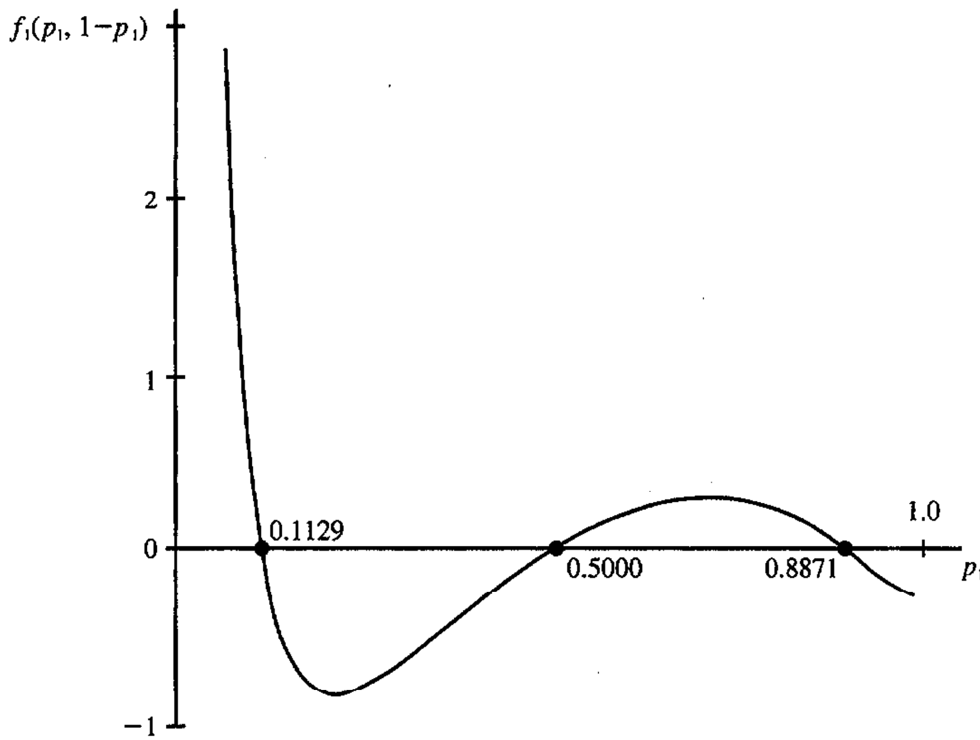

Figure 3.2 Nonuniqueness example in an excess demand diagram

2.4 GROSS SUBSTITUTABILITY

Two assumptions have played significant roles in discussions of uniqueness of equilibrium since the time of Wald (1936): gross substitutability and the weak axiom of revealed preference. We consider each of these two assumptions in turn, and explain how they are violated in our example of multiple equilibria.

Gross substitutability says that, if $p \geq q$ and $p_i = q_i$ for some i, then $f_i(p) \geq f_i(q)$ and, if $f(p) = f(q)$, then $p = q$. (This condition actually combines two conditions often known as "weak gross substitutability" and "indecomposability".) If $f(p)$ is continuously differentiable, then $\partial f_i(p)/\partial p_j > 0$ for $i \neq j$ is sufficient for gross substitutability, and $\partial f_i(\hat{p})/\partial p_j \geq 0$ is necessary. The argument that gross substitutability implies uniqueness is straightforward. Suppose that an excess demand function satisfies gross substitutability, but that there are two vectors p and q, such that $f(p) \leq 0$ and $f(q) \leq 0$. Then it must be the case that p and q are both strictly positive and that $f(p) = f(q) = 0$; otherwise, for example, $p_i = 0$, $2p \geq p$, and $f(2p) = f(p)$ would imply $2p = p$. If we set $\theta = \max [q_1/p_1, \ldots, q_n/p_n]$, then θ satisfies $\theta p \geq q$, $\theta p_i = q_i$ for some i. Consequently,

$f(\theta p) = f(p) = f(q) = 0$ and gross substitutability imply that $\theta p = q$ and that p and q are not distinct equilibria.

Our example of nonuniqueness violates gross substitutability because at $\hat{p} = (0.5, 0.5)$, for example, the Jacobian matrix of $f(p)$ is

$$Df(\hat{p}) = \begin{bmatrix} 1.28 & -1.28 \\ -1.28 & 1.28 \end{bmatrix},$$

and $\partial f_1(\hat{p})/\partial p_2 < 0$. It is easy to check that a sufficient condition for gross substitutability in economies where consumers have C.E.S. utility is that every consumer's curvature parameter satisfies $b_i \geq 0$; in other words, that every elasticity of substitution satisfies $\eta_i \geq 1$.

2.5 THE WEAK AXIOM

The weak axiom of revealed preference says that, if $p \cdot f(q) \leq 0$ and $f(p) \neq f(q)$, then $q \cdot f(p) > 0$. This condition implies that the set of equilibria is convex. Suppose that there are two vectors p and q such that $f(p) \leq 0$ and $f(q) \leq 0$. Observe first that $f(p) = f(q)$; otherwise $p \cdot f(q) \leq 0$ and $f(p) \neq f(q)$ would imply $q \cdot f(p) > 0$, which would contradict $f(p) \leq 0$ and $q \geq 0$. Let

$$p(\theta) = \theta p + (1 - \theta)q, 0 \leq \theta \leq 1.$$

Then $p(\theta) \cdot f(p) \leq 0$, and $p(\theta) \cdot f(q) \leq 0$. Furthermore, Walras's law says that $(\theta p + (1 - \theta)q) \cdot f(p(\theta)) = 0$. Consequently, it cannot be the case that $p \cdot f(p(\theta)) > 0$; otherwise $q \cdot f(p(\theta)) < 0$, which would contradict the weak axiom. We therefore conclude that $p \cdot f(p(\theta)) \leq 0$, which implies that $f(p(\theta)) = f(p)$. Therefore, unless the graph of $f_1(p_1, 1 - p_1)$ were to become tangent to the axis in figure 3.2, there is a unique equilibrium.

Wald (1936) actually employed a condition slightly stronger than the weak axiom: if $p \cdot f(q) \leq 0$ and $p \neq \theta q$, then $q \cdot f(p) > 0$. This condition implies uniqueness directly. The attractive feature of our weaker condition is that it is satisfied by any excess demand function derived from the maximum of a strictly concave utility function by a single consumer. (The term "axiom of revealed preference" was proposed by Samuelson (1938, 1948) as the basis for a characterization of individual demand functions that is an alternative to utility maximization.) That $p \cdot (x^i(q) - w^i) \leq 0$ means that $u_i(x^i(p)) \geq u_i(x^i(q))$ because $x^i(q)$ is affordable at prices p. Similarly, $q \cdot (x^i(p) - w^i) \leq 0$ means that $u_i(x^i(q)) \geq u_i(x^i(p))$. Therefore $x^i(p) = x^i(q)$, because otherwise the strict concavity of utility would imply that

$$u_i(\theta x^i(p) + (1 - \theta)x^i(q)) > u_i(x^i(p)), 0 < \theta < 1,$$

and, since $p \cdot (\theta x^i(p) + (1 - \theta)x^i(q) - w^i) \leq 0$, this would contradict $x^i(p)$ being utility-maximizing.

In our example, although both the individual excess demand functions $(x^1(p) - w^1)$ and $(x^2(p) - w^2)$ satisfy the weak axiom of revealed preference, their sum, $f(p)$, obviously does not: otherwise there could not be three isolated equilibria, $f(\hat{p}) = 0$. It is this lack of aggregatability that makes the weak axiom an unattractive condition compared to gross substitutability. If two excess demand functions satisfy gross substitutability, then so does their sum. By contrast, if two excess demand functions satisfy the weak axiom, then their sum may not, as our example demonstrates.

In fact, a series of results due to Sonnenschein (1973), Mantel (1974), and Debreu (1974) indicate that without fairly strong restrictions, the aggregate excess demand function is arbitrary: specifically, for any excess demand function for n goods that is homogeneous of degree zero and obeys Walras's law, and for any compact set of prices on which this function is continuous, there is an economy of n consumers who, in maximizing utility subject to their budget constraints, generate this aggregate excess demand function. Shafer and Sonnenschein (1982) provide a survey of these, and related, results.

There are two notable cases where we know that the sum of individual excess demand functions satisfies the weak axiom: first, where utility functions are homothetic and identical, but where endowment vectors are arbitrary, and, second, where utility functions are homothetic but possibly different, and endowment vectors are proportional to each other. In each of these cases, in fact, the aggregate excess demand function has all the properties of an individual excess demand function, in that it satisfies the strong axiom of revealed preference, which implies that there exists a utility function and endowment vector such that the excess demand function solves the consumer's maximization problem. (The strong axiom says that for any finite set of price vectors, $p^1, p^2, \ldots, p^k$, the conditions $p^2 \cdot f(p^1) \leq 0$, $p^3 \cdot f(p^2) \leq 0$, $\ldots$, $p^1 \cdot f(p^k) \leq 0$ cannot hold unless $f(p^1) = f(p^2) = \ldots = f(p^k)$; the weak axiom is this condition only for pairs of such price vectors; see Houthakker 1950 and Richter 1966.)

The case of identical homothetic utility functions was considered by Antonelli (1886), Gorman (1953), and Nataf (1953). It is easy to show that the excess demand function in this case can be derived by solving

$$\begin{aligned} &\max u(x) \\ &\text{s.t. } p \cdot x \leq \sum_{i=1}^{m} p \cdot w^i \\ &x \geq 0 \end{aligned}$$

and setting $f(p) = (x(p) - \sum_{i=1}^{m} w^i)$. Here, of course, $u(x)$ is the common utility function. The aggregate excess demand function $f(p)$ obviously satisfies the weak axiom (and the strong axiom) because it is the individual demand function of the consumer with utility function $u(x)$ and endowment vector $\sum_{i=1}^{m} w^i$. We can disaggregate consumption decisions by setting

$$x^i(p) = \left(p \cdot w^i / \sum_{j=1}^{m} p \cdot w^j\right) x(p).$$

The case of different homothetic utility functions but proportional endowments was considered by Eisenberg (1961) and Chipman (1974). In this case, the distribution of income is independent of prices, since $p \cdot w^i / \sum_{j=1}^m p \cdot w^j = \theta_i$, where θ_i is the proportionality factor such that $w^i = \theta_i \sum_{j=1}^m w^j$. It is easy to show that $(x^1(p), \ldots, x^m(p))$ solves

$$\begin{aligned} &\max \sum_{i=1}^{m} \theta_i \log u_i(x^i) \\ &\text{s.t.} \sum_{i=1}^{m} p \cdot x^i \leq \sum_{i=1}^{m} p \cdot w^i \\ &\qquad x^i \geq 0. \end{aligned}$$

Here u_i is the homogeneous-of-degree-one representation of the utility function of consumer i. To see that the excess demand function of such a group of consumers satisfies the weak axiom of revealed preference, suppose, to the contrary, that $p^1 \cdot f(p^2) \leq 0$, $p^2 \cdot f(p^1) \leq 0$, and $f(p^1) \neq f(p^2)$ for some $p^1, p^2 \in R^n_+ \backslash \{0\}$. Then

$$p^1 \cdot f(p^2) = \sum_{i=1}^{m} p^1 \cdot x^i(p^2) - \sum_{i=1}^{m} p^1 \cdot w^i \leq 0;$$

in other words, $(x^1(p^2), \ldots, x^m(p^2))$ is affordable at prices p^1, and

$$f(p^1) = \sum_{i=1}^{m} (x^i(p^1) - w^i) \neq \sum_{i=1}^{m} (x^i(p^2) - w^i) = f(p^2);$$

in other words, $(x^1(p^2), \ldots, x^m(p^2))$ is not the solution to the maximization problem at prices p^1. Consequently,

$$\sum_{i=1}^{m} \theta_i \log u_i(x^i(p^1)) > \sum_{i=1}^{m} \theta_i \log u_i(x^i(p^2)),$$

since the strict concavity of the objective function implies that there is a unique solution to the maximization problem. Reversing the roles of p^1 and p^2, we can reverse this inequality and generate a contradiction, establishing that $f(p)$ does indeed satisfy the weak axiom. In fact, this logic can be extended to show that $f(p)$ satisfies the strong axiom and can be derived from utility maximization by a single consumer, although it may be difficult to derive this consumer's utility function and endowment.

These two sets of restrictions that imply the weak axiom are unattractive, because they both rely on homothetic utility functions. Following the lead of Hildenbrand (1983), economists have searched for more general restrictions on preferences and endowments that imply the weak axiom for aggregate excess demand. These sorts of restrictions are especially important in the context of economies with production; we shall return to this issue later.

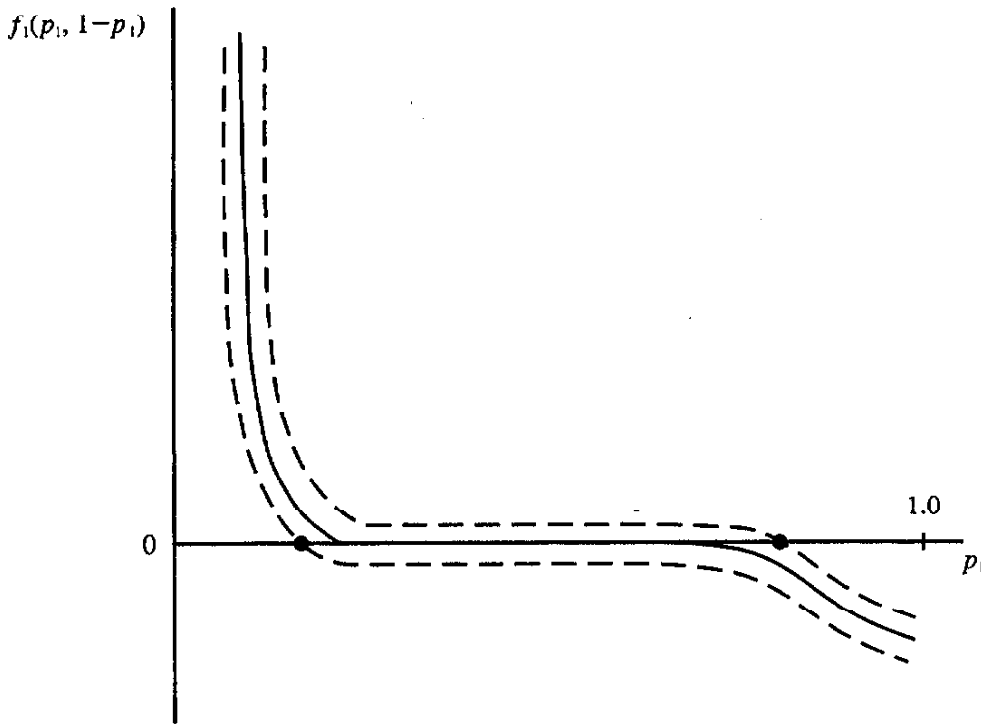

Figure 3.3 Degeneracy of an economy with a continuum of equilibria

3 Regularity and the Index Theorem

The weak axiom of revealed preference implies not that the equilibrium price vector of an exchange economy is unique, but that it is an element of a convex set of equilibria. Intuitively, however, we view situations like that depicted in figure 3.3, where there is a continuum of equilibria, as degenerate: a small perturbation in $f(p)$ – or in the underlying characteristics of the economy, $u_i(x)$ and w^i – should eliminate the continuum of equilibria.

3.1 REGULAR ECONOMIES

Debreu (1970) formalizes this intuitive notion of using small perturbations to rule out degenerate situations like that in figure 3.3 with his concept of a regular economy. A regular equilibrium is a price vector $\hat{p}$ such that $f(\hat{p}) = 0$ and the $(n-1) \times (n-1)$ matrix formed by deleting the last row and column from the Jacobian matrix $Df(\hat{p})$ is nonsingular. A regular economy is one for which every equilibrium is a regular equilibrium. Regular economies are attractive for four reasons. First, a regular economy has a finite number of equilibria. (A regular economy that satisfies the weak axiom therefore has a unique equilibrium.) Second, each equilibrium of a regular economy varies continuously with the underlying characteristics of the economy. Third, in the set of all possible economies given an appropriate topological structure, almost all economies are regular. Fourth, we can use a fixed point index theorem to develop necessary and sufficient conditions for uniqueness of equilibria of regular economies.

To understand the first property of a regular economy, that it has a finite number of equilibria, consider an economy in which strict monotonicity of utility rules out free goods in equilibrium. We can write out the equilibrium conditions as

$$f_1\left(p_1, \ldots, p_{n-1}, \left(1 - \sum_{j=1}^{n-1} p_j\right)\right) = 0$$

$$\vdots$$

$$f_{n-1}\left(p_1, \ldots, p_{n-1}, \left(1 - \sum_{j=1}^{n-1} p_j\right)\right) = 0.$$

As a solution $\hat{p}$, the $(n-1) \times (n-1)$ Jacobian matrix of the functions on the left-hand side of these equations is

$$\begin{bmatrix} \frac{\partial f_1}{\partial p_1}(\hat{p}) - \frac{\partial f_1}{\partial p_n}(\hat{p}) & \cdots & \frac{\partial f_1}{\partial p_{n-1}}(\hat{p}) - \frac{\partial f_1}{\partial p_n}(\hat{p}) \\ \vdots & & \vdots \\ \frac{\partial f_{n-1}}{\partial p_1}(\hat{p}) - \frac{\partial f_{n-1}}{\partial p_n}(\hat{p}) & \cdots & \frac{\partial f_{n-1}}{\partial p_{n-1}}(\hat{p}) - \frac{\partial f_{n-1}}{\partial p_n}(\hat{p}) \end{bmatrix}.$$

The determinant of this matrix is equal to

$$\det \begin{bmatrix} \frac{\partial f_1}{\partial p_1}(\hat{p}) - \frac{\partial f_1}{\partial p_n}(\hat{p}) & \cdots & \frac{\partial f_1}{\partial p_{n-1}}(\hat{p}) - \frac{\partial f_1}{\partial p_n}(\hat{p}) & 0 \\ \vdots & & \vdots & \vdots \\ \frac{\partial f_{n-1}}{\partial p_1}(\hat{p}) - \frac{\partial f_{n-1}}{\partial p_n}(\hat{p}) & \cdots & \frac{\partial f_{n-1}}{\partial p_{n-1}}(\hat{p}) - \frac{\partial f_{n-1}}{\partial p_n}(\hat{p}) & 0 \\ 1 & \cdots & 1 & 1 \end{bmatrix}.$$

We can multiply the final row of the above matrix by $\partial f_i(\hat{p})/\partial p_n$ and add it to each row $i = 1, \ldots, n-1$, without changing its determinant,

$$\det \begin{bmatrix} \frac{\partial f_1}{\partial p_1}(\hat{p}) & \cdots & \frac{\partial f_1}{\partial p_{n-1}}(\hat{p}) & \frac{\partial f_1}{\partial p_n}(\hat{p}) \\ \vdots & & \vdots & \vdots \\ \frac{\partial f_{n-1}}{\partial p_1}(\hat{p}) & \cdots & \frac{\partial f_{n-1}}{\partial p_{n-1}}(\hat{p}) & \frac{\partial f_{n-1}}{\partial p_n}(\hat{p}) \\ 1 & \cdots & 1 & 1 \end{bmatrix}.$$

The homogeneity of $f_i(p)$ implies that $\sum_{j=1}^{n} \hat{p}_j \partial f_i(\hat{p})/\partial p_j = 0$. Consequently, multiplying each column of the above matrix by $\hat{p}_j$ and adding to the final column multiplied by $\hat{p}_n$, we obtain

$$(1/\hat{p}_n)\det\begin{bmatrix} \frac{\partial f_1}{\partial p_1}(\hat{p}) & \cdots & \frac{\partial f_1}{\partial p_{n-1}}(\hat{p}) & 0 \\ \vdots & & \vdots & \vdots \\ \frac{\partial f_{n-1}}{\partial p_1}(\hat{p}) & \cdots & \frac{\partial f_{n-1}}{\partial p_{n-1}}(\hat{p}) & 0 \\ 1 & \cdots & 1 & 1 \end{bmatrix},$$

which has the same sign as the determinant of

$$\bar{J} = \begin{bmatrix} \frac{\partial f_1}{\partial p_1}(\hat{p}) & \cdots & \frac{\partial f_1}{\partial p_{n-1}}(\hat{p}) \\ \vdots & & \vdots \\ \frac{\partial f_{n-1}}{\partial p_1}(\hat{p}) & \cdots & \frac{\partial f_{n-1}}{\partial p_{n-1}}(\hat{p}) \end{bmatrix}.$$

A regular equilibrium is one where this matrix is nonsingular. (A similar argument shows that $\bar{J}$ is nonsingular if and only if the $(n-1) \times (n-1)$ matrix formed by deleting any row and any column from $Df(\hat{p})$ is nonsingular.)

If $\hat{p}$ is a regular equilibrium, then the inverse function theorem implies that there is some open neighborhood of $(p_1, \ldots, \hat{p}_{n-1})$ in R_{++}^{n-1} such that the $n-1$ functions that we have set to 0 to produce the equilibrium can be inverted, and that $(\hat{p}_1, \ldots, \hat{p}_{n-1}) = f^{-1}(0)$. A regular equilibrium $\hat{p}$ is, therefore, locally unique, in that there is a relatively open subset of the unit simplex

$$S = \left\{p \in R^n \mid \sum_{j=1}^{n} p_j = 1, p_j \geq 0\right\}$$

that contains $\hat{p}$ and no other equilibrium.

To see that this implies that a regular economy has a finite number of equilibria, suppose instead that there is an infinite number, $p^1, p^2, \ldots$. Since the set S is compact, this sequence has a convergent subsequence, which converges to, say, p^*. We know that $f(p)$ is continuous, that the subsequence of price vectors p^k converges to p^*, and that $f(p^k) = 0$. Consequently, $f(p^*) = 0$. Since the subsequence of price vectors p^k converges to p^*, however, every open set that contains p^* must contain an infinite number of other equilibria. Therefore, p^* is not a regular equilibrium, and the economy cannot be regular if it has an infinite number of equilibria.

To understand the second property of a regular economy, that its equilibria vary continuously with its underlying characteristics, suppose that we parameterize the excess demand function with a finite number of parameters $b = (b_1, \ldots, b_k)$. We write

$$f_1\left(p_1, \ldots, p_n, \left(1 - \sum_{j=1}^{n-1} p_j\right); b\right) = 0$$
$$\vdots$$
$$f_{n-1}\left(p_1, \ldots, p_n, \left(1 - \sum_{j=1}^{n-1} p_j\right); b\right) = 0.$$

The implicit function theorem says that if $f_i(p, b)$ is continuously differentiable in the price variables and the parameters, and if the $(n-1) \times (n-1)$ Jacobian matrix of partial derivatives with respect to the price variables is nonsingular – that is, if $\hat{p}$ is a regular equilibrium – then in some neighborhood of a solution $f(\hat{p}, b^0) = 0$ there is a function $p(b)$ such that $p(b^0) = \hat{p}$, and

$$f(p(b), b) = 0.$$

Dierker (1982) explains how these results can be extended to situations where the space of parameters is not finite-dimensional and where $f(p, b)$ is not necessarily continuously differentiable in the parameters.

The third property of regular economies is that they represent what is, in some sense, the generic situation. The most restrictive condition that a regular economy needs to satisfy is that its excess demand function be continuously differentiable. Debreu (1972) and Mas-Colell (1974) have argued, however, that arbitrarily small perturbations to the characteristics of an economy whose excess demand function is not continuously differentiable can make it continuously differentiable, at least if we rule out corner solutions to the consumers' maximization problem. Furthermore, Debreu (1970) has shown that if we fix the utility functions of all consumers and the endowments of all but one of the consumers in an economy where excess demand is continuously differentiable, then, for an open set of full Lebesgue measure of endowments of the remaining consumer, the corresponding economy is regular. Dierker (1982) and Mas-Colell (1985: ch. 8) survey generalizations of this result that allow all of the characteristics of the economy to vary. The most general results say that, in a topological space of utility functions and endowments that satisfy conditions for generating continuously differentiable excess demand, regular economies are open and dense. Furthermore, on suitably chosen finite-dimensional subspaces of parameters, regular economies satisfy these properties and the additional one of having full Lebesgue measure. That regular economies are open means that any small enough perturbation to a regular economy yields another regular economy. That regular economies are dense means that if an economy is not regular, then an arbitrarily small perturbation will make it regular. That regular economies have full Lebesgue measure means that if we choose the parameters of an economy from a large enough finite-dimensional set – say, the parameters a_j^i, b_i, and w_j^i in example 1 – at random, then with probability 1 the resulting economy will be regular.

It is possible to argue that, among economies that satisfy conditions like gross substitutability and the weak axiom, regular economies are also generic. The crucial step is to show that in the topology on the space of economies the subset that satisfies one of these conditions is big enough – the closure of an open set. If so, then the intersection of the subset of economies that satisfy the condition with the subset of economies that are regular consists of almost all economies that satisfy the condition.

Mas-Colell (1985: ch. 8) shows how these regularity results can be extended to economies where consumer's utility maximization has corner solutions: although the excess demand function may not be continuously differentiable, price vectors where continuous differentiability fails are almost never equilibria. Similarly, Kehoe (1980, 1982) shows how these regularity results can be extended to economies with free goods. Allowing for free goods is thought of most easily as allowing for production – in this case, disposal of the free good. We therefore postpone discussion of free goods until we have discussed production economies.

3.2 THE INDEX THEOREM

The fourth property of regular economies is that we can use an index theorem to develop necessary and sufficient conditions for uniqueness of their equilibria. To understand this property, it will help to recall how Brouwer's fixed point theorem can be used to demonstrate the existence of equilibrium.

Brouwer's fixed point theorem *Suppose that* $g(p) = (g_1(p), \ldots, g_n(p))$ *is continuous, and maps the unit simplex* $S = \{p \in \mathbb{R}^n \mid \sum_{j=1}^n p_j = 1, p_j \geq 0\}$ *into itself in the sense that* $\sum_{j=1}^n g_j(p) = 1$, $g_j(p) \geq 0$, *whenever* $\sum_{j=1}^n p_j = 1$, $p_j \geq 0$. *Then there exists a fixed point* $\hat{p} = g(\hat{p})$.

We need to find a function $g(p)$ defined on S such that $g(p) \in S$ and that $\hat{p} = g(\hat{p})$ if and only if $f(\hat{p}) \leq 0$. The existence of equilibrium then follows from Brouwer's fixed point theorem.

Figure 3.4 depicts a version of this theorem for the case where $n = 2$. Notice in this figure that we can say something more: if there is no fixed point at $p_1 = 0$ or $p_1 = 1$ and the graph of $g_1(p_1, 1 - p_1)$ never becomes tangent to the diagonal, then the graph of $g_1(p_1, 1 - p_1)$ must always cross the diagonal one more time from above than it does from below. Suppose we define

$$\text{index}(\hat{p}) = \text{sgn}\left[1 - \left(\frac{\partial g_1}{\partial p_1}(\hat{p}_1, 1 - \hat{p}_1)\right) + \left(\frac{\partial g_1}{\partial p_2}(\hat{p}_1, 1 - p_1)\right)\right],$$

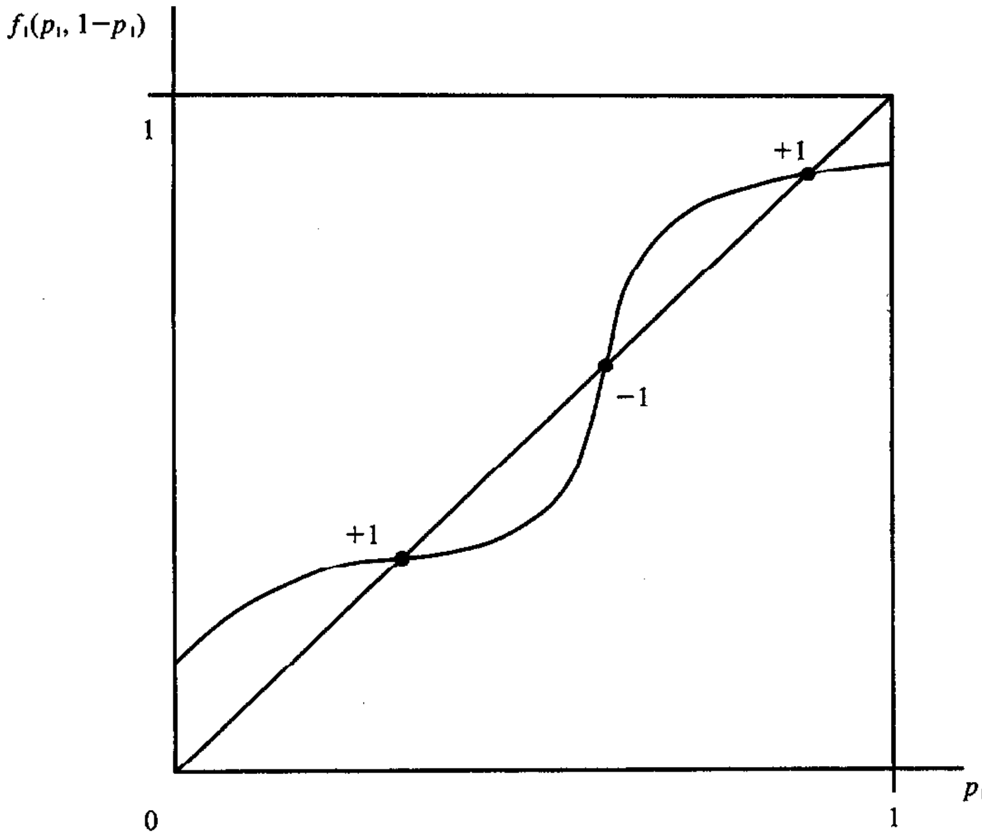

Figure 3.4 The index theorem when n $= 2$

that is, index($\hat{p}$) = +1 at a crossing from above, and index($\hat{p}$) = −1 at a crossing from below. Then

$$\sum_{p=g(p)} \text{index}(p) = +1.$$

This example is a special case of a more general index theorem originally introduced into economics by Dierker (1972) (see Mas-Colell 1985: ch. 1 for references).

Index theorem *Suppose that* g(p) *maps* S *into* S, *that* g(p) *is continuously differentiable at its fixed points, and that all fixed points of* g(p) *are interior to* S, *and such that [*I − Dg($\hat{p}$)*] is nonsingular. If we define*

$$index(\hat{p}) = sgn(det[I - Dg(\hat{p})]),$$

then

$$\sum_{p=g(p)} index(p) = +1.$$

The most obvious way to calculate the index of a fixed point is to locally extend $g(p)$ to a function from R^n to S and to calculate ordinary partial derivatives. Notice that in the case where $n = 2$, the formula in our example agrees with that in the theorem:

$$\det\begin{bmatrix} 1-\dfrac{\partial g_1}{\partial p_1}(\hat{p}) & -\dfrac{\partial g_1}{\partial p_2}(\hat{p}) \\ -\dfrac{\partial g_2}{\partial p_1}(\hat{p}) & 1-\dfrac{\partial g_2}{\partial p_2}(\hat{p}) \end{bmatrix} = \det\begin{bmatrix} 1-\dfrac{\partial g_1}{\partial p_1}(\hat{p}) & -\dfrac{\partial g_1}{\partial p_2}(\hat{p}) \\ 1 & 1 \end{bmatrix}$$

$$= 1 - \frac{\partial g_1}{\partial p_1}(\hat{p}) + \frac{\partial g_2}{\partial p_2}(\hat{p}).$$

The first equality follows from differentiating $g_1(p_1, p_2) + g_2(p_1, p_2) = 1$ with respect to p_1 to show that $\partial g_1(p)/\partial p_1 + \partial g_2(p)/\partial p_1 = 0$, then adding the first row of the matrix to the second.

The index theorem provides a sufficient condition for uniqueness of a fixed point:

$$\det[I - Dg(\hat{p})] > 0$$

at every fixed point. It also provides a necessary condition: if

$$\det[I - Dg(\hat{p})] < 0$$

at a fixed point, then there are necessarily multiple fixed points.

3.3 UNIQUENESS OF EQUILIBRIUM

To make economic sense of the index theorem, and to see that the condition that $[I - Dg(\hat{p})]$ be nonsingular at every fixed point is, in fact, the condition that the economy is regular, we need to differentiate a function $g(p)$ whose fixed points are equilibria. Suppose, for example, that we let $g(p)$ be the point in S that is closest in terms of Euclidean distance to $p + f(p)$; in other words, $g(p)$ is the vector $g \in R^n$ that solves

$$\min\ (1/2)\sum_{j=1}^{n}(g_j - p_j - f_j(p))^2$$

$$\text{s.t. } \sum_{j=1}^{n} g_j = 1$$

$$g_j \geq 0.$$

It is easy to show that $\hat{p} = g(\hat{p})$ if and only if $f(\hat{p}) \leq 0$, and that

$$\operatorname{sgn}(\det[I - Dg(\hat{p})]) = \operatorname{sgn}(\det[-\bar{J}])$$

if $\hat{p}$ is strictly positive (see Kehoe 1980, 1991). (Here, as before, $\bar{J}$ is the $(n-1) \times (n-1)$ matrix formed by deleting the last row and column from $Df(\hat{p})$.) We merely differentiate

$$g_j(p) = p_j + f_j(p) + (1/n)\left(1 - \sum_{\ell=1}^{n}(p_\ell + f_\ell(p))\right)$$

and perform elementary row and column operations on $[I - Dg(\hat{p})]$ that do not change the sign of its determinant.

Restricting our attention to economies with continuously differentiable excess demand functions, we see that

$$\det[-\bar{J}] > 0$$

at every equilibrium is sufficient for uniqueness, and is necessary if the economy is regular. Consequently, any conditions that imply uniqueness, such as gross substitutability and the weak axiom, must imply that $\det[-\bar{J}]$ is nonnegative and, in general, positive.

Gross substitutability, for example, implies that the diagonal elements of $-\bar{J}$ are nonnegative, and the off-diagonal elements are nonpositive. Since the homogeneity of $f(p)$ implies that $\sum_{j=1}^{n} \hat{p}_j \partial f_i(\hat{p})/\partial p_j = 0$, we know that, unless some row of $-\bar{J}$ is all zero and $\hat{p}$ is not a regular equilibrium, the diagonal elements of $-\bar{J}$ are actually positive. Furthermore, since

$$\begin{bmatrix} -\dfrac{\partial f_1}{\partial p_1}(\hat{p}) & \cdots & -\dfrac{\partial f_1}{\partial p_{n-1}}(\hat{p}) \\ \vdots & & \vdots \\ -\dfrac{\partial f_{n-1}}{\partial p_1}(\hat{p}) & \cdots & -\dfrac{\partial f_{n-1}}{\partial p_{n-1}}(\hat{p}) \end{bmatrix} \begin{bmatrix} \hat{p}_1 \\ \vdots \\ \hat{p}_{n-1} \end{bmatrix} = \begin{bmatrix} \hat{p}_n \dfrac{\partial f_1}{\partial p_n}(\hat{p}) \\ \vdots \\ \hat{p}_n \dfrac{\partial f_{n-1}}{\partial f_n}(\hat{p}) \end{bmatrix}$$

and the vector on the right-hand side has all elements nonnegative and not all zero (unless $\hat{p}$ is not a regular equilibrium), we know that $-\bar{J}$ is, in fact, a P matrix, a matrix with all of its principal minors – and hence its determinant – positive (see Metzler 1945, Hahn 1958, and McKenzie 1960).

Similarly, the weak axiom of revealed preference implies that $\det[-\bar{J}]$ is nonnegative and, in general, positive. Kihlstrom, Mas-Colell, and Sonnenschein (1976) prove that a necessary condition for the weak axiom is that

$$v \cdot Df(p)v \leqq 0$$

for all $v \in R^n$ such that $v \cdot f(p) = 0$, and that a sufficient condition is that the inequality be strict for all v not proportional to p. In the case where $f(\hat{p}) = 0$, this requires that $Df(\hat{p})$ be negative semi-definite, which implies that $-\bar{J}$ is positive semi-definite, and that $\det[-\bar{J}] \geq 0$.

3.4 EXAMPLE 1 (CONTINUED)

Our example with multiple equilibria is a regular economy, since, in this case, the $(n-1) \times (n-1)$ matrix $\bar{J}$ is just the number $\partial f_1(\hat{p})/\partial p_1$ which is nonzero at every equilibrium:

$$\frac{\partial f_1}{\partial p_1}(0.1129, 0.8871) = -15.6229.$$

$$\frac{\partial f_1}{\partial p_1}(0.5000, 0.5000) = 1.2800.$$

$$\frac{\partial f_1}{\partial p_1}(0.8871, 0.1129) = -0.3652.$$

Notice that, since $\text{index}(\hat{p}) = \text{sgn}(-\partial f_1(\hat{p})/\partial p_1)$, equilibria 1 and 3 have index $+1$ while equilibrium 2 has index -1. In fact, this example has been constructed by choosing the symmetric parameters a_j^i, b_i, and w_j^i so that at the symmetric equilibrium where $\hat{p}_1 = \hat{p}_2$ the index is -1. The index theorem then tells us that the economy has multiple equilibria; the other two equilibria have been located by a numerical method.

Since the economy in the example is regular, its equilibria vary continuously with the underlying parameters a_j^i, b_i, and w_j^i. For small changes in the parameters, there are only small changes in the equilibrium, and, in particular, there are three equilibria. Nevertheless, if we make large changes in the parameters, we can pass through a critical economy – an economy that is not regular – and arrive at a regular economy with a unique equilibrium. Figure 3.5 depicts the changes in the graph of $f_1(p_1, 1-p_1)$ as we change the parameters. In the figure, changing the parameters first results in another regular economy with three equilibria, then a critical economy with a unique equilibrium, but where the graph of $f_1(p_1, 1-p_1)$ is tangent to the axis, and finally a regular economy with a unique equilibrium.

Our discussion of gross substitutability and the weak axiom of revealed preference tells us in what direction we need to change the parameters to obtain an economy with a unique equilibrium. As we increase $b_1 = b_2$ from -4 towards $b_1 = b_2 = 0$, we move in the direction of a regular economy whose excess demand function exhibits gross substitutability, and that therefore has a unique equilibrium. (Of course, we should verify that $\partial f_1(\hat{p})/\partial p_1 \neq 0$ when $b_1 = b_2 = 0$; not surprisingly, it is.) Maintaining the symmetry of the parameters, this equilibrium is at $\hat{p} = (0.5, 0.5)$. Let us fix a_j^i and w_j^i. Somewhere between $b_1 = b_2 = -4$ and $b_1 = b_2 = 0$, the economy

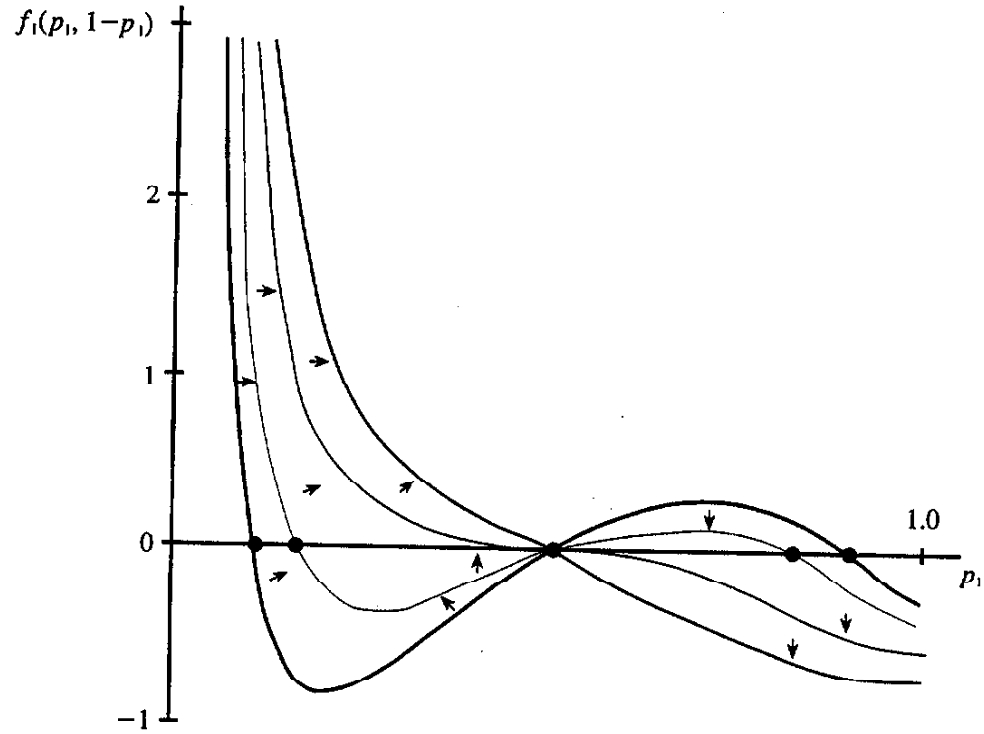

Figure 3.5 Passing through critical economy as the number of equilibria changes

becomes a critical economy where $\partial f_1(0.5, 0.5)/\partial p_1 = 0$, and the number of equilibria changes abruptly. This abrupt change is what is known as a mathematical catastrophe. It occurs at (approximately) $b_1 = b_2 = -3.5078$: for $b_1 = b_2 < -3.5078$ the economy is regular and has three equilibria, and for $b_1 = b_2 > -3.5078$ the economy is regular and has one equilibrium.

We also know two conditions that ensure that the aggregate excess demand function satisfies the weak axiom and, therefore, that the economy has a unique equilibrium. First, if $a_1^1 = a_2^1$ and $a_2^1 = a_2^2$ (and, of course, $b_1 = b_2 = -4$), then the consumers have identical homothetic preferences. Maintaining the symmetry of the parameters and normalizing $a_2^1 = a_1^2 = 1$, we know that when $a_1^1 = a_2^2 = 1$ there is a regular economy with a unique equilibrium. Fixing b_i and w_j^i at their original values, as we decrease $a_1^1 = a_2^2$ from 1024 to 1, we should pass through a critical economy. We do so at (approximately) $a_1^1 = a_2^2 = 41.6597$: for $a_1^1 = a_2^2 > 41.6597$, the economy is regular and has three equilibria, and for $a_1^1 = a_2^2 < 41.6597$, the economy is regular and has a unique equilibrium. Second, if $w_1^1 = w_2^1$ and $w_2^1 = w_2^2$, then the consumers have proportional – in this case identical – endowments and homothetic preferences. Maintaining the symmetry of the parameters and fixing $w_2^1 = w_1^2 = 5$, we know that when $w_1^1 = w_2^2 = 5$, there is a regular economy with a unique equilibrium. Fixing a_j^i and b_i at their original values, as we decrease $w_1^1 = w_2^2$ from 60 to 5, we pass through a critical economy at (approximately) $w_1^1 = w_2^2 = 48.5714$: for $w_1^1 = w_2^2 > 48.5714$, the economy is regular and has three equilibria, and for $w_1^1 = w_2^2 < 48.5714$, the economy is regular and has a unique equilibrium.

4 Production Economies

We now turn our attention to economies with production. We begin with economies whose technologies are generated by an activity analysis matrix, and later explain how the results obtained can be extended to economies with more general technologies.

4.1 ECONOMIES WITH ACTIVITY ANALYSIS PRODUCTION

In an economy with n goods consider an $n \times k$ activity analysis matrix

$$A = \begin{bmatrix} -1 & 0 & \dots & 0 & a_{1n+1} & \dots & a_{1k} \\ 0 & -1 & \dots & 0 & a_{2n+1} & \dots & a_{2k} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & -1 & a_{nn+1} & \dots & a_{nk} \end{bmatrix},$$

of which a column $a_j \in R^n$ is an activity, or feasible production plan, with positive entries denoting outputs and negative numbers denoting inputs. The first n columns are disposal activities which indicate that it is possible to costlessly dispose of all goods. (See Koopmans 1951 for an exposition of activity analysis; Kehoe (1982) explains how to relax the assumption of free disposal in the context of the results presented in this section.) We generate the entire set of feasible production plans by considering all sums of nonnegative multiples of the columns of A: a vector $x \in R^n$ is a feasible production plan if

$$x = Ay \text{ for some } y \geqslant 0.$$

The elements of the nonnegative vector $y = (y_1, \dots, y_k)$ are called activity levels.

The set of all feasible production plans is the closed, convex cone spanned by the columns of A:

$$Y = \{x \in R^n \mid x = Ay \text{ for some } y \geq 0\}.$$

We assume that no inputs are possible without outputs, that

$$Y \cap R^n_+ = \{0\}.$$

Combining this specification of the production side of the economy with the utility functions and endowments that specify the consumer side, we define an equilibrium as a price vector $\hat{p}$, and allocation $(\hat{x}^1, \ldots, \hat{x}^m)$, and a vector of activity levels $\hat{y}$ such that

- given $\hat{p}$ consumer i chooses $\hat{x}^i$ to solve
  $\max u_i(x)$
  s.t. $\hat{p} \cdot x \leq \hat{p} \cdot w^i$
  $x \geq 0$;
- $\hat{p} \cdot A \leq 0, \hat{p} \cdot A\hat{y} = 0$;
- $\sum_{i=1}^{m} \hat{x}^i = A\hat{y} + \sum_{i=1}^{m} w^i$.

The second condition is the familiar profit maximization condition for a constant returns production technology: given $\hat{p}$, no feasible production plan Ay, $y \geq 0$, can make positive profits, and the equilibrium production plan $A\hat{y}$ makes zero profit. The third condition is simply the feasibility condition where explicitly modeling disposal activities allows us to write the condition as an equality.

If we specify the consumption side of the economy using an aggregate excess demand function $f(p)$, then an equilibrium is a price vector $\hat{p}$ and a vector of activity levels $\hat{y}$ such that

- $\hat{p} \cdot A \leq 0, \hat{p} \cdot A\hat{y} = 0$;
- $f(\hat{p}) = A\hat{y}$.

To extend the use of the theory of regular economies and the index theorem of the previous section to economies with production, we define a function $g(p)$ that continuously maps the simplex into itself, whose fixed points are equilibria, and which is almost always continuously differentiable at its fixed points. Since $f(p)$ is homogeneous of degree zero, once again we are permitted a price normalization: $\theta\hat{p}$, for $\theta > 0$, satisfies other equilibrium conditions if $\hat{p}$ does. Again normalizing $\sum_{j=1}^{n} \hat{p}_j = 1$, we know that any equilibrium price vector must be an element of the set

$$S_A = \{p \in R^n \mid p \cdot A \leq 0, p \cdot e = 1\},$$

where $e = (1, \ldots, 1)$, so that $p \cdot e = \sum_{j=1}^{n} p_j$. The inclusion of the disposal activities in A insures that all p in S_A are nonnegative. Using the assumption that no outputs are possible without inputs and the separating hyperplane theorem, we can show that S_A is nonempty. It is easy to see that S_A is also closed and convex. In fact, it is a convex polygon with sides of the form $\sum_{i=1}^{n} a_{ij} p_i = 0$ (see figure 3.6). Define $g(p)$ to be the closest point in S_A

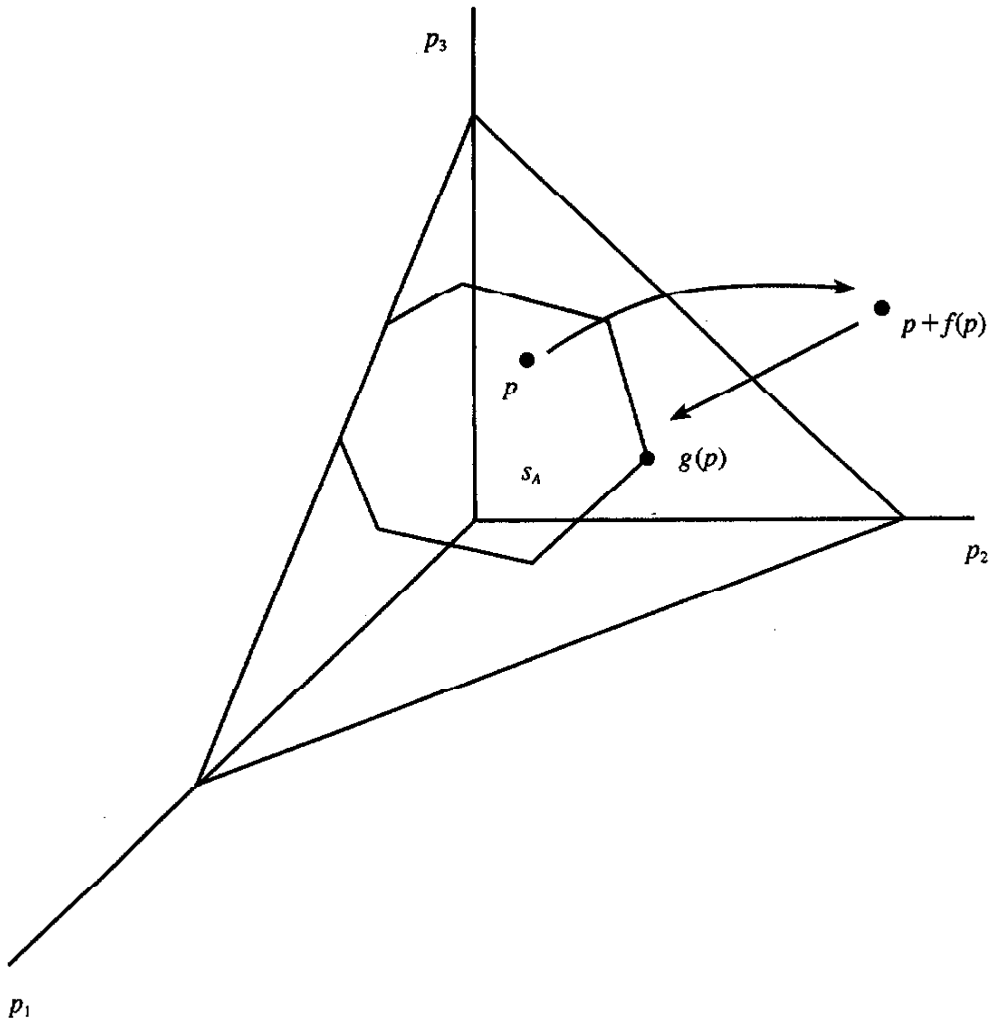

Figure 3.6 The fixed point function for production economies

to $p + f(p)$, in terms of Euclidean distance; that is, we first add $f(p)$ to p, then project the sum onto S_A. Since S_A is closed and convex, the projection is continuous. Consequently, since $f(p)$ is continuous, so is $g(p)$. In fact, $g(p)$ is the vector $g \in R^n$ that solves

$$\begin{aligned} &\min\ (1/2)(g - p - f(p)) \cdot (g - p - f(p)) \\ &\text{s.t. } g \cdot A \leq 0 \\ &\qquad g \cdot e = 1. \end{aligned}$$

To see that $\hat{p} = g(\hat{p})$ if and only if there exists $\hat{y} \geq 0$ such that $(\hat{p}, \hat{y})$ is an equilibrium of $(f(p), A)$, we write out the necessary and sufficient conditions for $g(p)$ to solve the problem:

$$g(p) - p - f(p) + Ay + \lambda e = 0$$
$$g(p) \cdot A \leq 0, g(p) \cdot Ay = 0$$
$$g(p) \cdot e = 1$$

for some vector of Lagrange multipliers $y \in R^k_+$ and some Lagrange multiplier $\lambda \in R$. First, notice that if $(\hat{p}, \hat{y})$ is an equilibrium, then, setting $y = \hat{y}$ and $\lambda = 0$, we see that $g(\hat{p}) = \hat{p}$ solves the problem. Second, notice that, if $g(\hat{p}) = \hat{p}$, then

$$-f(\hat{p}) + Ay + \lambda e = 0.$$

Premultiplying by $\hat{p}$, we obtain

$$-\hat{p} \cdot f(\hat{p}) + \hat{p} \cdot Ay + \lambda \hat{p} \cdot e = \lambda = 0$$

because of Walras's law and the condition $\hat{p} \cdot Ay = g(\hat{p}) \cdot Ay = 0$. Consequently, $(\hat{p}, \hat{y})$ is an equilibrium where $\hat{y} = y$ is the equilibrium vector of activity levels.

To ensure that $g(p)$ is differentiable at its fixed points, we assume that if B is the $n \times \ell$ matrix of activities that earn zero profits at an equilibrium $(\hat{p}, \hat{y})$, then, first, the columns of B are linearly independent, and, second, the vector $\hat{y} = (\hat{y}_1, \ldots, \hat{y}_\ell)$ such that

$$f(\hat{p}) = B\hat{y}$$

is strictly positive. Kehoe (1980) proves that almost all economies satisfy these nondegeneracy conditions.

4.2 CALCULATION OF THE INDEX AND UNIQUENESS OF EQUILIBRIUM

Performing matrix manipulations that do not change the sign of the determinant of $[I - Dg(\hat{p})]$, Kehoe (1980) shows that

$$\operatorname{sgn}(\det[I - Dg(\hat{p})]) = \operatorname{sgn}\left(\det\begin{bmatrix} -Df(\hat{p}) & B & e \\ -B^T & 0 & 0 \\ -e^T & 0 & 0 \end{bmatrix}\right)$$

$$= \operatorname{sgn}\left(\det\begin{bmatrix} -\bar{J} & \bar{B} \\ -\bar{B}^T & 0 \end{bmatrix}\right).$$

Here $\bar{B}$ is the $(n-1) \times \ell$ matrix obtained by deleting the last row of B. A regular economy is one for which this expression is nonzero at every equilibrium. If we define

$$\text{index}(\hat{p}) = \text{sgn}(\det[I - Dg(\hat{p})]),$$

then the index theorem says that for a regular economy

$$\sum_{p=g(p)} \text{index}(p) = +1,$$

and once again we have a sufficient condition for uniqueness of equilibrium, that the determinant whose sign determines the index of a regular equilibrium be positive at every equilibrium, and a necessary condition, that it be nonnegative.

That

$$\begin{bmatrix} -\bar{J} & \bar{B} \\ -\bar{B}^T & 0 \end{bmatrix}$$

be nonsingular at every equilibrium for $(f(p), \hat{A})$ to be a regular economy makes intuitive sense: renormalizing prices so that $p_n = \hat{p}_n$, we see that this matrix is just the Jacobian matrix of the functions whose zero is the equilibrium,

$$-f_1(p_1, \ldots, p_{n-1}, \hat{p}_n) + \sum_{j=1}^{\ell} b_{1j}y_j = 0$$

$$\vdots$$

$$-f_{n-1}(p_1, \ldots, p_{n-1}, \hat{p}_n) + \sum_{j=1}^{\ell} b_{n-1,j}y_j = 0$$

$$\vdots$$

$$-\sum_{i=1}^{n-1} p_i b_{i\ell} - \hat{p}_n b_{n\ell} = 0$$

$$\vdots$$

$$-\sum_{i=1}^{n-1} p_i b_{i\ell} - \hat{p}_n b_{n\ell} = 0.$$

Since Walras's law allows us to ignore the feasibility condition for good n, we have a system of $(n-1) + \ell$ equations in the $n - 1$ prices and the ℓ activity levels. Kehoe (1980, 1982) shows that, for almost all economies $(f(p), A)$, these equations are independent at all equilibria: that is, that the economy is regular.

Incidentally, by extending our regularity theory and index theorem to economies with production, we have also extended it to economies with the possibility of free goods. (There is a minor technicality in that the domain of excess demand must be extended to a set that includes S in its interior; see Kehoe 1980.) If a good j is free, then B contains the disposal activity for good j. This means that one of the final ℓ columns of

$$\begin{bmatrix} -\bar{J} & \bar{B} \\ -\bar{B}^T & 0 \end{bmatrix}$$

corresponds to the disposal activity for good j; that is, it has its element j equal to -1 and every other element equal to 0. Furthermore, the corresponding row of the matrix has its element j equal to 1 and every other element equal to 0. Consequently, we can eliminate column j and row j from $\bar{J}$, eliminate row j from $\bar{B}$, and eliminate the column of $\bar{B}$ corresponding to the disposal activity without changing the determinant. In other words, at any equilibrium where a good is free, we can simply ignore that good in the definitions of a regular equilibrium and of the index of the equilibrium.

The weak axiom continues to imply uniqueness of regular equilibria in production economies. Suppose that $f(p^1) = Ay^1$, $p^1 \cdot A \leq 0$ and $f(p^2) = Ay^2$, $p^2 \cdot A \leq 0$. Then, as in the exchange economy case, $p^2 \cdot f(p^1) \leq 0$ and $p^1 \cdot f(p^2) \leq 0$, which implies that $f(p^1) = f(p^2)$ and that $\theta p^1 + (1 - \theta)p^2$ for $0 \leq \theta \leq 1$ is an equilibrium. Consequently, either p^1 and p^2 are proportional, or the equilibria are critical.

It is easy to show that the weak axiom implies that the determinant whose sign determines the index of a regular equilibrium is nonnegative and, in general, positive. Let $C = [B\ e]$ be the $n \times (\ell + 1)$ matrix whose columns are the ℓ activities in use at the equilibrium and a column of all ones. Our nondegeneracy assumption on B, $\hat{p} \cdot B = 0$, and $\hat{p} \cdot e = 1$ imply that C has rank $\ell + 1 \leq n$. Suppose that $\ell + 1 < n$, and let V be an $n \times (n - \ell - 1)$ matrix whose columns span the null space of C. Then index($\hat{p}$) is determined by the sign of the determinant of

$$\begin{bmatrix} -Df(\hat{p}) & C \\ -C^T & 0 \end{bmatrix}.$$

Postmultiplying this matrix by a nonsingular $(n + \ell) \times (n + \ell)$ matrix and premultiplying it by the transpose of the same matrix does not change the sign of the determinant. Consequently, the index is determined by the sign of the determinant of

$$\begin{bmatrix} V^T & 0 \\ C^T & 0 \\ 0 & (C^TC)^{-1} \end{bmatrix} \begin{bmatrix} -Df(\hat{p}) & C \\ -C^T & 0 \end{bmatrix} \begin{bmatrix} V & C & 0 \\ 0 & 0 & (C^TC)^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} -V^TDf(\hat{p})V & -V^TDf(\hat{p})C & 0 \\ -C^TDf(\hat{p})V & -C^TDf(\hat{p})C & I \\ 0 & -I & 0 \end{bmatrix},$$

which is equal to

$$\det[-V^TDf(\hat{p})V].$$

The weak axiom implies that for any $v \in R^n$ such that $v \cdot f(\hat{p}) = 0$ and v is not proportional to p, $v \cdot Df(\hat{p}) v < 0$. For any $w \in R^{n-\ell-1}$, $Vw \in R^n$ satisfies $w \cdot V^Tf(\hat{p}) = w \cdot V^TB\hat{y} = 0$ and $w \cdot V^Te = 0 \neq \hat{p} \cdot e$. Consequently,

$-V^T Df(\hat{p})V$ is positive semi-definite and hence has a nonnegative and, in general positive, determinant.

Notice that if $\ell + 1 = n$ and C is a nonsingular $n \times n$ matrix – that is, if $n - 1$ activities are used in equilibrium, then the index is necessarily $+1$. Consequently, any restrictions on an economy $(f(p), A)$ that imply that any possible equilibrium has $n - 1$ activities in use imply uniqueness of equilibrium. One well-known set of conditions that imply that there must be $n - 1$ activities in use at equilibrium are those of the nonsubstitution theorem of input–output analysis:

- there is one nonproduced good, say, good n: $a_{nj} \leq 0$ for $j = 1, \dots, k$;
- excess demand for the other goods is always positive: $f_i(p) > 0$ for $i = 1, \dots, n - 1$;
- there is no joint production: $a_{ij} > 0$ for at most one i for $j = 1, \dots, k$;
- there exists a nonnegative vector of activity levels y that yields positive outputs of the produced goods: $\sum_{j=1}^{k} a_{ij} y_j > 0$, $i = 1, \dots, n - 1$.

One way to insure that the second condition holds is to allow initial endowments only of the nonproduced good, usually called labor. Since all $n - 1$ produced goods must then be produced, and since we rule out joint production, $n - 1$ activities must be run at positive levels in any equilibrium. The nonsubstitution theorem itself says something stronger than that the equilibrium is unique: it says that the efficient combination of activities B, and hence the equilibrium prices, are determined solely by the technology represented by the matrix A (see, for example, Samuelson 1951).

Scarf has shown that if we can impose conditions only on $f(p)$, and not on the production technology, then the weak axiom is the weakest condition that implies uniqueness of equilibrium (see Kehoe 1985b). Suppose that $p^2 f \cdot (p^1) \leq 0$, $p^1 f \cdot (p^2) \leq 0$, and $f(p^1) \neq f(p^2)$. Letting

$$A = [-I \; f(p^1) \; f(p^2)],$$

we see that p^1 and p^2 are both distinct equilibria. In other words, if $f(p)$ does not satisfy the weak axiom, we can invent a production technology so that the economy has multiple equilibria.

4.3 EXAMPLE 2

Arrow, Block, and Hurwicz (1959) have shown that gross substitutability in $f(p)$ implies that the weak axiom holds at least in comparisons between the equilibrium price vector of an exchange economy and any nonequilibrium price vector. Unfortunately, gross substitutability does not imply that the weak axiom holds in general, as the following example demonstrates.

Consider a static production economy with two consumers and four goods. Consumer i, $i = 1, 2$, has the utility function

$$u_i(x_1, x_2, x_3, x_4) = \sum_{j=1}^{4} a_j^i \log x_j$$

where $a_j^i \geq 0$. Suppose that the two consumers have the symmetric parameters $a_1^1 = a_2^2 = 0.75$, $a_2^1 = a_1^2 = 0.25$, $a_3^1 = a_4^1 = a_3^2 = a_4^2 = 0$, $w_1^1 = w_2^1 = w_1^2 = w_2^2 = 0$, $w_3^1 = w_4^2 = 5$, $w_4^1 = w_3^2 = 1$. In other words, the consumers derive utility from consuming goods 1 and 2 and have endowments of goods 3 and 4. It is a straightforward exercise to verify that the aggregate excess demand function $f(p)$ exhibits gross substitutability. Calculating the Jacobian matrix of the aggregate excess demand function at $\hat{p} = (0.25, 0.25, 0.25, 0.25)$, we obtain

$$Df(\hat{p}) = \begin{bmatrix} -24 & 0 & 16 & 8 \\ 0 & -24 & 8 & 16 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Notice that even though $f(p)$ exhibits gross substitutability, it violates the weak axiom: Walras's law implies that $-\hat{p} \cdot Df(\hat{p}) = f(\hat{p}) = (6, 6, -6, -6)$. Setting $v = (1, 0, 3, -2)$, we see that $v \cdot f(\hat{p}) = 0$, and v is not proportional to $\hat{p}$, but that $v \cdot Df(\hat{p})v = 8 > 0$. Therefore, we should be able to construct a production technology for which the economy $(f(p), A)$ has multiple equilibria.

Suppose that we set

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 & 4 & -1 \\ 0 & -1 & 0 & 0 & -1 & 4 \\ 0 & 0 & -1 & 0 & -2 & -1 \\ 0 & 0 & 0 & -1 & -1 & -2 \end{bmatrix}.$$

Then the economy has the three equilibria listed below.

Equilibrium 1

	$\hat{x}_1^i$	$\hat{x}_2^i$	$\hat{x}_3^i$	$\hat{x}_4^i$	u_i
$\hat{x}_j^1$	6.25	3.125	0.0	0.0	1.6593
$\hat{x}_j^2$	0.4167	1.875	0.0	0.0	0.2526
$\hat{p}_j$	0.3	0.2	0.5	0.0	

$\hat{y} = (0.0, 0.0, 0.0, 0.3333, 2.1111, 1.7778)$

Equilibrium 2

	$\hat{x}_1^i$	$\hat{x}_2^i$	$\hat{x}_3^i$	$\hat{x}_4^i$	u_i
$\hat{x}_j^1$	4.5	1.5	0.0	0.0	1.2294
$\hat{x}_j^2$	1.5	4.5	0.0	0.0	1.2294
$\hat{p}_j$	0.25	0.25	0.25	0.25	

$\hat{y} = (0.0, 0.0, 0.0, 0.0, 2.0, 2.0)$

Equilibrium 3

	$\hat{x}_1^i$	$\hat{x}_2^i$	$\hat{x}_3^i$	$\hat{x}_4^i$	u_i
$\hat{x}_j^1$	1.875	0.4167	0.0	0.0	0.2526
$\hat{x}_j^2$	3.125	6.25	0.0	0.0	1.6593
$\hat{p}_j$	0.2	0.3	0.0	0.5	

$$\hat{y} = (0.0, 0.0, 0.3333, 0.0, 1.7778, 2.1111)$$

This example has been constructed so that at equilibrium 2, the index is -1, because

$$\begin{bmatrix} -\bar{J} & \bar{B} \\ -\bar{B}^T & 0 \end{bmatrix} = \begin{bmatrix} 24 & 0 & -16 & 4 & -1 \\ 0 & 24 & -8 & -1 & 4 \\ 0 & 0 & 0 & -2 & -1 \\ -4 & 1 & 2 & .0 & 0 \\ 1 & -4 & 1 & 0 & 0 \end{bmatrix}$$

has determinant equal to -72.

This example can easily be perturbed to make all of the utility parameters a_j^i and all of the endowments w_j^i strictly positive, so that $\partial f_i(\hat{p})/\partial p_j > 0$ for all $j \neq i$. In economies with production, however, it is natural that there be some goods for which there are no final demands, some goods for which there are no endowments, and even some goods for which there are neither. Kehoe (1982) extends the theory of regular economies to such economies.

4.4 MONOTONICITY AND THE WEAK AXIOM

Examples like that presented above suggest that to guarantee the uniqueness we need either to develop conditions that guarantee that the weak axiom holds for aggregate excess demand or to develop joint conditions on aggregate excess demand and on the production technology to guarantee that the index is always positive in equilibrium. We discuss each of these approaches in turn.

Although the weak axiom itself does not aggregate, because the sum of two excess demand functions that satisfy the weak axiom may not satisfy it, there are stronger conditions that are easily shown to imply the weak axiom and do aggregate. Two such conditions are the monotonicity conditions (1) there exists $z \in R_+^n$, $z \neq 0$, such that, if $p \cdot z = q \cdot z$ and $f(p) \neq f(q)$, then $(p - q) \cdot (f(p) - f(q)) < 0$, and (2) there exists $z \in R_+^n$, $z \neq 0$, such that, if $v \cdot z = 0$ and $v \neq 0$, then $v \cdot Df(p)v < 0$. In fact, the second condition is just the differentiable version of the first: it is easy to show that $v \cdot Df(p)v < 0$ implies $(p - q) \cdot (f(p) - f(q)) < 0$, and that $(p - q) \cdot (f(p) - f(q)) < 0$ implies $v \cdot Df(p)v \leq 0$ (see Mas-Colell 1985: ch. 5). It is essential to notice the importance of the vector z that normalizes prices in each case: the sum of two excess demand functions that are monotone with respect to the same normalizing vector is also monotone with respect to that normalizing vector, but the sum of two excess demand functions that are monotone with respect to different normalizing vectors need not have any monotonicity properties.

Hildenbrand (1983) considers an economy with a continuum of consumers distributed over the interval [0, 1]. Suppose that each of these consumers has the same utility function and an endowment that is proportional to the aggregate endowment. If the utility function is homothetic, then, as we have seen in section 2.5, the aggregate excess demand function satisfies the strong axiom. For arbitrary utility functions, however, Mantel (1976) demonstrates that the aggregate excess demand function is essentially arbitrary. Hildenbrand considers the case where income is distributed with nonincreasing density: let vw be the endowment of consumer of income level v, $v \in [0, 1]$, where $w \in R^n_+$; let μ be the density of consumers over income levels, and suppose that μ is nonincreasing. Let $z(p, y)$ be the solution to the problem

$$\begin{aligned} &\max u(x) \\ &\text{s.t. } p \cdot x = y \\ &\qquad x \geq 0. \end{aligned}$$

Then the aggregate – in this case, the mean – excess demand function is

$$f(p) = \int_0^1 \mu(v)(z(p, vp \cdot w) - vw)dv.$$

Hildenbrand shows that this function satisfies $(p - q) \cdot (f(p) - f(q)) < 0$ for $p \cdot w = q \cdot w$ and $f(p) \neq f(q)$.

Suppose now that we have an economy made up a finite number of types of consumers where there is a continuum distributed over [0, 1] of each type, each type has a different utility function, but all consumers have endowments that are proportional, and incomes are distributed with nonincreasing density within each type. Since the aggregate excess demand function is now the sum of the mean excess demand functions over types, and these mean excess demand functions are all monotone with respect to the same normalizing vector w, it satisfies the weak axiom.

Mitiushin and Polterovich (see Mas-Colell 1991) examine conditions under which an individual excess demand function $(x^i(p) - w^i)$ is monotone with respect to the endowment vector w^i. They find that a sufficient condition for a twice continuously differentiable, concave, and monotonically increasing utility function $u_i(x)$ to generate a monotone excess demand function $(x^i(p) - w^i)$ is that

$$-\frac{x \cdot D^2u_i(x)x}{Du_i(x)x} < 4 \text{ for all } x \in R^n_{++}.$$

In an economy where all consumers have proportional endowments and all utility functions satisfy this criterion, the aggregate excess demand function is monotone with respect to the aggregate endowment vector, and hence satisfies the weak axiom. The Mitiushin–Polterovich condition generalizes considerably the Eisenberg–Chipman requirement that utility functions be homothetic: when we use the homogeneous-of-degree-one representation of utility – $u_i(\theta x) \equiv \theta u_i(x)$ for all $\theta > 0, x \in R^n_{++}$ – we obtain

$$-\frac{x \cdot D^2 u_i(x) x}{D u_i(x) x} = 0.$$

Other significant results concerning aggregate excess demand functions that satisfy the weak axiom have been obtained by Freixas and Mas-Colell (1987), who consider conditions on individual Engel curves, and by Grandmont (1992), who considers conditions on distributions of utility functions.

There is another curious condition that implies monotonicity of excess demand and may help us appreciate the example of nonuniqueness in the previous section. In that example there are four goods, and the excess demand function satisfies gross substitutability, but violates the weak axiom. Kehoe and Mas-Colell (1984) argue that no such example can be constructed if there are fewer than four goods. If $n = 2$, gross substitutability can be trivially shown to imply the strong axiom. If $n = 3$, although gross substitutability does not imply the strong axiom, it does imply a monotonicity condition that implies the weak axiom. Specifically, gross substitutability implies that for any price vectors $p, q \in R^3_{++}$, that are not proportional, there exists $\theta > 0$ such that $(p - \theta q) \cdot (f(p) - f(q) < 0$. Walras's law implies that if this condition holds, then $p \cdot f(q) \leq 0$ implies $g \cdot f(p) \geq 0$.

To see that gross substitutability implies the monotonicity condition, consider two price vectors $p, q \in R^3_{++}$, that are not proportional, and suppose, without loss of generality, that $p_1/q_1 \geq p_2/q_2 \geq p_3/q_3$ with at least one inequality strict. Applying the definition of gross substitutability twice, we see that $f_1(p) \leq f_1(q)$, $f_3(p) \geq f_3(q)$, and $f(p) \neq f(q)$. There are four cases to be considered:

- when either $f_1(p) < f_1(q)$ and $p_1/q_1 > p_2/q_2$ or $f_3(p) < f_3(p)$ and $p_2/q_2 > p_3/q_3$, we set $\theta = p_2/q_2$ and calculate $(p - \theta q) \cdot (f(p)) - f(q))$ as

$$(p_1 - \theta q_1)(f_1(p) - f_1(q)) + (p_3 - \theta q_3)(f_3(p) - f_3(q)) < 0;$$

- when $f_3(p) = f_3(q)$ and $p_1/q_1 = p_2/q_2$, we know that $f_1(p) \leq f_1(q)$, and $f_2(p) \leq f_2(q)$, with one inequality strict, and we set $\theta = p_3/q_3$;
- similarly, when $f_1(p) = f_1(q)$ and $p_2/q_2 = p_3/q_3$, we set $\theta = p_1/q_1$;
- finally, when $f_1(p) = f_1(q)$ and $f_3(p) = f_3(q)$, but $p_1/q_1 > p_2/q_2 > p_3/q_3$, we set $\theta = p_3/q_3$ if $f_2(p) < f_2(q)$, and we set $\theta = p_1/q_1$ if $f_2(p) > f_2(q)$.

4.5 EQUILIBRIUM IN FACTOR MARKETS

If we cannot guarantee that the aggregate excess demand function $f(p)$ satisfies the weak axiom, we must look for combinations of conditions on $f(p)$ and on the production technology A if we want to ensure uniqueness of equilibrium in a production economy: it is easy to show that if there is more than one $p \in S$ such that $p \cdot A \leq 0$, then there exists $f(p)$ such that the economy $(f(p), A)$ has multiple equilibria (see Kehoe 1983, 1985b); in other words, no condition on the production technology alone – except for complete reversibility, that there is only one $p \in S$ for which $p \cdot A \leq 0$ – can guarantee uniqueness. As we have seen, one example of a combination of conditions on consumption and production that guarantees uniqueness is the conditions of the nonsubstitution theorem of input–output analysis.

There are two ways in which the conditions of the nonsubstitution theorem can be generalized. Both involve reduction of the equilibrium conditions to equilibrium in factor markets. In each case we assume that the economy has the following generalized input–output structure:

- there are $h < n$ factors of production: $a_{ij} \leq 0$ for $i = 1, \ldots, h$ and $j = 1, \ldots, k$;
- excess demand for the other $n - h$ goods is always positive;
- there is no joint production;
- production of positive amounts of the $n - h$ produced goods is possible.

The first approach to reducing the equilibrium conditions to equilibrium in factor markets utilizes the equilibrium zero profit condition (see Kehoe 1984). Let $p_1 = (p_{11}, \ldots, p_{1,n-h})$ now be the vector of prices of produced goods, and let $p_2 = (p_{21}, \ldots, p_{2h})$ be the vector of factor prices. Partition A into

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix},$$

where A_1 is $(n - h) \times k$ and A_2 is $h \times k$. Similarly, partition $f(p_1, p_2)$ into $(f_1(p_1, p_2), f_2(p_1, p_2))$.

First, consider the case where A consists of $2n - h$ activities: n disposal activities and an $n \times (n - h)$ matrix with one activity to produce each of the produced goods. The $(n - h) \times (n - h)$ matrix B_1 is a productive Leontief matrix; under a mild decomposability assumption, B_1^{-1} is strictly positive (see, for example, Debreu and Herstein 1953). The zero profit condition $\hat{p}_1 \cdot B_1 + \hat{p}_2 \cdot B_2 = 0$ implies that $\hat{p}_1 = -(B_2 B_1^{-1})^T p_2$. Market clearing for produced goods, $B_1 \hat{y} = f_1(\hat{p}_1, \hat{p}_2)$, implies that $\hat{y} = B_1^{-1} f_1(\hat{p}_1, \hat{p}_2)$. The remaining equilibrium condition is market clearing for factors $B_2 \hat{y} = f_2(\hat{p}_1, \hat{p}_2)$. We define an imputed excess demand function for factors $\phi(p_2)$ by the rule

$$\phi(p_2) = f_2(-(B_2B_1^{-1})^T p_2, p_2) - B_2B_1^{-1} f_1(-(B_2B_1^{-1})^T p_2, p_2).$$

It is easy to verify that $\phi(p_2)$ is continuous, is homogeneous of degree zero, and obeys Walras's law. Furthermore, $\phi(\hat{p}_2) \leq 0$ is equivalent to

$$f_2(\hat{p}_1, \hat{p}_2) \leq B_2B_1^{-1} f_1(\hat{p}_1, \hat{p}_2) = B_2\hat{y}.$$

Consequently, $\hat{p}_2$ is an equilibrium of the h-good exchange economy $\phi(p_2)$ if and only if $(\hat{p}_1, \hat{p}_2, \hat{y})$ is an equilibrium of the n-good production economy $(f(p), A)$.

When there is more than one possible activity for producing each produced good, the situation is slightly more complicated. To calculate $\phi(p_2)$, we start by solving the linear programming problem

$$\begin{aligned} &\min -p_2 \cdot A_2 y \\ &\text{s.t. } A_1 y = e \\ &\qquad\; y = 0. \end{aligned}$$

The nonsubstitution theorem says that for any vector of factor prices p_2 there is an efficient set of $n - h$ activities that does not vary as the right-hand side of the constraint varies. Letting these $n - h$ activities be the matrix B, we proceed as before. Kehoe (1984) shows that when the linear programming problem becomes degenerate and there are more than $n - h$ efficient activities, the excess demand for factors becomes a convex-valued, upper hemi-continuous correspondence. Equilibrium in factor markets still assures equilibrium in the original economy.

Any condition, such as gross substitutability in $\phi(p_2)$, that guarantees uniqueness in the factor market equilibrium guarantees uniqueness of equilibrium in the original economy. The major advantage of the reduction of the equilibrium conditions to factor market clearing is that it is a technique frequently used in applied models, and the great reduction in dimensionality that it allows can sometimes permit an exhaustive search over factor price-space to guarantee uniqueness of equilibrium in an applied model that satisfies no known analytical conditions sufficient for uniqueness. Kehoe and Whalley (1985) exploit a reduction in dimension of this sort to ensure that two large-scale applied general equilibrium models have unique equilibria.

The second technique for reducing the equilibrium conditions to factor market clearing deals with the specification of the consumption side of the economy in terms of utility functions and endowment (see, for example, Taylor 1938 and Rader 1972b: ch. 9). There are m consumers with utility functions $u_i(x_1, x_2)$ and endowments $w^i = (0, w_2^i)$, where once again we have partitioned x and w^i into vectors of length $(n - h)$ and h. We define an imputed utility function for factors $v_i(z)$ as the solution to

$$\begin{aligned}&\max u_i(x_1, x_2)\\ &\text{s.t. } x_1 = Ay\\ &\quad x_2 = Ay + z\\ &x_1, x_2 \geqslant 0.\end{aligned}$$

It is easy to show that a price vector $\hat{p}_2$ and an allocation $\hat{z}^1, \ldots, \hat{z}^m$ is an equilibrium of the h-good exchange economy with utility functions $v_i(z)$ and endowments w_2^i if and only if there is a corresponding equilibrium of the original n-good production economy. In section 4.8 we present conditions developed by Mas-Colell (1991) that ensure that this reduced exchange economy exhibits gross substitutability. To do this, however, we must first allow for more general production technologies.

4.6 EXAMPLE 2 (CONTINUED)

Our example with activity analysis production in section 4.3 has the generalized input–output analysis structure discussed in the previous section where $n - h = h = 2$. Given a vector of factor prices (p_{21}, p_{22}), we can use the zero-profit condition to calculate the prices of the produced goods:

$$\begin{bmatrix} p_{11} \\ p_{12} \end{bmatrix} = -\begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix}^{-1} \begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} p_{21} \\ p_{22} \end{bmatrix} = \begin{bmatrix} 0.6p_{21} + 0.4p_{22} \\ 0.4p_{21} + 0.6p_{22} \end{bmatrix}.$$

Excess demands for the two produced goods are

$$\begin{bmatrix} f_{11}(p) \\ f_{12}(p) \end{bmatrix} = \begin{bmatrix} \dfrac{4p_{21} + 2p_{22}}{p_{11}} \\[2ex] \dfrac{2p_{21} + 4p_{22}}{p_{12}} \end{bmatrix} = \begin{bmatrix} \dfrac{20p_{21} + 10p_{22}}{3p_{21} + 2p_{22}} \\[2ex] \dfrac{10p_{21} + 20p_{22}}{2p_{21} + 3p_{22}} \end{bmatrix}.$$

The equilibrium activity levels are

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix}^{-1} \begin{bmatrix} \dfrac{20p_{21} + 10p_{22}}{3p_{21} + 2p_{22}} \\[2ex] \dfrac{10p_{21} + 20p_{22}}{2p_{21} + 3p_{22}} \end{bmatrix} = \begin{bmatrix} \dfrac{16p_{21} + 8p_{22}}{9p_{21} + 6p_{22}} + \dfrac{2p_{21} + 4p_{22}}{6p_{21} + 9p_{22}} \\[2ex] \dfrac{4p_{21} + 2p_{22}}{9p_{21} + 6p_{22}} + \dfrac{8p_{21} + 16p_{22}}{6p_{21} + 9p_{22}} \end{bmatrix}.$$

Consequently, the imputed excess demand for the two factors is

$$\begin{bmatrix} \phi_1(p_{21}, p_{22}) \\ \phi_2(p_{21}, p_{22}) \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \dfrac{16p_{21} + 8p_{22}}{9p_{21} + 6p_{22}} + \dfrac{2p_{21} + 4p_{22}}{6p_{21} + 9p_{22}} \\[2ex] \dfrac{4p_{21} + 2p_{22}}{9p_{21} + 6p_{22}} + \dfrac{8p_{21} + 16p_{22}}{6p_{21} + 9p_{22}} \end{bmatrix} - \begin{bmatrix} 6 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{12p_{21} + 6p_{22}}{3p_{21} + 2p_{22}} + \dfrac{4p_{21} + 8p_{22}}{2p_{21} + 3p_{22}} - 6 \\ \dfrac{8p_{21} + 4p_{22}}{3p_{21} + 2p_{22}} + \dfrac{6p_{21} + 12p_{22}}{2p_{21} + 3p_{22}} - 6 \end{bmatrix}.$$

As before, there are three equilibria: $(\hat{p}_{21}, \hat{p}_{22}) = (1, 0)$, $(\hat{p}_{21}, \hat{p}_{22}) = (0.5, 0.5)$, and $(\hat{p}_{21}, \hat{p}_{22}) = (0, 1)$.

4.7 GENERAL PRODUCTION TECHNOLOGIES

Consider an economy in which any vector that satisfies the constraints

$$\begin{aligned} &f(x) = 0 \\ &x_i \geq 0, i = 1, \ldots, h \\ &x_i \leq 0, i = h + 1, \ldots, n, \end{aligned}$$

is a feasible net-output combination. Here f is a constant-returns production function that produces the first h commodities as outputs and employs the final $n - h$ commodities as inputs. We assume that f is homogeneous at degree one and concave. For example,

$$f(x_1, x_2, x_3) = \theta(-x_2)^{\alpha}(-x_3)^{1-\alpha} - x_1$$

is the familiar Cobb–Douglas production function where $\theta > 0, 1 \geq \alpha \geq 0$.

For many purposes, we find it convenient to specify production technologies in terms of profit functions rather than production functions. For any price vector p, let $x(p)$ be the vector that maximizes $p \cdot x$ subject to feasibility and some sort of constraint on the level of production, such as $x_1 = 1$ or $x \cdot x = 1$. Given our assumption of constant returns to scale, such a constraint is necessary to keep profits from becoming unbounded. Define the profit function $a(p) = p \cdot x(p)$. It is well known that $a(p)$ is homogeneous of degree one, convex, and continuous even if $x(p)$ is not unique. If $a(p)$ is continuously differentiable, Hotelling's lemma says that $Da(p) = x(p)^T$ (see, for example, Diewert 1982).

Specifying the production technology in terms of k profit functions is a generalization of the activity analysis specification where $a_j(p) = \sum_{i=1}^{n} a_{ij} p_i$. Let $A(p)$ be the $n \times k$ matrix whose columns are the gradients of the profit functions. Hotelling's lemma and constant returns allow us to interpret $A(p)$ as a matrix of activities. Notice that $a(p) = A(p)^T p$. We define an equilibrium as price vector $\hat{p}$ and a vector of activity levels $\hat{y}$ such that

- $a(\hat{p}) \leq 0, a(\hat{p}) \cdot \hat{y} = 0$
- $f(\hat{p}) = A(\hat{p})\hat{y}$.

If we define $B(\hat{p})$ as the $n \times \ell$ matrix whose columns are the gradient vectors of the ℓ profit functions that earn zero profits at $\hat{p}$, we can construct a function $g(p)$ whose fixed points are equilibria, and define a regular economy and the index of an equilibrium as before. Let $H(\hat{p})$ be the $n \times n$ weighted sum of the Hessian matrices of the ℓ profit functions that earn zero profits at $\hat{p}$; the weights are the appropriate activity levels,

$$H(\hat{p}) = \sum_{j=1}^{\ell} D^2 b_j(\hat{p})\hat{y}_j.$$

Kehoe (1983) calculates the index of an equilibrium as

$$\begin{aligned}\text{index}(\hat{p}) &= \text{sgn}\left(\det\begin{bmatrix} -Df(\hat{p}) + H(\hat{p}) & B(p) & e \\ -B(p)^T & 0 & 0 \\ -e^T & 0 & 0 \end{bmatrix}\right) \\ &= \text{sgn}\left(\det\begin{bmatrix} -\bar{J} + \bar{H} & \bar{B} \\ -\bar{B}^T & 0 \end{bmatrix}\right).\end{aligned}$$

Here $\bar{J}$ and $\bar{H}$ are the $(n-1) \times (n-1)$ matrices obtained by deleting the last row and column from $Df(\hat{p})$ and $H(\hat{p})$, and $\bar{B}$ is the $(n-1) \times \ell$ matrix obtained by deleting the last row from $B(\hat{p})$.

An advantage to this general approach is that it can easily be extended to economies with production technologies that exhibit decreasing returns. In such an environment we need to specify production functions for individual firms and to distribute the profits of these firms to consumers. The situation can then be treated as a special case of constant-returns production where we define an addition good as a primary input to account for the profits of each firm, and endow consumers with a total of one unit of this input in proportions equal to their shares of profits; see McKenzie (1959) for details of this construction. For an economy with decreasing returns production, Kehoe (1983, 1985b) calculates the index of an equilibrium as

$$\text{sgn}\left(\det\begin{bmatrix} -D_1 f(\hat{p}, \hat{\pi}) - D_2 f(\hat{p}, \hat{\pi})A(\hat{p})^T + H(\hat{p}) & e \\ -e^T & 0 \end{bmatrix}\right).$$

Here $\pi = a(p)$ is the vector of profits of the k firms – alternatively thought of as the prices of the k primary inputs that convert the decreasing returns technologies to constant returns – and $D_2 f(\hat{p}, \hat{\pi})$ is the $n \times k$ matrix of derivatives of consumer excess demands with respect to this vector. In this context the $n \times n$ matrix $H(\hat{p})$ can be thought of as the Jacobian matrix of the excess supply function $B(p)e$.

Suppose that $D_1 f(\hat{p}, \hat{\pi}) + D_2 f(\hat{p}, \hat{\pi})A(\hat{p})^T$ has all of its off-diagonal elements positive. Kehoe (1985b) argues that in this case, $D_1 f(\hat{p}, \hat{\pi}) + D_2 f(\hat{p}, \hat{\pi})A(\hat{p})^T$ is negative semi-definite. Since each of the profit functions is convex, we already know that $H(\hat{p})$ is positive semi-definite. This implies that the formula for the index is nonnegative and, in general, positive.

Rader (1972a) originally noticed this result that gross substitutability in $f(p, a(p))$ implies uniqueness of equilibrium. Unfortunately, $D_2f(p, a(p))A(p)^T$ does not depend on consumer's utility and endowments alone; it involves a complex interaction of income effects in consumption and production decisions. It may be possible, however, to develop conditions that ensure that it has the required sign pattern.

The results of section 4.5 concerning the reduction of equilibrium to factor market clearing can be easily extended to the more general production technologies discussed in this section. Mas-Colell (1991) considers economies that have the generalized input–output structure of the previous section and satisfy the additional restriction that consumers have no utility for factors of production. (By defining additional goods and production activities, we can always insure that this condition holds.) He uses the second reduction, the one that defines induced utility functions for factor supplies, to show that in an economy in which all utility functions and production functions are Cobb–Douglas, the reduced exchange economy also has Cobb–Douglas utility functions, which implies that it exhibits gross substitutability and hence has a unique equilibrium. We provide an example that illustrates this result in the next section. Mas-Colell (1991) also generalizes this result to economies in which utility functions and production functions are super-Cobb–Douglas. The super-Cobb–Douglas condition is that the functions locally exhibit at least as much substitutability as a Cobb–Douglas function. In the case of a utility function $u(x)$, for example, the requirement is that for every $\bar{x} \in R^n_+$ there is a Cobb–Douglas function $u_{\bar{x}}(x)$ and a neighborhood $U_{\bar{x}} \subset R^n_+$, $\bar{x} \in U_{\bar{x}}$, such that $u_{\bar{x}}(\bar{x}) = u(\bar{x})$ and $u_{\bar{x}}(x) \leq u(x)$ for all $x \in U_{\bar{x}}$ (see figure 3.7).

4.8 EXAMPLE 3

Consider an economy in which there are two consumers with the same utility functions and endowments as in example 2, but where production possibilities are described by six profit functions $a_j(p) = -p_j$, $j = 1, 2, 3, 4$, and

$$a_5(p) = 4p_1 - 4p_2^{0.25}p_3^{0.5}p_4^{0.25}$$

$$a_6(p) = 4p_2 - 4p_1^{0.25}p_2^{0.25}p_4^{0.5}.$$

These are the profit functions corresponding to the Cobb–Douglas production functions

$$x_1 = 8^{0.5}(-x_2)^{0.25}(-x_3)^{0.5}(-x_4)^{0.25}$$

$$x_2 = 8^{0.5}(-x_1)^{0.25}(-x_3)^{0.25}(-x_4)^{0.5}.$$

The parameters of these functions have been chosen so that at $\hat{p} = (0.25, 0.25, 0.25, 0.25)$, $a_j(\hat{p}) < 0$, $j = 1, 2, 3, 4$, $a_5(\hat{p}) = a_6(\hat{p}) = 0$, and

$$B(\hat{p}) = \begin{bmatrix} 4 & -1 \\ -1 & 4 \\ -2 & -1 \\ -1 & -2 \end{bmatrix}.$$

Consequently, as in example 2, $\hat{p}$ and $\hat{y} = (0, 0, 0, 0, 2, 2)$ is an equilibrium.
We can find index($\hat{p}$) by calculating

$$D^2a_5(\hat{p}) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 3 & -2 & -1 \\ 0 & -2 & 4 & -2 \\ 0 & -1 & -2 & 3 \end{bmatrix}, D^2a_6(\hat{p}) = \begin{bmatrix} 3 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 3 & -2 \\ -2 & 0 & -2 & 4 \end{bmatrix}.$$

multiplying by activity levels $\hat{y}_5 = 2$ and $\hat{y}_6 = 2$, summing to obtain

$$H(\hat{p}) = \begin{bmatrix} 6 & 0 & -2 & -4 \\ 0 & 6 & -4 & -2 \\ -2 & -4 & 14 & -8 \\ -4 & -2 & -8 & 14 \end{bmatrix},$$

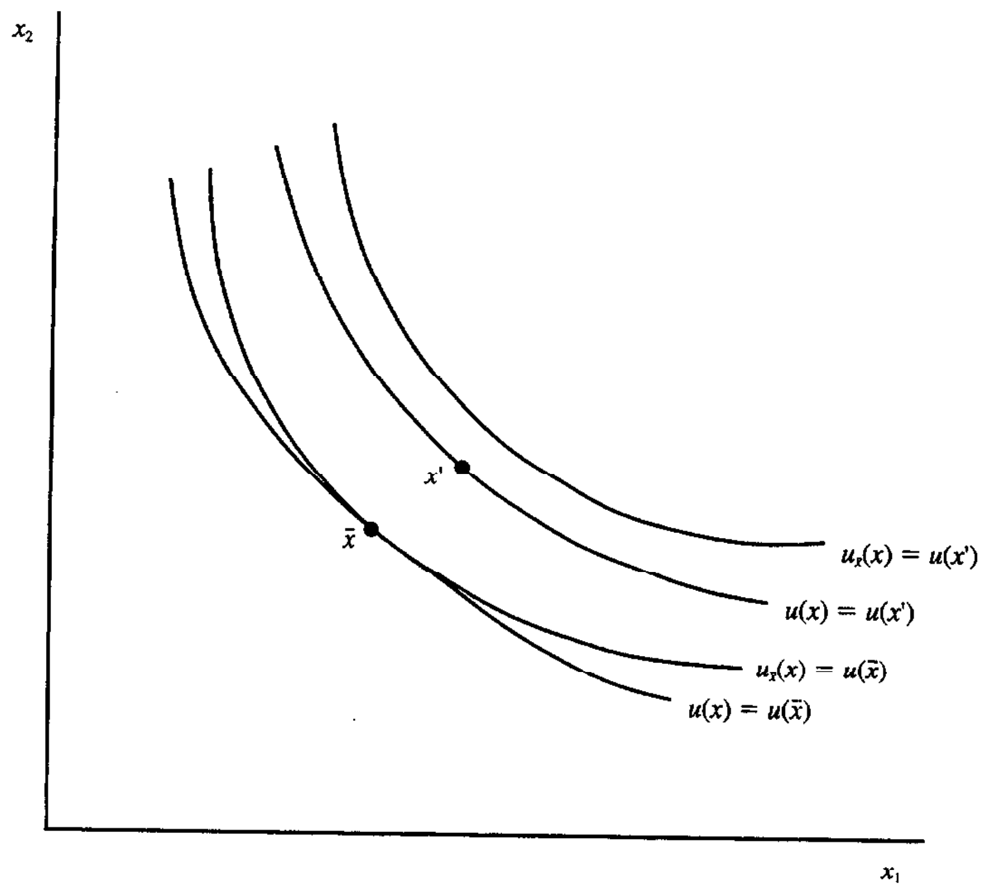

Figure 3.7 Mas-Colell's super-Cobb–Douglas condition

and then plugging this expression into the matrix whose determinant determines the index,

$$\begin{bmatrix} -\bar{J}+\bar{H} & \bar{B} \\ -\bar{B}^T & 0 \end{bmatrix} = \begin{bmatrix} 30 & 0 & -18 & 4 & -1 \\ 0 & 30 & -12 & -1 & 4 \\ -2 & -4 & 14 & -2 & -1 \\ -4 & 1 & 2 & 0 & 0 \\ 1 & -4 & 1 & 0 & 0 \end{bmatrix}.$$

Since the determinant of this matrix is 2520, index($\hat{p}$) = +1.

In fact, this economy has a unique equilibrium. One way to see this is to use Mas-Colell's (1991) argument that reduces the economy to a two-good exchange economy in the space of factors. For a vector of factor prices (p_3, p_4) we consider a solution to the problem

$$\begin{aligned} &\max v_i(z_3, z_4) \\ &\text{s.t. } p_3 z_3 + p_4 z_4 \leq p_3 w_3^i + p_4 w_4^i \\ &\qquad z_3, z_4 \geq 0. \end{aligned}$$

(Here we have left goods with their usual numbering to simplify notation. Our definition of $v_i(z)$ implies that this problem can be rewritten as

$$\begin{aligned} &\max a_1^i \log x_1 + a_2^i \log x_2 \\ &\text{s.t. } x_1 + v_{12} \leq 8^{0.5} v_{21}^{0.25} v_{31}^{0.5} v_{41}^{0.25} \\ &\qquad x_2 + v_{21} \leq 8^{0.5} v_{12}^{0.25} v_{32}^{0.25} v_{42}^{0.5} \\ &p_3(v_{31} + v_{32}) + p_4(v_{41} + v_{42}) \leq p_3 w_3^i + p_4 w_4^i \\ &\qquad\qquad x_j, v_{ij} \geq 0, \end{aligned}$$

where v_{ij} is the amount of good i used in the production of good j, and $z_i = v_{i1} + v_{i2}$, $i = 3, 4$. The solution to this problem is associated with Lagrange multipliers p_1, p_2, and λ for the three constraints. We are justified in setting $\lambda = 1$ because, if we solve this problem once, rescale p_3 and p_4 to be λp_3 and λp_4, then re-solve, we find that $\lambda = 1$. Letting y_j denote the total production of good j, we find that

$$\begin{bmatrix} 1 & -0.25 \\ -0.25 & 1 \end{bmatrix} \begin{bmatrix} p_1 y_1 \\ p_2 y_2 \end{bmatrix} = \begin{bmatrix} p_1 x_1 \\ p_2 x_2 \end{bmatrix} = (p_3 w_3^i + p_4 w_4^i) \begin{bmatrix} a_1^i \\ a_2^i \end{bmatrix}.$$

Similarly,

$$\begin{bmatrix} p_3 z_3 \\ p_4 z_4 \end{bmatrix} = \begin{bmatrix} p_3(v_{31} + v_{32}) \\ p_4(v_{41} + v_{42}) \end{bmatrix} = \begin{bmatrix} 0.5 & 0.25 \\ 0.25 & 0.5 \end{bmatrix} \begin{bmatrix} p_1 y_1 \\ p_2 y_2 \end{bmatrix}.$$

$$\begin{aligned} \begin{bmatrix} p_3 z_3 \\ p_4 z_4 \end{bmatrix} &= (p_3 w_3^i + p_4 w_4^i) \begin{bmatrix} 0.5 & 0.25 \\ 0.25 & 0.5 \end{bmatrix} \begin{bmatrix} 1 & -0.25 \\ -0.25 & 1 \end{bmatrix}^{-1} \begin{bmatrix} a_1^i \\ a_2^i \end{bmatrix} \\ &= (p_3 w_3^i + p_4 w_4^i) \begin{bmatrix} 0.6 a_1^i + 0.4 a_2^i \\ 0.4 a_1^i + 0.6 a_2^i \end{bmatrix}. \end{aligned}$$

This says that the imputed demands for factors are those of the Cobb–Douglas utility function

$$v_i(z_3, z_4) = (0.6a_1^i + 0.4a_2^i) \log z_3 + (0.4a_1^i + 0.6a_2^i) \log z_4.$$

The excess demands for factors in this economy are

$$\begin{bmatrix} \dfrac{0.55(5p_3 + p_4)}{p_3} + \dfrac{0.45(p_3 + 5p_4)}{p_3} - 6 \\[2ex] \dfrac{0.45(5p_3 + p_4)}{p_4} + \dfrac{0.55(p_3 + 5p_4)}{p_4} - 6 \end{bmatrix} = \begin{bmatrix} \dfrac{-14p_3 + 14p_4}{5p_3} \\[2ex] \dfrac{-14p_3 + 14p_4}{5p_4} \end{bmatrix}$$

The unique equilibrium of this two-good exchange economy is $(\hat{p}_3, \hat{p}_4) = (0.5, 0.5)$, which corresponds to the equilibrium that we have already found for the four-good production economy.

5 Economies with Tax Distortions

In this section, we generalize our analysis to economies that allow such distortions as taxes and tariffs. This sort of generalization is essential if our results are to be useful for applied work, since most applied general equilibrium models are designed to analyze such distortions (see Shoven and Whalley 1984 for a survey).

5.1 EQUILIBRIUM IN AN ECONOMY WITH TAXES

Consider an economy in which consumer i solves the problem

$$\begin{aligned} &\max u_i(x) \\ &\text{s.t.} \sum_{j=1}^{n} p_j(1 + \tau_j)x_j \leq p \cdot w^i + \theta_i r \\ &\qquad x \geq 0. \end{aligned}$$

Here $\tau_j \geq 0$ is the *ad valorem* tax on the consumption of good j and $\theta_i \geq 0$, $\sum_{i=1}^{m} \theta_i = 1$, is the share of total tax revenue that is rebated to consumer i as a lump sum. In many models $\theta_i = 0$, $i = 1, \ldots, m - 1$, and $\theta_m = 1$, where consumer m is the government.

We specify production possibilities using an $n \times k$ activity analysis matrix A that allows free disposal but no output without inputs. Production taxes are specified by another $n \times k$ matrix A^* where

$$a_{ij}^* = a_{ij} - \sigma_{ij} \mid a_{ij} \mid,$$

where $\sigma_{ij} \geq 0$ is the *ad valorem* tax on the input or output of commodity i in activity j. With this notation, the vector of after-tax profits, one for each activity, is $p \cdot A^*$, and the tax revenue generated by the production plan Ay is $p \cdot (A - A^*)y$.

In this economy an equilibrium is a price vector $\hat{p}$, a level of tax revenues $\hat{r}$, an allocation $\hat{x}^1, \ldots, \hat{x}^m$, and a vector of activity levels $\hat{y}$ such that

- Given $\hat{p}$ and $\hat{r}$, consumer i chooses $\hat{x}^i$ to solve the problem of maximizing utility subject to the budget constraint;
- $\hat{p} \cdot A^* \leq 0, \hat{p} \cdot A^*\hat{y} = 0$;
- $\sum_{i=1}^m \hat{x}^i = A\hat{y} + \sum_{i=1}^m w^i$;
- $\hat{r} = \sum_{j=1}^n \hat{p}_j(1 + \tau_j)\sum_{i=1}^m \hat{x}_j^i + \hat{p} \cdot (A - A^*)\hat{y}$.

The final condition requires that the amount of tax revenues that the consumers, including the government, take into account when making spending decisions is equal to the total amount collected as a result of these decisions.

In an economy involving several countries, we would model the same good available in different countries as different goods. A tariff would be a tax on the output of the activity that transforms the good in one country to the same good in another country. Kehoe (1985a) shows further how this framework can be extended to include specific, as well as *ad valorem*, taxes and subsidies and income taxes with any degree of progressivity.

In the specification of this economy that relies on an aggregate excess demand function, we set

$$f(p, r) = \sum_{i=1}^{m} (x^i(p, r) - w^i).$$

We also need to define a function that tells us the total taxes paid by consumers,

$$t(p, r) = \sum_{j=1}^{n} p_j(1 + \tau_j)\sum_{i=1}^{m} x_j^i(p, r).$$

The two functions $f(p, r)$ and $t(p, r)$ are continuous, at least for strictly positive p and nonnegative r; $f(p, r)$ is homogeneous of degree zero and $t(p, r)$ is homogeneous of degree one; together, $f(p, r)$ and $t(p, r)$ satisfy a modified version of Walras's law:

$$p \cdot f(p, r) + t(p, r) \equiv r.$$

An equilibrium is now $(\hat{p}, \hat{r}, \hat{y})$ such that

- $\hat{p} \cdot A^* \leq 0,\ \hat{p} \cdot A^*\hat{y} = 0$;
- $f(\hat{p}, \hat{r}) = A\hat{y}$;
- $\hat{r} = t(\hat{p}, \hat{r}) + \hat{p} \cdot (A - A^*)\hat{y}$.

5.2 REGULAR ECONOMIES AND THE INDEX THEOREM

Kehoe (1985a) extends the theory of regular economies and the index theorem to economies with tax distortions by constructing a function $g(p, r)$ that continuously maps a compact, convex set into itself, whose fixed points are equilibria, and which is almost always continuously differentiable at its fixed points.

If we restrict attention to economies where $\hat{r}$ is always positive, a regular economy is one for which the matrix

$$\begin{bmatrix} -D_1 f(\hat{p}, \hat{r}) & B \\ -B^{*T} & 0 \end{bmatrix}$$

is nonsingular at every equilibrium. (See Kehoe 1985a for the case where $\hat{r} = 0$.) Here $D_1 f(p, r)$ is the $n \times n$ matrix of partial derivatives of $f(p, r)$ with respect to p; once again B is the $n \times \ell$ submatrix of A whose columns are those activities used at positive levels in equilibrium; and B^* is the corresponding submatrix of A^*.

We can think of this matrix as the Jacobian matrix of the functions whose zero is an equilibrium:

$$\begin{aligned} -f(p, r) + By &= 0 \\ -B^{*T}p &= 0 \\ t(p, r) - r + p \cdot (B - B^*)y &= 0. \end{aligned}$$

We use homogeneity to normalize $r = \hat{r}$, and we use Walras's law to drop the final equation.

The index theorem says that if we set

$$\text{index}(\hat{p}, \hat{r}) = \text{sgn}\left(\det\begin{bmatrix} -D_1 f(\hat{p}, \hat{r}) & B \\ -B^{*T} & 0 \end{bmatrix}\right),$$

then

$$\sum_{(p,r) = g(p,r)} \text{index}(p, r) = +1.$$

In the more general case of production technologies described by profit functions, the crucial matrix becomes

$$\begin{bmatrix} -D_1 f(\hat{p}, \hat{r}) + H(\hat{p}) & B(\hat{p}) \\ -B^{*T}(\hat{p}) & 0 \end{bmatrix},$$

where $H(\hat{p})$ is defined as in section 4.7.

5.3 EXAMPLE 4

Unfortunately, even the extremely restrictive assumptions of the nonsubstitution theorem and of a representative consumer – each of which are sufficient separately to imply uniqueness in economies without distortions – are not sufficient to imply uniqueness with distortions. The point has been made by Foster and Sonnenschein (1970) and Hatta (1977), who present graphic examples of economies with representative consumers and input–output structures but with multiple equilibria.

To see the possibility of a multiplicity of equilibria even in an economy that satisfies these restrictive assumptions, consider an economy in which the consumer has utility for consumption of two produced goods and endowment only of the third, nonproduced good. Specifically, the utility function is

$$u(x_1, x_2, x_3) = \begin{cases} -(14 - x)^2(x_2 + 1)^{-1} & \text{if } x_1 \leq 14 \\ x_1 - 14 & \text{if } x_1 \geq 14 \end{cases}$$

and the endowment vector is $w = (0, 0, 4)$. Suppose that production possibilities are given by the activity analysis matrix

$$A = \begin{bmatrix} -1 & 0 & 0 & 9 & -1 \\ 0 & -1 & 0 & -1 & 3 \\ 0 & 0 & -1 & -2 & -2 \end{bmatrix}.$$

We assume that the representative consumer pays no taxes, but that producers pay an *ad valorem* tax $\sigma_{14} = 2/3$ on output of good 1:

$$A^* = \begin{bmatrix} -1 & 0 & 0 & 3 & -1 \\ 0 & -1 & 0 & -1 & 3 \\ 0 & 0 & -1 & -2 & -2 \end{bmatrix}.$$

(Equivalently, we could set $a^*_{14} = 9$ and $a^*_{15} = -3$ and have consumers pay a tax $\tau_1 = 2$ on purchases of good 1.)

The excess demand function is

$$f_1(p, r) = \begin{cases} 0 & \text{if } 8p_3 + 2r \leq 14p_1 - 2p_2 \\ (-14p_1 + 2p_2 + 8p_3 + 2r)/p_1 & \text{if } 4p_3 + r \leq 14p_1 - 2p_2 \leq 8p_3 + 2r \\ (4p_3 + r)/p_1 & \text{if } 14p_1 - 2p_2 \leq 4p_3 + r \end{cases}$$

$$f_1(p, r) = \begin{cases} (4p_3 + r)/p_2 & \text{if } 8p_3 + 2r \leq 14p_1 - 2p_2 \\ (14p_1 - 2p_2 - 4p_3 - r)/p_2 & \text{if } 4p_3 + r \leq 14p_1 - 2p_2 \leq 8p_3 + 2r \\ 0 & \text{if } 14p_1 - 2p_2 \leq 4p_3 + r \end{cases}$$

$$f_3(p, r) = -4.$$

This economy has three equilibria, which are listed below and depicted in figure 3.8.

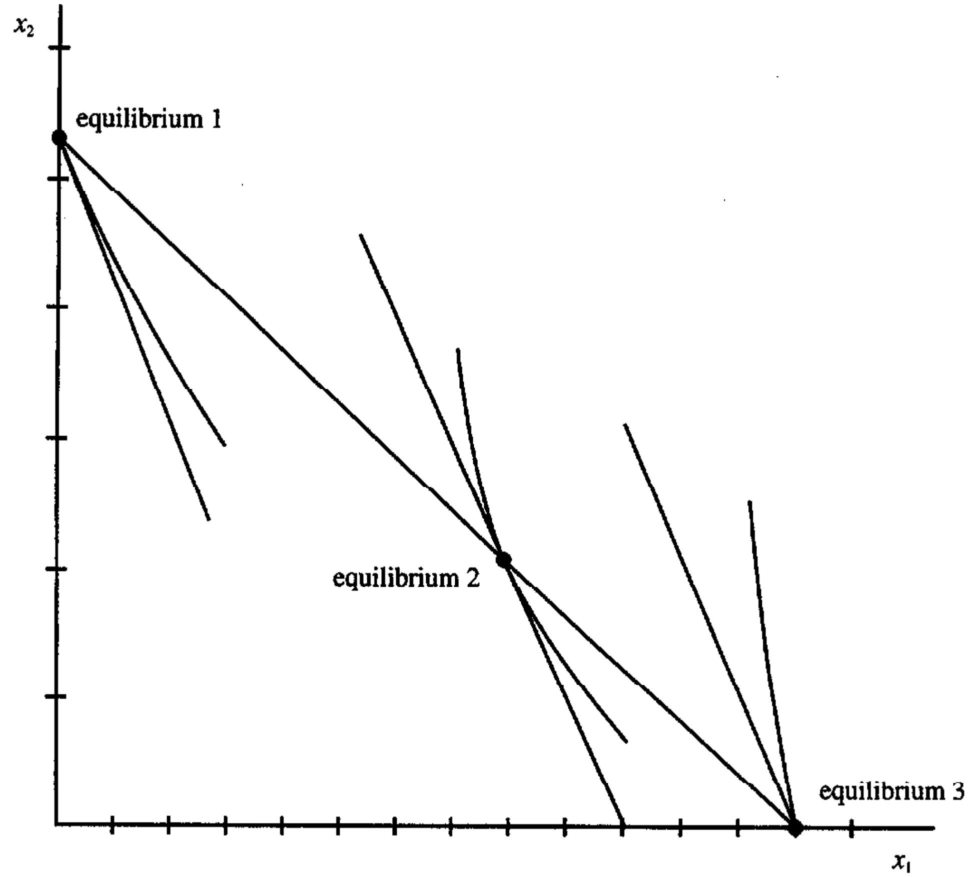

Figure 3.8 Nonuniqueness in the example with distortionary taxes

Equilibrium 1

	1	2	3
$\hat{p}_j$	0.3333	0.33333	0.3333
$\hat{x}_j$	0	5.2	0

$\hat{r} = 0.4$, $u = -31.6129$

$\hat{y} = (0, 0, 0, 0.2, 1.8)$

Equilibrium 2

	1	2	3
$\hat{p}_j$	0.3333	0.3333	0.3333
$\hat{x}_j$	8	2	0

$\hat{r} = 2$, $u = -21.3333$

$\hat{y} = (0, 0, 0, 1, 1)$

Equilibrium 3

	1	2	3
$\hat{p}_j$	0.3333	0.3333	0.3333
$\hat{x}_j$	13	0	0

$\hat{r} = 3, u = -1.0000$
$\hat{y} = (0, 0, 0, 1.5, 0.5)$

This example has been constructed by choosing parameters so that at equilibrium 2 index$(\hat{p}) = -1$.

$$\begin{bmatrix} -D_1 f(\hat{p}, \hat{r}) & B \\ -\bar{B}^{*T} & 0 \end{bmatrix} = \begin{bmatrix} 66 & -6 & -24 & 9 & -1 \\ -42 & 12 & 12 & -1 & 3 \\ 0 & 0 & 0 & -2 & -2 \\ -3 & 1 & 2 & 0 & 0 \\ 1 & -3 & 2 & 0 & 0 \end{bmatrix}$$

has determinant -192. Consequently, the economy necessarily has multiple equilibria. Notice that these equilibria are Pareto-ranked.

5.4 A SUFFICIENT CONDITION FOR UNIQUENESS

We have a mathematical condition for uniqueness of equilibrium in economies with distortionary taxes that is sufficient and almost always necessary – that the index be positive at every equilibrium. The problem is in translating this mathematical condition into easy-to-check economic conditions.

In this section we present a set of such conditions that rule out the multiple equilibria of the example in the previous section. As Foster and Sonnenschein (1970) and Hatta (1977) point out, such an example depends on one of the goods being inferior, which means that $\partial f_i(\hat{p}, \hat{r})/\partial r > 0$ for some i. Kehoe (1985a) employs this observation and the index theorem to develop a set of sufficient conditions for uniqueness:

- suppose that the economy $(f(p), t(p), A, A^*)$ satisfies the conditions of the nonsubstitution theorem presented in section 4.2;
- letting B^* be the matrix of $(n - 1)$ efficient activities guaranteed by the nonsubstitution theorem and letting $\bar{p} \in S$ be the unique vector of efficiency prices associated with the production matrix B in the sense that $\bar{p} \cdot B = 0$, suppose that $\bar{p} \cdot D_2 f(p, r) > 0$ for any choice of p and r.

To see why these conditions work, we partition the matrix whose determinant determines the index,

$$\begin{bmatrix} -D_1 f(\hat{p},\hat{r}) & B \\ -\bar{B}^{*T} & 0 \end{bmatrix} = \begin{bmatrix} -D_{11} & -d_{12} & B_1 \\ -d_{21} & -d_{22} & b_2 \\ -B_1^{*T} & -b_2^{*T} & 0 \end{bmatrix}$$

where D_{11} is $(n-1) \times (n-1)$, d_{12} is $(n-1) \times 1$, d_{21} is $1 \times (n-1)$, d_{22} is 1×1, and so on. We have numbered goods, so that the last is the nonproduced factor of production. For production to be feasible, the $(n-1) \times (n-1)$ matrix B_1 must be a productive Leontief matrix and, hence, have a positive determinant. Moreover, $(\hat{p}_1, \ldots, \hat{p}_{n-1}) \cdot B_1^* = -\hat{p}_n b_2^* \geqq 0$ implies that B_1^* must also be a productive Leontief matrix. Notice too that the homogeneity of degree zero of $f(p, r)$ implies that

$$D_1 f(\hat{p},\hat{r})\hat{p} + D_2 f(\hat{p},\hat{r})\hat{r} = 0.$$

Multiplying each column $j = 1, \ldots, n-1$ of the matrix whose determinant determines the index by $\hat{p}_j$ and adding it to column n multiplied by $\hat{p}_n$, we see that the determinant of the original matrix in multiplied by $\hat{p}_n$, and column n becomes

$$\begin{bmatrix} D_2 f(\hat{p},\hat{r})\hat{r} \\ 0 \end{bmatrix}.$$

Now multiplying each row, $i = 1, \ldots, n-1$ of the matrix by $\bar{p}_i$, adding it to row n multiplied by $\bar{p}_n$, then dividing row n by $\hat{r}$, we see that

$$\det\begin{bmatrix} -D_1 f(\hat{p},\hat{r}) & B \\ -\bar{B}^{*T} & 0 \end{bmatrix} = \frac{\hat{r}}{\hat{p}_n \bar{p}_n} \det\begin{bmatrix} * & * & B_1 \\ * & \bar{p} \cdot D_2 f(\hat{p},\hat{r}) & 0 \\ -B_1^{*T} & 0 & 0 \end{bmatrix}$$

$$= \left(\frac{\hat{r}}{\hat{p}_n \bar{p}_n}\right)(\bar{p} \cdot D_2 f(\hat{p},\hat{r})(\det[B_1])(\det[B_1^*]),$$

whose sign depends solely on that of $\bar{p} \cdot D_2 f(\hat{p},\hat{r})$. (The elements marked * in the matrix are of no consequence.) Our uniqueness condition is that a weighted sum of income effects be positive, where the weights are the efficiency prices uniquely determined by the production technology. To see that this condition is violated by the example of the previous section, we use

$$[\bar{p}_1 \ \bar{p}_2 \ \bar{p}_3] \cdot \begin{bmatrix} 9 & -1 \\ -1 & 3 \\ -2 & -2 \end{bmatrix} = [0 \quad 0]$$

and the price normalization $\sum_{j=1}^{3} \bar{p}_j = 1$ to calculate $(\bar{p}_1, \bar{p}_2, \bar{p}_3) = (0.1481, 0.3704, 0.5185)$. We then calculate

0.4815

$$[\bar{p}_1\ \bar{p}_2\ \bar{p}_3] \cdot D_2 f(\hat{p}, \hat{r}) = [0.1481\ 0.3704\ 0.5185] \begin{bmatrix} 6 \\ -3 \\ 0 \end{bmatrix} = -0.2222.$$

6 Concluding Remarks

Given the importance of the topic, it is not surprising to find that a large amount of effort has gone into developing conditions that guarantee uniqueness of general equilibrium. Unfortunately, the conditions developed so far are probably too restrictive to be immediately relevant for applied work. In particular, there has been too little effort devoted to research on uniqueness in production economies, and there has been even less effort devoted to research on uniqueness in economies with distortionary taxes.

Applied models usually involve productions, distortionary taxes and tariffs, and even more features that complicate the uniqueness issue. Models of international trade, for example, often include increasing returns and imperfect competition (see Kehoe and Kehoe 1995 for a survey). Mercenier (1995) reports a problem with non-uniqueness of equilibrium in such an applied model designed to analyze economic integration in Europe.

Increasingly, applied models are also including time and uncertainty in such a way that, at least in principle, there are infinite numbers of goods. (In practice, these models are often truncated and solved on a computer.) Unlike economies with finite numbers of goods, which almost always have a finite number of equilibria, there are robust examples of economies with infinite numbers of goods that have continua of equilibria. Kehoe and Levine (1985, 1990) discuss this indeterminacy of equilibrium and show its relationship to sensitivity to terminal conditions in truncated versions of such economies. Furthermore, Kehoe, Levine, and Romer (1992) show that even in economies where there are usually a finite number of equilibria – such as economies with a finite number of consumers – the addition of distortionary taxes or externalities can lead to robust indeterminacy.

Economies with time and uncertainty that have infinite numbers of goods clearly present another set of issues that need to be studied. There is some hope, however, that many of the results developed for models with infinite numbers of goods may be relevant. Kehoe, Levine, Mas-Colell, and Woodford (1991), for example, find that gross substitutability has strong implications for uniqueness in exchange economies with infinite numbers of goods. It would be interesting to see if results like that of Mas-Colell (1991) on the equilibrium in factor markets of a generalized input–output model in which all utility functions and production functions are super-Cobb–Douglas can be extended to economies with infinite numbers of goods.

There is one result in Kehoe, Levine, Mas-Colell, and Woodford (1991) that is so simple and powerful that it is worth giving here. Suppose that

$f_i(p_1, p_2, \ldots)$, $i = 1, 2, \ldots$, is the excess demand function for an economy with an infinite number of goods. We assume that $f(p)$ is homogeneous of degree zero and obeys a version of Walras's law that says that $p \cdot f(p) = 0$ whenever $p \cdot w < \infty$, where $w = (w_1, w_2, \ldots)$ is the aggregate endowment vector. (We interpret inner products like $p \cdot w$ to be

$$p \cdot w = \lim_{t \to \infty} \sum_{i=1}^{t} p_i w_i.)$$

Assume that $f(p)$ exhibits gross substitutability: if $p \geq q$ and $p_i = q_i$ for some i, then $f_i(p) \geq f_i(q)$, and if $f(p) = f(q)$, then $p = q$. Suppose that there exists $\hat{p}$ such that $f(\hat{p}) = 0$ and $\hat{p} \cdot w < \infty$. Then $\hat{p}$ is the unique equilibrium of $f(p)$.

To see why this is so, assume that $f(p) = f(q) = 0$ and $p \cdot w < \infty$. Normalize prices so that $p_1 = q_1 = 1$, and define a new price sequence $\bar{p} = (\bar{p}_1, \bar{p}_2, \ldots)$ by setting $\bar{p}_i = \min(p_1, q_1)$ for all i. If $\bar{p}_i = p_i$, then gross substitutability implies that $f_i(\bar{p}) \leq f_i(p) = 0$. Similarly, if $\bar{p}_i = q_i$, then $f_i(\bar{p}) \leq f_i(q) = 0$. Therefore $f(\bar{p}) \leq 0$. Furthermore, since $\bar{p} \leq p$, $\bar{p} \cdot w \leq p \cdot w < \infty$. Consequently, Walras's law implies that $\bar{p} \cdot f(\bar{p}) = 0$, which implies that $f(\bar{p}) = 0$. Gross substitutability now implies that $\bar{p} = p$ and that $p = q$.

Kehoe, Levine, Mas-Colell, and Woodford (1991) present further results for economies like overlapping generations economies, where there need not be any equilibrium that satisfies $\hat{p} \cdot w < \infty$ and where, because of fiat money, the excess demand function need not obey Walras's law. Even in these economies, gross substitutability has strong implications.

At the same time that we extend the analysis of uniqueness to more general economies, we need to develop new techniques. The index theorem seems to be the most general tool available, but so far it has been mostly used to generalize previously known results and to generate counterexamples. It needs to be put to better use.

One direction to go in would be to use regularity theory to develop conditions dealing with the distance between economies. Not only do we know that if a regular production economy has an aggregate demand function that satisfies the weak axiom, then it has a unique equilibrium, for example; we also know that if another economy is close enough to it, then it too has a unique equilibrium. Similarly, if the tax distortions are small enough in an economy that otherwise satisfies conditions that guarantee uniqueness, then it has a unique equilibrium.

Another approach to studying the uniqueness question especially relevant in applied work would be to develop algorithms capable of checking if a given equilibrium is unique. Garcia and Zangwill (1981: ch. 18) discuss a method that is based on approximating the function $g(p)$ whose fixed points are equilibria by a polynomial function and that is capable, at least in principle, of calculating all equilibria (for discussion and references, see Kehoe 1991). Dakhlia (1995) presents some preliminary economic applications. This too is an area that deserves more research.

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