

## The model

The representative household solves

$$(1) \quad \sum_{t=T_0}^{\infty} \beta^t \left( \gamma \log C_t + (1-\gamma) \log(\bar{h}N_t - L_t) \right)$$

subject to a sequence of budget constraints,

$$(2) \quad C_t + K_{t+1} = w_t L_t + (1 - \delta + r_t) K_t,$$

nonnegativity constraints on  $C_t$  and  $K_t$ , and a constraint on the initial stock of capital,  $\bar{K}_{T_0}$ .

Firms operate in a perfectly competitive market, using a constant returns to scale technology, which we assume to be Cobb-Douglas:

$$(3) \quad Y_t = A_t K_t^\alpha L_t^{1-\alpha}.$$

The conditions that firms earn zero profits and minimize costs provide expressions for the factor prices:

$$(4) \quad w_t = (1 - \alpha) A_t K_t^\alpha L_t^{-\alpha}$$

$$(5) \quad r_t = \alpha A_t K_t^{\alpha-1} L_t^{1-\alpha}.$$

The feasibility constraint is

$$(6) \quad C_t + K_{t+1} - (1 - \delta) K_t = A_t K_t^\alpha L_t^{1-\alpha}.$$

## Computation of Equilibrium

The paths of TFP and population are exogenously given, and the agent has perfect foresight over their values. We start the model at date  $T_0 = 1980$  and let time run out to infinity.

**Definition.** Given sequences of productivity,  $A_t$ , and working-age population,  $N_t$ ,  $t = T_0, T_0 + 1, \dots$ , and the initial capital stock,  $\bar{K}_{T_0}$ , an *equilibrium* is sequences of wages,  $w_t$ , interest rates,  $r_t$ , consumption,  $C_t$ , labor,  $L_t$ , and capital stocks,  $K_t$ , such that

1. given the wages and interest rates, the representative household chooses consumption, labor, and capital to maximize the utility

function (1) subject to the budget constraints (2), appropriate nonnegativity constraints, and the constraint on  $\bar{K}_{T_0}$ ;

2. the wages and interest rates, together with the firms' choices of labor and capital, satisfy the cost minimization and zero profit conditions, (4) and (5); and
3. consumption, labor, and capital satisfy the feasibility condition (6).

We turn these equilibrium conditions into a system of equations that can be solved to find the equilibrium of the model. We begin by taking first-order conditions of the household's problem of maximizing the utility function (1) subject to the budget constraint (2) to obtain

$$(7) \quad w_t (\bar{h}N_t - L_t) = \frac{1-\gamma}{\gamma} C_t$$

$$(8) \quad \frac{C_{t+1}}{C_t} = \beta(1 - \delta + r_{t+1}).$$

Combining the household's optimality conditions (7) and (8), the firm optimality conditions (4) and (5), and the feasibility condition (6), we can specify a system of equations that can be solved to find the equilibrium of the model.

Before explaining how to calculate the whole equilibrium path, let us explain how to calculate a balanced-growth path for this model.

**Definition.** Suppose that productivity,  $A_t$ , grows at the constant rate  $g^{1-\alpha} - 1$  and that working-age population grows at the constant rate  $n - 1$ , then a *balanced-growth path* is levels of the wage,  $\hat{w}$ , the interest rate,  $\hat{r}$ , consumption,  $\hat{C}$ , labor,  $\hat{L}$ , the capital stock,  $\hat{K}$ , and output,  $\hat{Y}$ , such that  $w_t = g^{t-T_0} \hat{w}$ ,  $r_t = \hat{r}$ ,  $C_t = (gn)^{t-T_0} \hat{C}$ ,  $L_t = n^{t-T_0} \hat{L}$ ,  $K_t = (gn)^{t-T_0} \hat{K}$ ,  $Y_t = (gn)^{t-T_0} \hat{Y}$  satisfy the conditions for an equilibrium when the initial capital stock is  $K_{T_0} = \hat{K}$ .

To solve for the balanced-growth path, we use (5) and (8) to solve for the capital-output ratio  $\hat{K} / \hat{Y}$ ,

$$(9) \quad g = \beta \left( 1 + \alpha \frac{\hat{Y}}{\hat{K}} - \delta \right).$$

We then use (4) and (6) to rewrite (7) as

$$(10) \quad (1 - \alpha) \left( \bar{h} \frac{N_{T_0}}{\hat{L}} - 1 \right) = \frac{1 - \gamma}{\gamma} \left( 1 - (gn - 1 + \delta) \frac{\hat{K}}{\hat{Y}} \right)$$

and use this equation to calculate labor,  $\hat{L}$ . We can then use the production function (3) to solve for  $\hat{K}$  and  $\hat{Y}$ . Using the feasibility condition (6), we can then solve for  $\hat{C}$ , and, using the firm optimality conditions (4) and (5), we can solve for  $\hat{w}$  and  $\hat{r}$ .

We now return to the calculation of the equilibrium path. Plugging the prices (4) and (5) into the household's optimality conditions (7) and

(8), and using the feasibility condition (6), we obtain the system of equations

$$(11) \quad (1-\alpha)A_t K_t^\alpha L_t^{-\alpha} (\bar{h}N_t - L_t) = \frac{1-\gamma}{\gamma} C_t$$

$$(12) \quad \frac{C_{t+1}}{C_t} = \beta \left( 1 - \delta + \alpha A_{t+1} K_{t+1}^{\alpha-1} L_{t+1}^{1-\alpha} \right)$$

$$(13) \quad C_t + K_{t+1} - (1-\delta)K_t = A_t K_t^\alpha L_t^{1-\alpha}.$$

Solving for an equilibrium involves choosing sequences of consumption, capital stocks, and hours worked such that these equations are satisfied, given the initial condition  $K_{T_0}$  and final condition, the transversality condition,

$$(14) \quad \lim_{t \rightarrow \infty} \beta^t \frac{\gamma}{C_t} K_{t+1} = 0.$$

In principle, the system of equations that characterize the equilibrium, (11)–(13), involves an infinite number of equations and unknowns. To make the computation of an equilibrium tractable, we assume that the economy converges to the balanced-growth path at some date  $T_1$ , which allows us to truncate the system of equations. Using the feasibility condition (6) to solve for  $C_t$ , we can write these equations as

$$(15) \quad (1 - \alpha) A_t K_t^\alpha L_t^{-\alpha} (\bar{h} N_t - L_t) = \frac{1 - \gamma}{\gamma} (A_t K_t^\alpha L_t^{1 - \alpha} - K_{t+1} + (1 - \delta) K_t),$$

$$t = T_0, T_0 + 1, \dots, T_1$$

$$(16) \quad \frac{A_{t+1}K_{t+1}^\alpha L_{t+1}^{1-\alpha} - K_{t+2} + (1-\delta)K_{t+1}}{A_t K_t^\alpha L_t^{1-\alpha} - K_{t+1} + (1-\delta)K_t} = \beta \left( 1 - \delta + \alpha A_{t+1} K_{t+1}^{\alpha-1} L_{t+1}^{1-\alpha} \right),$$

$$t = T_0, T_0 + 1, \dots, T_1 - 1,$$

where  $K_{T_1+1} = gnK_{T_1}$ .

We choose  $T_1$  so that  $T_1 - T_0$  is large, say 60, so that we are solving the model over the period 1980–2040. We then construct the exogenous variables. The exogenous variables  $A_t$ ,  $N_t$  for 1980–2005 are as they are in the data. For 2006–40, we assume that TFP grows at a constant rate equal to the average growth rate of TFP over the period 1980–2005 and that the working-age population grows at the same rate as in 2004–5.

These are the growth rates  $g^{1-\alpha}$  and  $n$  in the specification of the balanced-growth path.

Solving the model now consists of choosing  $K_{T_0+1}, K_{T_0+2}, \dots, K_{T_1}$ , and  $L_{T_0}, L_{T_0+1}, \dots, L_{T_1}$  to solve the system of equations (15) and (16). This system of  $2(T_1 - T_0) - 1$  nonlinear equations in  $2(T_1 - T_0) - 1$  unknowns can be solved relatively quickly using numerical methods. A set of MATLAB programs for solving this model are available at [www.greatdepressionsbook.com](http://www.greatdepressionsbook.com).