Numerical Methods for Ph. D Students in Economics

Homework 1

1. (Numerical derivatives) Write a pseudocode and a subroutine that computes partial derivatives (or, in other words, a gradient) of one-dimensional functions and apply it to the following functions:

   (a) $f(x) = (x-1)(x^2-2)(x^3-3)$

   (b) $f(x,y) = 3x^2 + xy^2$

   (c) $f(x,y,z) = (\log x + 2\log y)/e^z$

   (d) $f(x;a,b) = ax^2 + bx$, for $(a,b) = (1,1), (-3,5), (2,3)$.

   (Suggestion: First write a subroutine which is applicable to the first three functions and then extend it to the last case.)

2. (Bisection method) Write a pseudocode and a subroutine that implements bisection method and use it for the following equations:

   (a) $x^2 - 1 = 0$, $0 \leq x \leq 4$.

   (b) $\sqrt{x} - 2^{-x} = 0$, $0 \leq x \leq 7$.

   (c) $x^4 - 4x^3 - 23x^2 + 54x + 72 = 0$, $0 \leq x \leq 5$.

   (d) $f(x;a) = ax - e^{-(x-1)} = 0$, $0 \leq x \leq 5$ for $a = 1,2,3$.

3. (Newton's method) Write a pseudocode and a subroutine that implements Newton's method for one-dimensional equations and apply it to the following:

   (a) $x^2 - 1 = 0$.

   (b) $f(x) = 0$, where $f(x) = \begin{cases} x-2 & \text{for } x < -1 \\ -x^3 + 4x & \text{for } -1 \leq x \leq 1 \\ x+2 & \text{for } x > 1 \end{cases}$

   Does the iteration converge when you choose $x_0 = 0.5$? What about $x_0 = 2$?

   (c) $f(x;a) = ax - e^{-(x-1)} = 0$, for $a = 1,2$.

4. (Optional: Jacobian) Write a pseudocode and a subroutine that computes Jacobian matrices of vector-valued functions and apply it to the following:

   (a) $f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \end{bmatrix}$, $f_1(x) = x^3$, $f_2(x) = x^2$, $f_3(x) = x$.

   (b) $f(x,y) = \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \end{bmatrix}$, $f_1(x) = x^2y$, $f_2(x) = x + y^2$.

(c) $f(x, y; a, b) = \begin{bmatrix} f_1(x, y; a, b) \\ f_2(x, y; a, b) \end{bmatrix}$, $f_1(x) = x^a y,\ f_2(x) = x + by$.

5. (Optional: Lagrange interpolation) The Lagrange interpolation problem is to find N-th order simple polynomials ($g(x) = \sum_{i=0}^{N} \theta_i x^i$) so that for a given collection of data points $(x_i, y_i)_{i=1}^{N+1}$ (where x_i's are distinct) $g(x_i) = y_i$ for all i. Write a pseudocode and a subroutine that solves this problem (for the polynomial coefficients) and apply it to the following sets of data points.

   (a) $x_i \in \{0, 2, 3, 5\}$, $y_i = e^{x_i}$.

   (b) $x_i \in \{1, 2, 3, 5\}$, $y_i = \log x_i$.

   (c) Redo (a) and (b) for $N = 5$ (you need to add two more data points) and compare the results using figures.

6. (Optional: Orthogonal polynomials) Write a pseudocode and a subroutine that generates coefficients of orthogonal polynomials for $n = 0, 1, ..., N$ using recursion formulae.

   (a) (Chebyshev) $T_0(x) = 1$, $T_1(x) = x$, $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$. $N = 10$.

   (b) (Legendre) $P_0(x) = 1$, $P_1(x) = x$, $P_{n+1}(x) = \frac{2n+1}{n+1} x P_n(x) - \frac{n}{n+1} P_{n-1}(x)$. $N = 7$.